# Complexity of Oracles for Packing Dimension

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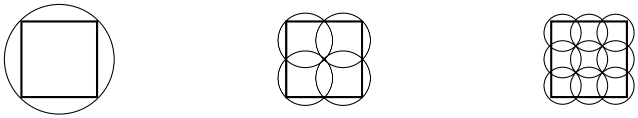
# Dimension

Question: How should we define the dimension of a set $A \subseteq \mathbb{R}^n$?

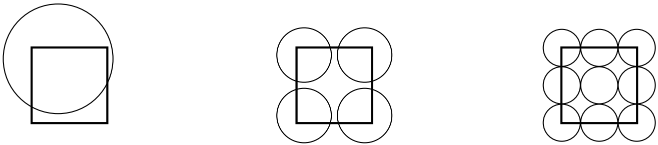
Hausdorff dimension: What is the minimum number of balls of radius $r$ needed to cover $A$?

Dimension of $A = d \implies \sim (1/r)^d$ balls needed



Packing dimension: What is the maximum number of disjoint balls of radius $r$ with centers in $A$?

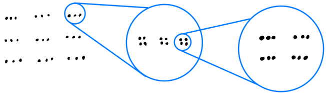
Dimension of $A = d \implies \sim (1/r)^d$ balls possible



Hausdorff dimension: What is the minimum number of balls of radius $r$ needed to cover $A$?

Packing dimension: What is the maximum number of disjoint balls of radius $r$ with centers in $A$?

Problem: What if this number doesn't scale with $r$ in a regular way?



Definition. A set $A$ has $H^d$-measure 0 if for all $\varepsilon > 0$, there is a countable collection of balls $\{B_n\}_{n\in\omega}$ such that

$$A \subseteq \bigcup_{n\in\mathbb{N}} B_n \qquad \text{and} \qquad \sum_{n\in\mathbb{N}} \text{radius}(B_n)^d < \varepsilon$$

Definition. $\dim_H(A) = \inf\{d \mid A \text{ has } H^d\text{-measure } 0\}$.

Packing dimension: Similar, but more annoying

Question. What is the difference between $\dim_H(A)$ and $\dim_P(A)$?

Answer. Intuitively,

$\dim_H(A) \approx$ what does the dimension of $A$ look like at the scales where it looks smallest?

$\dim_P(A) \approx$ what does the dimension of $A$ look like at the scales where it looks largest?

# Effective Dimension

Definition. For a finite string $\sigma \in 2^{<\omega}$, the Kolmogorov complexity of $\sigma$, denoted $C(\sigma)$, is the length of the shortest program that outputs $\sigma$.

Definition. For $x \in 2^{\omega}$:

- Effective Hausdorff dimension of $x$: $\text{edim}_H(x) = \liminf_{n\to\infty} \frac{C(x\upharpoonright n)}{n}$.
- Effective packing dimension of $x$: $\text{edim}_P(x) = \limsup_{n\to\infty} \frac{C(x\upharpoonright n)}{n}$.

**Informally:** $\text{edim}_H(x)$, $\text{edim}_P(x) \approx$ number of bits needed to describe the first $n$ bits of $x$ as a fraction of $n$

$\text{edim}_H(x)$: consider only initial segments of $x$ that are easiest to describe

$\text{edim}_P(x)$: consider only initial segments that are hardest to describe

Example. Let $x$ be a sequence such that all even bits are 0 and all odd bits are chosen by flipping a coin.

$$x = 0100000101010001000100010000000010\ldots$$

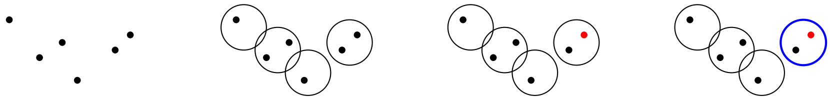
For any $n$, $C(x \upharpoonright n) \approx n/2 \implies \text{edim}_H(x) = \text{edim}_P(x) = 1/2$.

## Why is effective Hausdorff dimension similar to Hausdorff dimension?

**Another way to view Hausdorff dimension:** You are playing a game with your friend using the set $A$:

(1) First you pick an arbitrary point $x \in A$ and an arbitrary $r > 0$
(2) Your goal is to describe $x$ to your friend as concisely as possible
(3) More precisely: you need to give some information to your friend to allow them to guess a point that is within distance $r$ of $x$ and you want to give as little information as possible

**The point:** If $A$ can be covered by $(1/r)^d$ balls of radius $r$, you only need to give your friend $\log((1/r)^d) = d\log(1/r)$ bits of information



If $A \subseteq 2^\omega$ then guessing $x$ within distance $2^{-n}$ corresponds to guessing the first $n$ bits of $x$, which we can describe using at most $dn$ bits

# The Point-to-Set Principle

The connection between effective Hausdorff dimension and Hausdorff dimension is more than just conceptual.

