Borel diagonalization

Patrick Lutz

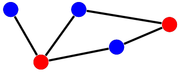
AMS 2025 Fall Central Sectional Meeting

Borel combinatorics
vs.
computable combinatorics

Borel combinatorics: When does a Borel instance of a combinatorial problem have a Borel solution?

Computable combinatorics: When does a computable instance of a combinatorial problem have a computable solution?

Example combinatorial problem.



Instance. Undirected graph G
Solution. 2-coloring of G

Well-known fact. G has a 2-coloring $\iff$ G has no odd cycles

Natural questions.

(1) Does every Borel graph with no odd cycles have a Borel 2-coloring?

(2) Does every computable graph with no odd cycles have a computable 2-coloring?

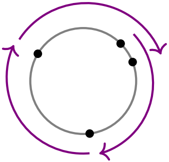
Answers. No and no

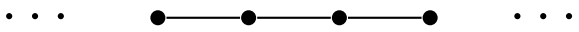
Fact. There is a Borel graph with no odd cycles but no Borel 2-coloring

Proof. Let $T\colon S^1 \to S^1$ be rotation by an irrational angle

A graph G on S^1: for each x, put an edge between x and $T(x)$

Irrational angle $\implies$ no cycles





Suppose there is a Borel 2-coloring

Let $A =$ red vertices

$B =$ blue vertices

Observe: $T(A) = B$ and $T(T(A)) = A$

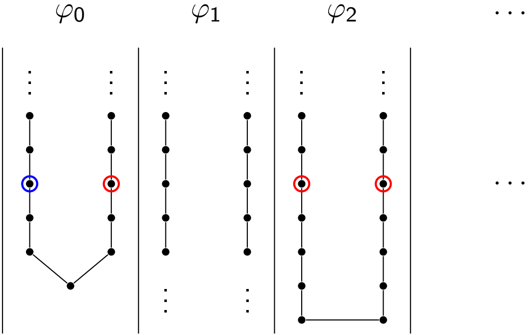
T is measure preserving $\implies \lambda(A) = \lambda(B) \implies \lambda(A) = 1/2$

T^2 is ergodic $\implies \lambda(A) = 0$ or $\lambda(A) = 1$

Fact. There is a computable graph with no odd cycles but no computable 2-coloring

Slightly more interesting: make all vertices have degree 2

Proof.



Question. Is it possible to construct counterexamples in Borel combinatorics using diagonalization?

Borel diagonalization

Question. Is it possible to construct counterexamples in Borel combinatorics using diagonalization?

Cardinality.

$|\text{Borel functions } 2^\omega \to 2| = 2^{\aleph_0}$
$|2^\omega| = 2^{\aleph_0}$

Complexity.

Fix an indexing of (partial) Borel functions, $\{f_a\}_{a \in 2^\omega}$

Idea. Define a graph on $2^\omega \times 2^\omega$ where the points of the form (a, x) are used to diagonalize against f_a

Problem. The relation "$f_a(x) = y$" is Π^1_1, not Δ^1_1

Problem. The relation "$f_a(x) = y$" is Π^1_1, not Δ^1_1

But wait...

In the computable version:

- We need to build a Δ^0_1 graph
- But the relation "$\varphi_e(n) = m$" is Σ^0_1, not Δ^0_1

What's going on. Witnesses to Σ^0_1 facts were encoded by the vertices of the graph

Idea. To construct counterexamples in Borel combinatorics:

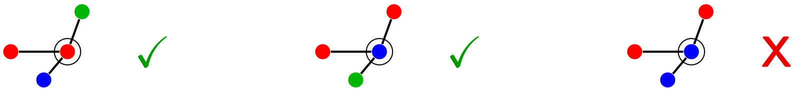
(1) Find a construction which diagonalizes against all Borel functions and which depends only on Σ^1_1 facts

(2) Convert this to a Borel graph by encoding witnesses to these Σ^1_1 facts into vertices/edges

Domatic partitions

Definition. A k-domatic partition of a graph G is a coloring of G with k colors such that every vertex sees every color

