PROBLEM SET 11 (POSTED ON THURSDAY, NOV 13)

(All Exercises are references to the September 8, 2024 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Recall that any scheme has a unique morphism to $\operatorname{Spec} \mathbb{Z}$, so can naturally be viewed as a $\mathbb{Z}$-scheme. Suppose that X is an A-scheme. Show that X is separated as an A-scheme if and only if X is separated as a $\mathbb{Z}$-scheme.

Problem 2. Suppose $\sigma : X \to Y$ is a section of some morphism $\tau : Y \to X$, i.e. $\tau \circ \sigma$ is the identity on X. First show that if τ is separated, then σ is a closed embedding. Then modify your argument to show that even if τ is not separated, σ is a locally closed embedding. (You may want to read section 9.2 on locally closed embeddings since they haven't come up in class, also you may want to use Proposition 11.2.1(b).)

Problem 3. Exercise 11.3.B (when are morphisms determined by where they send closed points? - you may want to read the preceding exercise/minor remarks and also use the result of Exercise 5.3.F (a consequence of Exercises 3.6.J, which was on an earlier homework).

Problem 4. Let $n \geq 2$ be an integer. Compute the (maximal) domain of definition of the generalized Cremona transformation

$$C : \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^n_{\mathbb{C}},$$

a rational map given by $[x_0 : \cdots : x_n] \mapsto [x_0^{-1} : \cdots : x_n^{-1}]$ (on closed points with $x_0 \cdots x_n \neq 0$ - your first task is to figure out how to construct such a map!). (Hint: in general showing that a rational map $X \dashrightarrow Y$ has domain of definition U has two parts: constructing a morphism $\pi : U \to Y$ representing the rational map and showing that for each point $p \in X \setminus U$ and each open neighborhood V of p containing U, there does not exist a morphism $V \to Y$ representing the rational map. Two possible approaches for this latter part: first, you can sometimes show that there is no continuous extension of π to V by just looking at multiple different subsets of U with closure containing the point p. Second, you can pick an affine open neighborhood $\operatorname{Spec} A$ of p as well as an affine open cover $\{\operatorname{Spec} B_i\}$ of Y, and then you can try to show that there is never any morphism $D_{\operatorname{Spec} A}(f) \to \operatorname{Spec} B_i$ agreeing with π, where $D_{\operatorname{Spec} A}(f)$ is any distinguished open containing p. This can then be translated into algebra.)