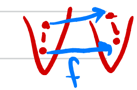
① Intro

Reminder: A Turing invariant function $f: 2^\omega \to 2^\omega$ is

- **order preserving** if $\forall x, y \in 2^\omega \, (x \geq_T y \Rightarrow f(x) \geq_T f(y))$ 
- **measure preserving** if $\forall z \underbrace{\exists y \forall x \, (x \geq_T y \Rightarrow f(x) \geq_T z)}$

  ↪ f gets above z on a cone

Thm (Slaman-Steel) Part 2 of Martin's conjecture holds for all order preserving functions which are not above the hyperjump

last week ↙

Thm Part 1 of Martin's conjecture holds for all measure preserving functions

~~$f(x)$ constant on a cone~~ or $f(x) \geq_T x$ on a cone

Thm Every order preserving function is either constant on a cone or measure preserving

today ↙

Cor Part 1 of Martin's conjecture holds for all order preserving functions

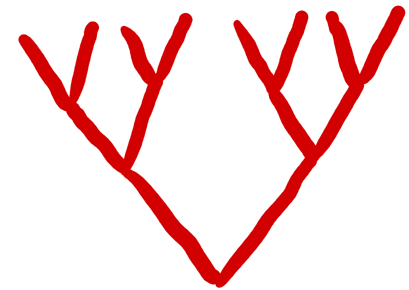
Goal for today: Every order preserving function is either constant on a cone or measure preserving

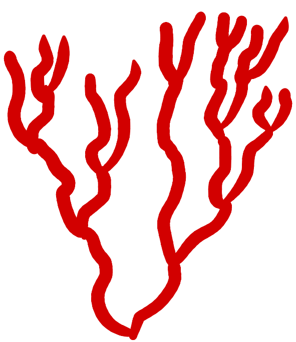
"order preserving ⇒ measure preserving"

Main tool: Basis theorem for perfect sets

1. Explain basis thm
2. Show order preserving ⇒ measure preserving
3. Prove basis thm

Perfect Sets and Martin's Conjecture for Order Preserving Functions.



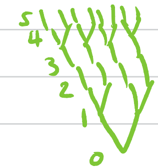


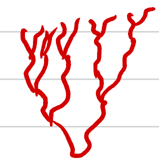
② A Basis Theorem for Perfect Sets

<u>Def</u> A set $A \subseteq 2^\omega$ is **perfect** if it is nonempty, closed and has no isolated points

↪ technically correct, but not very intuitive

<u>Def</u> A tree $T \subseteq 2^{<\omega}$ is **perfect** if every node has incomparable descendants





↪ Note: T has no dead ends

<u>Example</u> $T = \{ \sigma \in 2^{<\omega} \mid \sigma(p) = 0 \text{ for all primes } p \in \mathrm{dom}(\sigma) \}$

<u>Notation</u> If T is a tree, **$[T]$** denotes the set of infinite paths through T

<u>Prop</u> If T is perfect then $[T]$ is homeomorphic to 2^ω

<u>Prop</u> $A \subseteq 2^\omega$

A is perfect $\iff$ there is a perfect tree T such that $A = [T]$

<u>Basis thm, version 1</u> Suppose $A \subseteq 2^\omega$ is perfect and T is a perfect tree such that $A = [T]$. Then for all $x \in 2^\omega$ there is some $y \in A$ such that