Theorem (J. Lutz and N. Lutz). For any set $A \subseteq 2^\omega$

$$\dim_H(A) = \min_a \sup_{x \in A} \operatorname{edim}_H^a(x)$$
$$\dim_P(A) = \min_a \sup_{x \in A} \operatorname{edim}_P^a(x)$$

Comment. $\operatorname{edim}_H^a(x) / \operatorname{edim}_P^a(x)$ denote effective Hausdorff/packing dimension relative to an oracle $a$

Idea of the proof (for Hausdorff dimension).

$\geq$ Roughly the idea on the previous slide

$\leq$ For the appropriate $a$, $A \subseteq \{x \mid \operatorname{edim}_H^a(x) \leq d\}$, which can be proved to have dimension $d$

Idea. Use the point-to-set principle + reasoning about Kolmogorov complexity/computability theory to prove statements about Hausdorff and packing dimension

An impressive example, due to Don Stull:

Definition. For a set $A \subseteq \mathbb{R}^n$, the distance set of $A$ is

$$\Delta(A) = \{|x - y| \mid x, y \in A\}.$$

Falconer's conjecture. $\dim_H(A) > n/2 \implies \dim_H(\Delta(A)) = 1$.

Stull's result: For $A \subseteq \mathbb{R}^2$, $\dim_H(\Delta(A)) \geq 3/4$.

Best previous bound: $\approx .7017$.

A key point in Stull's proof:

Theorem (Hitchcock). If $A$ is computably compact ($\approx \Pi^0_1$) then the point-to-set principle holds without any oracle, i.e.

$$\dim_H(A) = \sup_{x \in A} \operatorname{edim}_H(x).$$

Theorem (Hitchcock). If $A$ is computably compact ($\approx \Pi^0_1$) then the point-to-set principle holds without any oracle, i.e.

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Question. Why is this useful?

Definition. For any $e \in S^{n-1}$ and $A \subseteq \mathbb{R}^n$, let $\pi_e(A)$ denote the orthogonal projection of $A$ onto the line spanned by $e$.

If $A$ is computably compact then for every $e$, $e$ can be used as the oracle in the point-to-set principle for $\pi_e(A)$. I.e.

Hitchcock's Theorem $\implies \dim_H(\pi_e(A)) = \sup_{x\in\pi_e(A)} \operatorname{edim}^e_H(x)$

Natural question. Does Hitchcock's Theorem hold for packing dimension?

Answer (Conidis). No.

# Packing Oracles

Definition. A packing oracle for a set $A$ is an oracle $a$ such that

$$\dim_P(A) = \sup_{x \in A} \operatorname{edim}_P^a(x).$$

Theorem (Conidis). There is a $\Pi^0_1$ set $A \subseteq 2^\omega$ with no computable packing oracle.

Question. How bad can it get? And how much of the consequences of Hitchcock's theorem can be salvaged for packing dimension?

Theorem (L.). There is a $\Pi^0_1$ set $A \subseteq 2^\omega$ with no hyperarithmetic packing oracle.

Theorem (L.). If $A \subseteq 2^\omega$ is $\Pi^0_1$ and $a$ computes every hyperarithmetic set then $a$ is a packing oracle for $A$.

The second theorem is enough to recover some consequences of Hitchcock's theorem

Theorem (L.). If $A \subseteq 2^\omega$ is $\Pi^0_1$ and $a$ computes every hyperarithmetic set then $a$ is a packing oracle for $A$.

**A consequence of this theorem:**

Suppose that $a$ computes every hyperarithmetic set and that $A \subseteq \mathbb{R}^n$ is computably compact

Then for almost every $e \in S^{n-1}$, $a \oplus e$ is a packing oracle for $\pi_e(A)$, i.e.

$$\dim_P(\pi_e(A)) = \sup_{x \in \pi_e(A)} \operatorname{edim}_P^{a \oplus e}(x)$$

The key point is that for almost every $e$, $a \oplus e$ computes all sets which are hyperarithmetic in $e$

Theorem (L.). There is a $\Pi^0_1$ set $A \subseteq 2^\omega$ with no hyperarithmetic packing oracle.

Easier to prove: For any hyperarithmetic $a$, there is some $\Pi^0_1$ set $A \subseteq 2^\omega$ such that $a$ is not a packing oracle for $A$

Fact. Every hyperarithmetic set is computed by some $\Pi^0_1$ singleton in $\omega^\omega$. I.e. if $a$ is hyperarithmetic then there is some $b \in \omega^\omega$ such that $b \geq_T a$ and $\{b\}$ is $\Pi^0_1$.

Idea. Take $b \in \omega^\omega$ to be a $\Pi^0_1$-singleton computing $a'$ and consider the element of $d \in 2^\omega$ given by

$$d = \underbrace{11\ldots1}_{n_0} 0\, \sigma_0\, \underbrace{11\ldots1}_{n_1} \sigma_1 \;\ldots$$

where $b = \langle n_0, n_1, n_2, \ldots\rangle$and $\sigma_0, \sigma_1, \ldots$ is a sequence of strings computed by $b$ which have high Kolmogorov complexity relative to $a$.

$\{d\}$ is not a $\Pi^0_1$ set, but it is contained in a countable $\Pi^0_1$ set, $A$

$A$ countable $\implies \dim_P(A) = 0$, but $\mathrm{edim}^a_P(d) = 1$