Lossless expansion and measure hyperfiniteness

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Joint work with Jan Grebík

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Specifically, work by Jan Grebík and myself on the structure of countable Borel equivalence relations using ideas about expander graphs

Borel reducibility of equivalence relations

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These questions are part of the motivation for the theory of Borel reducibility of equivalence relations.

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Idea. If $E \leq_B F$ then the classification problem for F is at least as hard as the classification problem for E

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Theorem (Hjorth). Conjugacy of (orientation preserving) homoemorphisms of $[0, 1]$ is classifiable by countable structures, but conjugacy of homeomorphisms of $[0, 1]^2$ is not

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Consequence. Every CBER is either smooth or is above all the smooth CBERs

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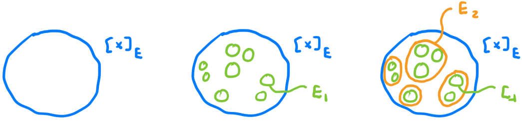

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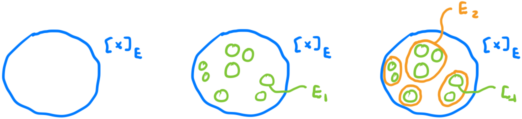

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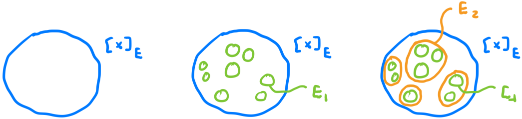

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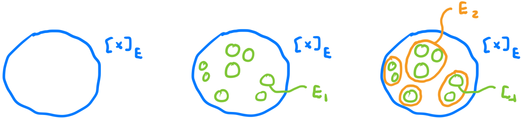

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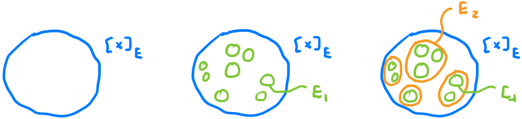

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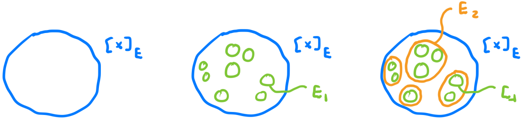

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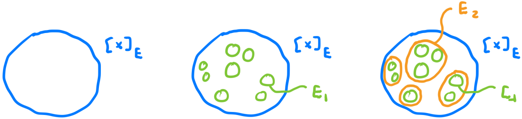

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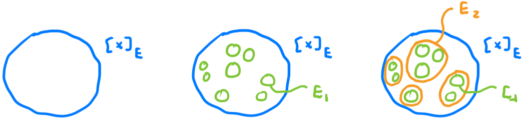

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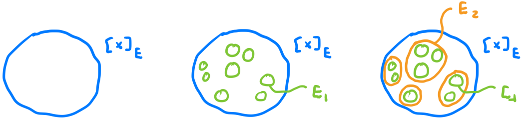

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I.e. hyperfinite equivalence relations are the simplest non-trivial CBERs

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- **Effectiveness.** Is hyperfiniteness effective? I.e. if E is Δ^1_1 and hyperfinite, is there a sequence of Δ^1_1 equivalence relations $E_1 \subseteq E_2 \subseteq \dots$ witnessing its hyperfiniteness?

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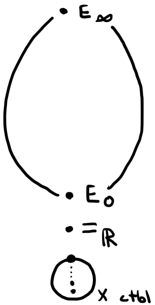

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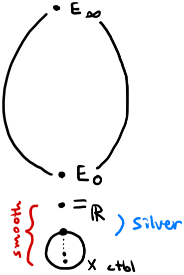

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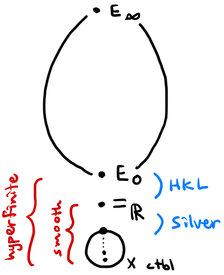

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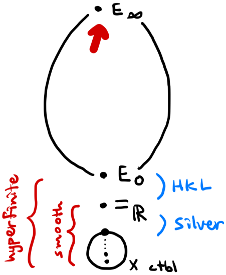

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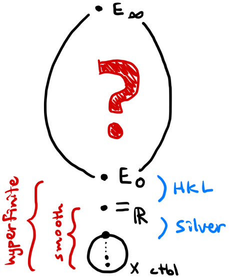

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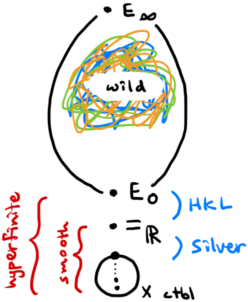

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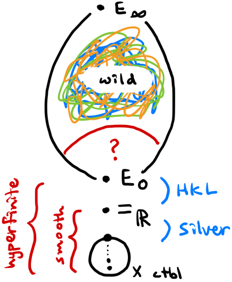

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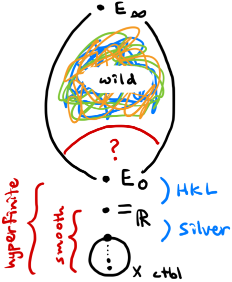

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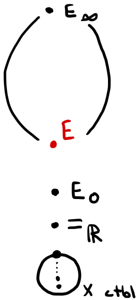

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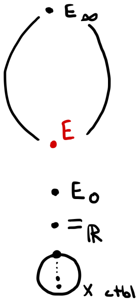

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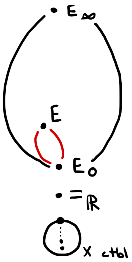

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This talk. The answer to question 2 is probably "yes."

