# Borel graphability and diameter

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ASL North American Annual Meeting 2026

# Two households, both alike in dignity

**Computability theory**

Studies complexity of definitions

... and constructivity of proofs

Loves counterexamples

Key technique: diagonalization

**Descriptive set theory**

Studies complexity of definitions

... and constructivity of proofs

Loves counterexamples

Key technique: measure theory

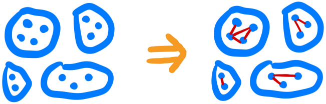
**This talk.** An example of using diagonalization (and other ideas from computability theory) in descriptive set theory

Also features a surprising (to me, at least) fact about computability theoretic genericity

# Borel graphability

**Definition.** A graphing of an equivalence relation $E$ on $X$ is a graph $G$ such that $E$ is the connectedness relation of $G$

I.e. $x \, E \, y \iff$ there is a path in $G$ between $x$ and $y$



**Definition.** An equivalence relation $E$ on a Polish space $X$ is Borel graphable if it has a graphing $G$ which is Borel as a subset of $X^2$

**Easy facts.** Borel $\subseteq$ Borel graphable $\subseteq$ Analytic

Borel $\neq$ Borel graphable

Borel graphable $\neq$ Analytic

**Question.** Which analytic equivalence relations are Borel graphable?

The answer can be complicated!

**Definition.** $F_{\omega_1}$ is the equivalence relation on $2^\omega$ defined by

$$x \, F_{\omega_1} \, y \iff \omega_1^x = \omega_1^y$$

**Theorem (Arant-Kechris-L.).** $F_{\omega_1}$ is Borel graphable if and only if there is a non-constructible real

In particular, whether $F_{\omega_1}$ is Borel graphable is independent of ZFC

**Proposition.** If $E$ is an analytic equivalence relation which only has one equivalence class of size more than one then $E$ is Borel graphable.

*Proof.* Let $A =$ equivalence class with size more than one.

$A$ is analytic $\implies$ $A(x) = \exists y\, B(x, y)$ for some Borel $B$

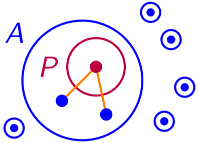
Perfect set theorem $\implies$ $A$ is countable or contains a perfect set

Assume $A$ contains a perfect set $P$

Fix a continuous surjection $\varphi\colon P \to 2^\omega$

Define a graph $G$: put an edge between $x$ and $y$ if

- $y \in P$ and $\varphi(y)$ computes $z$ such that $B(x, z)$
- or vice-versa




**Two key points.**

(1) $E$ has a Borel graphing of diameter 2

(2) Main step in showing that $E$ is Borel graphable: figure out how to code information into elements of equivalence classes of $E$

# Diameter

**Empirically.** Any time an analytic equivalence relation is proved to be Borel graphable, it is by finding a way to encode arbitrary information into all uncountable equivalence classes and it always results in a graphing of diameter 2

**Definition.** The distance between points $x$ and $y$ in a graph $G$, denoted $d_G(x, y)$, is the length of the shortest path between $x$ and $y$ (or 0 if $x = y$)

**Definition.** The diameter of a graphing $G$ of an equivalence relation $E$ is $\text{diam}(G) = \sup\{d_G(x, y) \mid x \, E \, y\}$.

i.e. the longest shortest path in $G$ between two $E$-equivalent points

**Question.** Does every Borel graphable equivalence relation have a Borel graphing of diameter at most 2?

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**Theorem (L.).** There is an analytic equivalence relation with a Borel graphing of diameter 3 but no Borel graphing of diameter 2

**Empirically.** Any time an analytic equivalence relation is proved to be Borel graphable, it is by finding a way to encode arbitrary information into all uncountable equivalence classes and it always results in a graphing of diameter 2

**Empirically.** ~~Any time~~ Most times an analytic equivalence relation is proved to be Borel graphable, it is by finding a way to encode arbitrary information into all uncountable equivalence classes and it ~~always~~ almost always results in a graphing of diameter 2