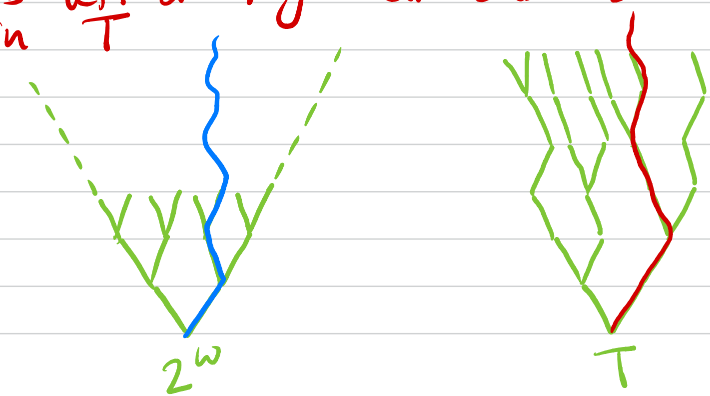
$$x \leq_T y \oplus T$$

<u>proof</u> Using T, compute a homeomorphism $f : A \to 2^\omega$ and pick $y \in A$ s.t. $f(y) = x$

What this means: Encode bits of x into decisions about whether y turns left or right at each branch point it encounters in T

$x = 101\ldots$

$y = 1\,1\,0\,0\,1\,0\ldots$



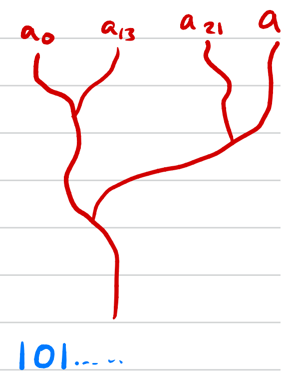
Basis thm, version 2 Suppose $A \subseteq 2^\omega$ is perfect and $\langle a_0, a_1, a_2, \dots \rangle$ is a countable dense subset of A. Then for all $x \in 2^\omega$ there is some $y \in A$ such that

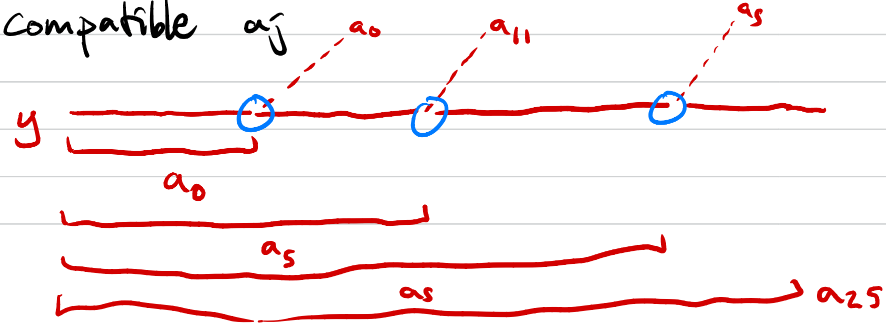
$$x \le_T y \oplus \left(\bigoplus_{n \in N} a_n \right)$$

Obstacle Don't know a tree T s.t. $[T] = A$ so don't know when y encounters branch pts

Solution Use $\langle a_n \rangle$ to compute a perfect subtree of T

What this means Encode the bits of x into decisions about whether to continue following current a_i or to switch to the next compatible a_j





Basis thm, version 3 Suppose $A \subseteq 2^\omega$ is perfect and $\langle a_0, a_1, a_2, \ldots \rangle$ contains a countable dense subset of A. Then for all $x \in 2^\omega$, there are $y_0, y_1, y_2 \in A$ such that

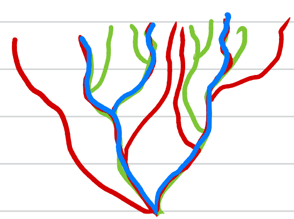
$$x \leq_T y_0 \oplus y_1 \oplus y_2 \oplus \left(\bigoplus_{n \in \mathbb{N}} a_n\right)$$

Groszek–Slaman (simplified version)

Application If there is a nonconstructible real then every perfect set contains a nonconstructible real

Obstacle How do you avoid a_i's which are not in A?

