396 DISCUSSION SECTION 6

WITH NOAH LUNTZLARA

Today we will not prove the Sylow Theorems, but we will try to get an understanding of how they help. In each section, you may use only the previous results.

1. First Sylow Theorem

For every prime factor p with multiplicity n of the order of a finite group G, there exists a Sylow p-subgroup of G, of order p^n.

(Ex. 1) Prove Cauchy's Theorem: Given a finite group G and a prime number p dividing the order of G, then there exists an element (and hence a subgroup) of order p in G.

(Ex. 2) Prove that if H is the unique p-Sylow of G, then H is normal.

(Ex. 3) Prove that the polynomial $x^2 + x + 1$ has a root in $\mathbb{F}_p$ if and only if $p \equiv 1 \pmod 3$.

2. Second Sylow Theorem

Given a finite group G and a prime number p, all Sylow p-subgroups of G are conjugate to each other, i.e. if H and K are Sylow p-subgroups of G, then there exists an element g in G with $g^1 Hg = K$.

(Ex. 1) Let G_1 and G_2 be finite groups, and f a homomorphism of G_1 into G_2. Let P_1 be a Sylow p-subgroup of G_1. Then there exists a Sylow p-subgroup P_2 of G_2 such that $f(P_1) \subseteq P_2$.

(Ex. 2) Every subgroup of G that is a p-group (i.e. its order is a power of p) is contained in a Sylow p-subgroup of G.

(Ex. 3) Let H be as subgroup of G and let P be a Sylow p-subgroup of H. Then there exists a Sylow p-subgroup P' of G such that $P = H \cap P'$.

(Ex. 4) Let P be a Sylow p-subgroup of G, let N be its normalizer, and let M be a subgroup of G that contains M. Then M is its own normalizer.

3. Third Sylow Theorem

Let p be a prime factor with multiplicity n of the order of a finite group G, so that the order of G can be written as $p^n m$, where $n > 0$ and p does not divide m. Let n_p be the number of Sylow p-subgroups of G. Then the following hold:

- n_p divides m, which is the index of the Sylow p-subgroup in G.
- $n_p \equiv 1 \pmod p$.
- $n_p = |G : N_G(P)|$, where P is any Sylow p-subgroup of G and N_G denotes the normalizer.

(Ex. 1) Show that if $|G| = 50$ then G has a normal subgroup.

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