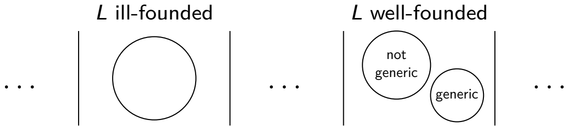
# The equivalence relation

**Recall.** LO = Polish space of linear orders with domain $\mathbb{N}$

For $r \in 2^\omega$ and $L \in$ LO well-founded,
$r^{(L)}$ = unique jump hierarchy on $L$ beginning with $r$

**Definition.** Let $X = \mathrm{LO} \times 2^\omega \times 2^\omega$ and let $E$ be the equivalence relation on $X$ where $(L, r, x)\, E\, (L, r, y)$ if any of the following hold

- $L$ is ill-founded
- $L$ is well-founded and both $x$ and $y$ are $r^{(L)}$-generic
- $L$ is well-ounded and neither $x$ nor $y$ are $r^{(L)}$-generic




**Idea.** Elements of the form $(L, r, -)$ diagonalize against $\Delta^0_\alpha(r)$ graphings (where $L$ is a presentation of $\alpha$)

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- $L$ is well-founded and both $x$ and $y$ are $r^{(L)}$-generic
- $L$ is well-ounded and neither $x$ nor $y$ are $r^{(L)}$-generic

**Theorem (Arant-Kechris-L.).** $E$ is Borel graphable with diameter $\leq 4$

*Proof idea.* First show that $E$ is analytic

Then define a graph $G$ by putting an edge between $(L, r, x)$ and $(L, r, y)$ if together they compute a witness to their equivalence

Fix $L$ and $r$

Key Lemma: If $x$ and $y$ are mutually $r^{(L)}$-generic then there is a path of length 2 between them

Starting with $x, y$ $r^{(L)}$-generic, first pick $z$ mutually $r^{(L)}$-generic with both $x$ and $y$

Paths of length 2 between $(L, r, x)$ and $(L, r, z)$ and between $(L, r, z)$ and $(L, r, y)$ $\implies$ path of length 4 between $(L, r, x)$ and $(L, r, y)$

# No graphing of diameter $2$

**Definition.** Let $X = \mathrm{LO} \times 2^\omega \times 2^\omega$ and let $E$ be the equivalence relation on $X$ where $(L, r, x)\, E\, (L, r, y)$ if any of the following hold

- $L$ is ill-founded
- $L$ is well-founded and both $x$ and $y$ are $r^{(L)}$-generic
- $L$ is well-ounded and neither $x$ nor $y$ are $r^{(L)}$-generic

**Theorem (L.).** $E$ has no Borel graphing of diameter 2

*Proof idea.* Suppose $G$ is a Borel graphing of $E$

For some $r$ and some $\alpha < \omega_1$, $G$ is $\Delta^0_\alpha(r)$. $L =$ some presentation of $\alpha$

Look at $G$ restricted to elements of the form $(L + 1, r, -)$

Key point: Can show that if $x$ and $y$ are mutually $r^{(L)}$-generic then $(L, r, x)$ and $(L, r, y)$ cannot be adjacent in $G$, because $G$ is too stupid to tell if $x$ and $y$ are $r^{(L+1)}$-generic or not

Crazy idea: Find $x, y$ which are $r^{(L+1)}$-generic such that any $z$ which is $r^{(L+1)}$-generic is either mutually $r^{(L)}$-generic with $x$ or mutually $r^{(L)}$-generic with $y$

Crazy idea: Find $x, y$ which are $r^{(L+1)}$-generic such that any $z$ which is $r^{(L+1)}$-generic is either mutually $r^{(L)}$-generic with $x$ or mutually $r^{(L)}$-generic with $y$

**(Surprising!) Lemma.** For any $r$, there are $x$ and $y$ such that

- $x$ and $y$ are both $r'$-generic
- Any $z$ which is $r'$ generic is either mutually $r$-generic with $x$ or mutually $r$-generic with $y$

**Question.** Is it the case that for every $n$, there is an equivalence relation with a Borel graphing of diameter $n+1$, but no Borel graphing of diameter $n$?

I only know how to handle $n = 1$ and $n = 2$

**Question.** Is there a Borel graphable equivalence relation with no Borel graphing of finite diameter?

Note that a positive answer to the first question implies a positive answer to the second question