Solution Use y_1, y_2 to code information about which a_i's to avoid



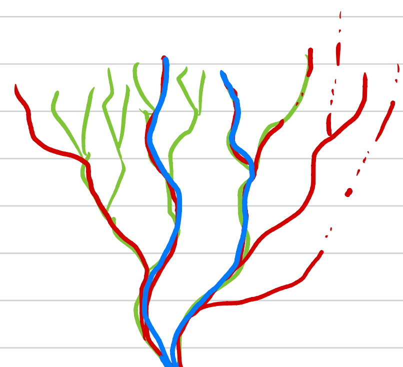
final version

<u>Basis thm, version 4</u> Suppose $A \subseteq 2^\omega$ is perfect and a computes every element of a countable dense subset of A. Then for all $x \in 2^\omega$ there are $y_0, y_1, y_2, y_3 \in A$ such that

$$x \leq_T y_0 \oplus y_1 \oplus y_2 \oplus y_3 \oplus a$$

<u>Obstacle</u> How do you avoid partial functions computable from a?

<u>Solution</u> Use y_3 to code information about convergence times



③ Order Preserving $\Rightarrow$ Measure Preserving

Thm Every order preserving function is either constant on a cone or measure preserving

Main Ingredients

① Basis theorem for perfect sets

② Perfect set theorem $ZF+AD \Rightarrow$ Every set $A \subseteq 2^\omega$ is either countable or contains a perfect set

Morton Davis

↪ Implies a version of CH holds in $ZF+AD$

proof Fix $f: 2^\omega \to 2^\omega$ order preserving
Either range(f) ctbl or range(f) contains a perfect set
We will show

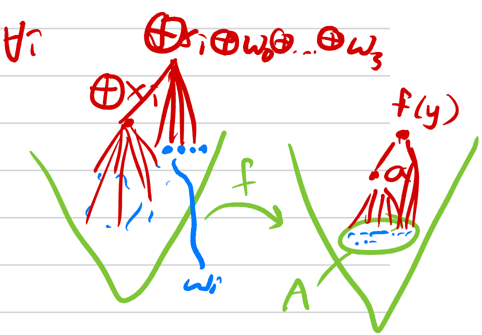
range(f) ctbl $\Rightarrow$ f constant on a cone
range(f) contains a perfect set $\Rightarrow$ f measure preserving

proof (continued)

Assume: $f : 2^\omega \to 2^\omega$ is order preserving
$A \subseteq \mathrm{range}(f)$ is a perfect set

Fix $z \in 2^\omega$
We want to show f is above x on a cone
Since f is order preserving, enough to show $\exists y\ f(y) \geq_T z$
$\hookrightarrow x \geq_T y \Rightarrow f(x) \geq_T f(y) \geq_T z$
Pick $\langle a_i \rangle \subseteq A$ ctbl dense subset
$\exists \langle x_i \rangle$ s.t. $f(x_i) = a_i$ $\quad A \subseteq \mathrm{range}(f)$ + countable choice

Set $a = f(\oplus x_i)$ $\quad f(\oplus x_i) \geq_T f(x_i) = a_i\ \forall i$
$\exists y_0, y_1, y_2, y_3 \in A$ $\quad x \leq_T y_0 \oplus y_1 \oplus y_2 \oplus y_3 \oplus a$
$\hookrightarrow$ Basis thm
$\exists w_0, w_1, w_2, w_3$ $\quad f(w_i) = y_i$ $\quad A \subseteq \mathrm{range}(f)$



$y = (\oplus x_i) \oplus w_0 \oplus w_1 \oplus w_2 \oplus w_3$
$f(y) \geq_T a \oplus \underset{\overset{\shortparallel}{f(w_0)}}{y_0} \oplus y_1 \oplus y_2 \oplus y_3 \geq_T z$ $\quad \square$
$y \geq_T w_0$ + f is order preserving

④ Proof of the Basis Theorem

Thm Suppose $A \subseteq 2^\omega$ is perfect and $a \in 2^\omega$ computes every element of a countable, dense subset of A. Then for every $x \in 2^\omega$ there are $y_0, y_1, y_2, y_3 \in A$ such that

$$x \leq_T y_0 \oplus y_1 \oplus y_2 \oplus y_3 \oplus a$$

proof

Coder: knows $x, a, T \rightsquigarrow$ ~~Computes~~ arbitrarily powerful y_0, y_1, y_2, y_3