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(2) Two plausible candidates for this combinatorial property

Expander graphs

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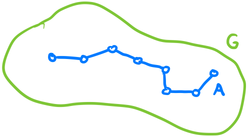

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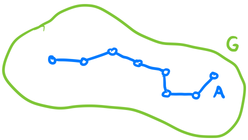

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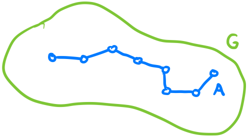

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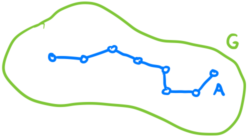

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"Theorem." A random d-regular graph is a lossless expander

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Comment. Actually, this is more similar to what some computer scientists have called an "ultra lossless expander"

Lossless expansion and measure hyperfiniteness

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(E.g. if Γ is non-amenable, acts freely and λ is Γ-invariant)

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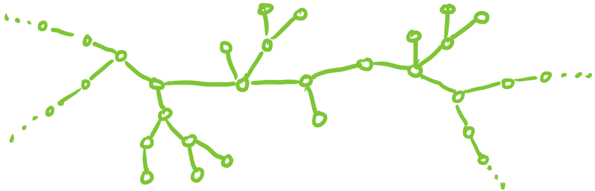

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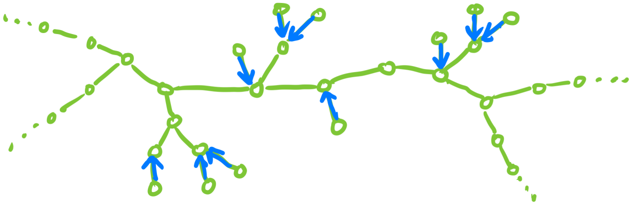

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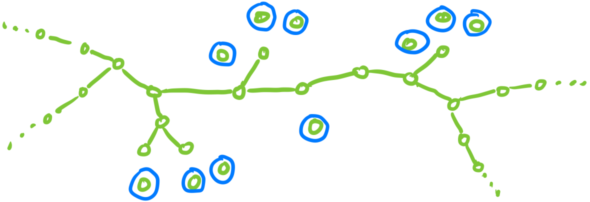

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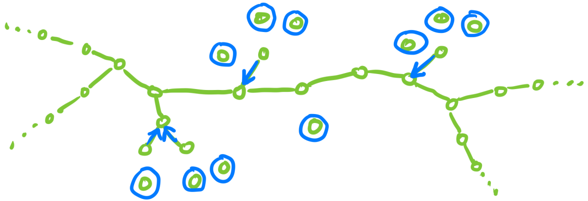

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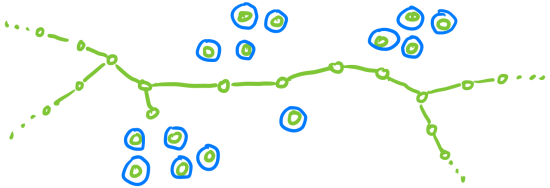

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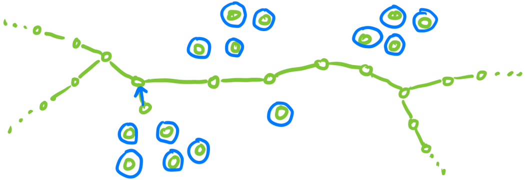

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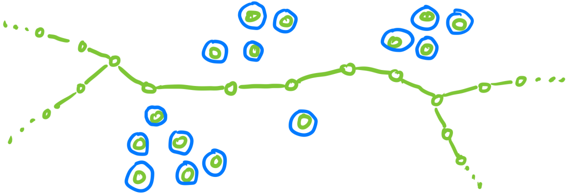

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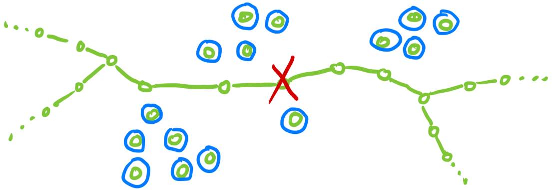

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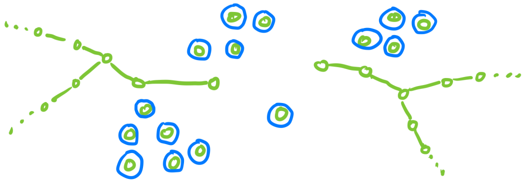

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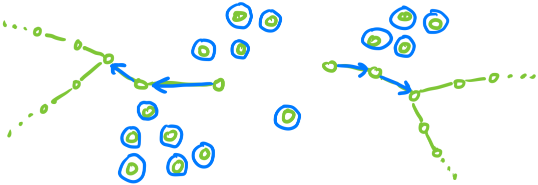

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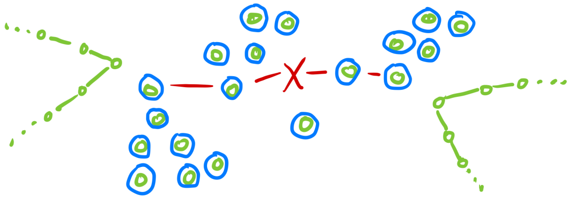

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- G, the Schreier graph of the action
- λ, a Borel probability measure on X such that $\text{supp}(\lambda) = X$

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- Limit of a sequence of random lifts of graphs

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It is also known that the Schreier graph will be an expander graph (but not necessarily a lossless expander)