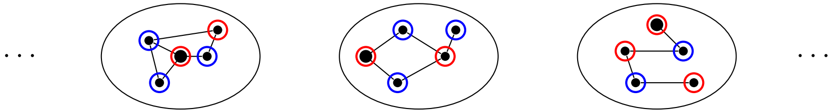
Important note. Every vertex can see itself



More formal definition. A k-domatic partition of a graph $G = (V, E)$ is a function $c\colon V \to \{1, 2, \ldots, k\}$ such that for each vertex v and color $i \leq k$, either $c(v) = i$ or for some neighbor u of v, $c(u) = i$

Fact 1. Every graph has a 1-domatic partition

Fact 2. G has a 2-domatic partition $\iff$ G has no isolated vertices



Definition. A *k-domatic partition* of a graph G is a coloring of G with k colors such that every vertex sees every color

Fact. G has a 2-domatic partition $\iff$ G has no isolated vertices

Question (Hou). Is there a Borel graph such that every vertex has uncountably many neighbors but there is no Borel 2-domatic partition?

Answer. Yes.

Strategy.

(1) Construct a Σ^1_1 equivalence relation E on X such that:
- Every equivalence class is uncountable
- For every Borel function $c\colon X \to \{1, 2\}$, there is some equivalence class on which c is constant

(2) Construct a Borel graph G where there is an edge between x and y whenever $x \oplus y$ computes a witness that x and y are E-equivalent

Comment. (2) is related to ideas in "Borel graphable equivalence relations" by Arant-Kechris-L.

Definition. Suppose L is a linear order on $\mathbb{N}$ and $x \in 2^\omega$. A jump hierarchy on L starting from x is a set $H \subseteq \mathbb{N} \times \mathbb{N}$ such that for all n,

$$H_n = \bigoplus_{m <_L n} (x \oplus H'_m).$$

Fact. If L is a well-order, then for any x there is a unique jump hierarchy on L starting from x, which we will denote $x^{(L)}$.

Fact. For every Borel function $f : 2^\omega \to 2$, there is some a, some well-order L computable from a and some e such that for all x,

$$f(x) = \Phi_e((x \oplus a)^{(L)}).$$

Fact. If L is a well-order computable from x and g is $x^{(L)}$-generic then $(x \oplus g)^{(L)} \equiv_T x^{(L)} \oplus g$.

Theorem. There is a Borel graph G such that every vertex has uncountably many neighbors but G has no Borel 2-domatic partition.

Strategy. Construct a Σ^1_1 equivalence relation E on X such that:

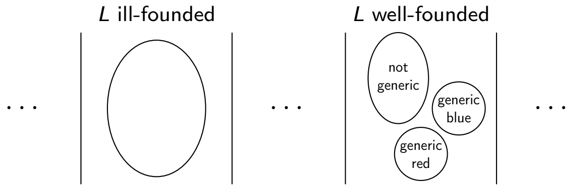
- Every equivalence class is uncountable
- For every Borel function $c\colon X \to \{1, 2\}$, there is some equivalence class on which c is constant

Let E be the equivalence relation on $2^\omega \times 2^\omega \times \mathbb{N} \times 2^\omega$ where $(L, a, e, x)\, E\, (L, a, e, y)$ holds if any of the following are true:

(1) L does not code a well-founded linear order computable from a

(2) L is well-founded and neither x nor y are $a^{(L)}$-generic.

(3) L is well-founded, both x and y are $a^{(L)}$-generic and $\Phi_e((L \oplus a \oplus e \oplus x)^{(L)}) = \Phi_e((L \oplus a \oplus e \oplus y)^{(L)})$



A different approach

Challenge of Borel diagonalization. How can you build an object of complexity α which diagonalizes against objects of complexity $\beta > \alpha$ for all β?

Answer? Priority arguments and overspill

Theorem (Harrington). There is a Π^0_1 class $A \subseteq \omega^\omega$ such that:

- A is uncountable
- For all distinct $x, y \in A$ and all $\alpha < \omega_1^{CK}$, $x \not\leq_T y^{(\alpha)}$