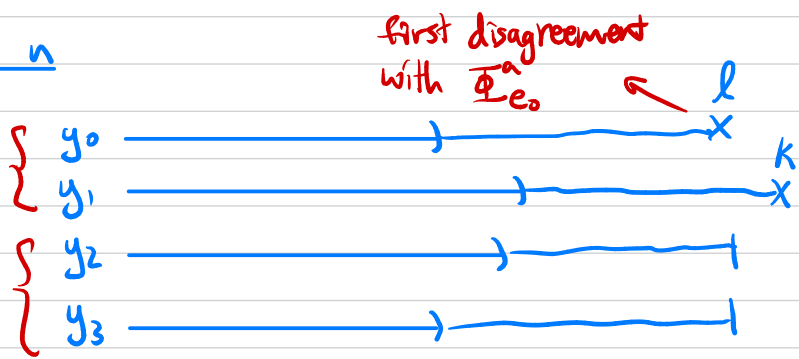
Decoder: knows $a, y_0, y_1, y_2, y_3 \rightsquigarrow$ computes x

Step n

alternates

helper mode

coding next bit of x



current guesses

$e_0 \to \Phi^a_{e_0}$ total & matches y_0 so far

e_1

$? \to$ New guess: least e which matches y_2 on 1^{st} ℓ bits after running k steps

$?$

If $e_2 > e_3$: $x(n) = 0$

If $e_3 \geq e_2$: $x(n) = 1$

⑤ Application 2: Sacks's Question

<u>Question</u> Which partial orders embed into $\mathcal{D}_T$?

<u>Obvious restrictions</u>

1. $\mathcal{D}_T$ has size $2^{\aleph_0}$
2. Every element of $\mathcal{D}_T$ has countably many predecessors

<u>Conjecture</u> (Sacks) Every locally countable partial order of size continuum embeds into $\mathcal{D}_T$

1. Provable in ZFC+CH ← Sacks
2. Independent of ZF ← ?
3. Borel version:

Me + Kojiro Higuchi

Every weight two locally countable Borel partial order embeds

(!!) Not every weight three locally countable Borel partial order embeds

Thm There is a locally countable Borel partial order of height three with no Borel embedding into the Turing degrees

$\hookrightarrow f : 2^\omega \to 2^\omega$ → usually called Borel reduction

$x \leq_P y \iff f(x) \leq_T f(y)$

proof $\leq_P$ on 2^ω sufficiently "free"

Suppose $f : 2^\omega \to 2^\omega$ Borel & reduces $\leq_P$ to $\leq_T$

$f(1^{st}$ level$)$ uncountable $\Rightarrow$ contains a perfect set A

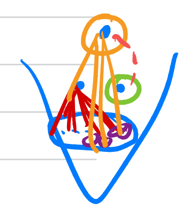
$a_0, a_1, a_2, \ldots$ in 1^{st} level of $\leq_P$ s.t. $f(a_0), f(a_1), f(a_2), \ldots$ ctbl dense in A

$a \geq_P a_0, a_1, a_2, \ldots$ upper bd on 2^{nd} level $\Rightarrow$ $f(a) \geq_T f(a_0), f(a_1), \ldots$

Pick b not below a

$y_0, y_1, y_2, y_3 \in A$ $f(b) \leq_T f(a) \oplus y_0 \oplus \ldots \oplus y_3$

$\exists\, c_0, c_1, c_2, c_3 \in 1^{st}$ level $f(c_i) = y_i$



Pick d above a, c_0, c_1, c_2, c_3 not above b

$f(d) \geq_T f(a), f(c_0), \ldots, f(c_3) \Rightarrow f(d) \geq_T f(b)$ $\rightarrow\leftarrow$