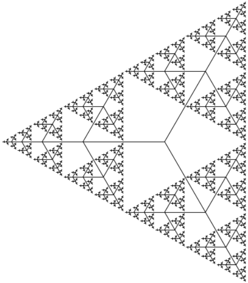
Free Groups and Infinite Trees



with

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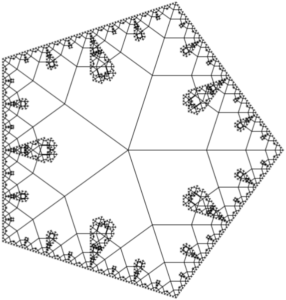
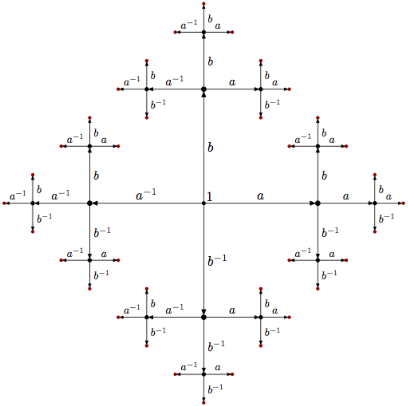


Honors 135.004
Fractals: Their Beauty and Topology
Connor Davis

Outline

- Group Theory
- Examples of Groups
- Free groups
- Cayley graphs



Group Theory

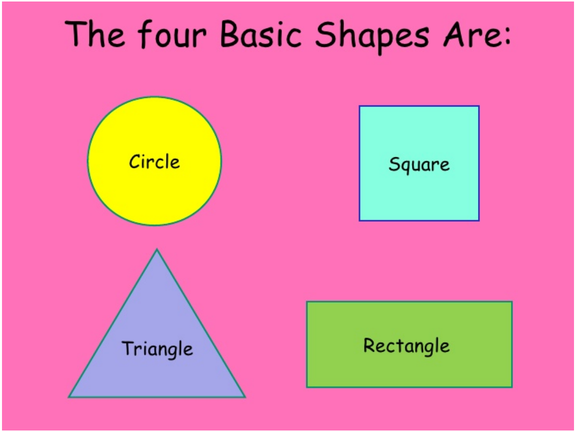
Group theory

Group theory

Group theory is the study of *symmetry*.

What is symmetry?

Symmetry



Which is the most symmetric? Which is the least?

Symmetry

Definition

A *symmetry* is a transformation that leaves something unchanged.

- Symmetries are always *reversible*.
- *Doing nothing* is always a symmetry.

Groups

Symmetries describe something concrete about an object. A group, on the other hand is a set of abstract symmetries.

Definition

A *group* is a pair $(G, *)$, where G is a set and $*$ is a multiplication rule, which satisfies:

- **Associativity** $g * (h * k) = (g * h) * k$ always holds.
- There is an **identity** e in G such that for any g in G,
$$e * g = g \text{ and } g * e = g.$$
- Every g in G has an **inverse** g^{-1} in G such that
$$g * g^{-1} = e \text{ and } g^{-1} * g = e.$$

Examples of Groups

Examples of Groups

What are some examples of groups?

Groups of Numbers

Associativity $g * (h * k) = (g * h) * k$.
Identity e such that $e * g = g * e = g$.
Inverses g^{-1} such that $g * g^{-1} = g^{-1} * g = e$.

You can make groups out of numbers.

- $(\mathbb{Z}, +)$, where $\mathbb{Z}$ is integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
  - **Associativity.** $a + (b + c) = (a + b) + c$.
  - **Identity.** $0 + a = a + 0 = a$.
  - **Inverses.** $a + (-a) = (-a) + a = 0$.
- Same thing with $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\cdots$
- $(\mathbb{R}, \cdot)$ is *not* a group!
  - **Associativity.** $r \cdot (s \cdot t) = (r \cdot s) \cdot t$.
  - **Identity.** $1 \cdot r = r \cdot 1 = r$.
  - **Inverses.** $0 \cdot (\text{anything}) = 0 \neq 1$.

Groups of Numbers

Associativity $g * (h * k) = (g * h) * k$.
Identity e such that $e * g = g * e = g$.
Inverses g^{-1} such that $g * g^{-1} = g^{-1} * g = e$.

- $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, $\cdots$
- $(\mathbb{R}, \cdot)$ is *not* a group!
  - **Inverses.** $0 \cdot (\text{anything}) = 0 \neq 1$.
- $(\mathbb{R} \setminus \{0\}, \cdot)$ is a group.
  - **Inverses.** $r \cdot (1/r) = (1/r) \cdot r = 1$.

Groups of Numbers

Associativity $g * (h * k) = (g * h) * k$.
Identity e such that $e * g = g * e = g$.
Inverses g^{-1} such that $g * g^{-1} = g^{-1} * g = e$.

- $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, $\cdots$
- $(\mathbb{R} \setminus \{0\}, \cdot)$, $(\mathbb{Q} \setminus \{0\}, \cdot)$, $(\mathbb{C} \setminus \{0\}, \cdot)$, not $(\mathbb{Z} \setminus \{0\}, \cdot)$, $\cdots$
- Matrix groups with matrix multiplication, like

$$GL_2(\mathbb{R}) = \{2 \cdot 2 \text{ matrices, entries in } \mathbb{R} \text{ and det} = 1\}.$$

N.B. For most pairs of matrices, $AB \neq BA$. For instance,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So generally $g * h = h * g$ isn't true in any group.

Definition

A group $(G, *)$ is *abelian* provided that $g * h = h * g$ for any elements g, h in G.

The operation is then said to be *commutative*.

Abelian groups.

- $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, $\cdots$
- $(\mathbb{R} \setminus \{0\}, \cdot)$, $(\mathbb{Q} \setminus \{0\}, \cdot)$, $(\mathbb{C} \setminus \{0\}, \cdot)$, $\cdots$

Non-abelian groups.

- $GL_2(\mathbb{R})$.

Symmetry Groups

Associativity $g * (h * k) = (g * h) * k$.
Identity e such that $e * g = g * e = g$.
Inverses g^{-1} such that $g * g^{-1} = g^{-1} * g = e$.

Theorem

The set of symmetries of any object where the operation is *composition* forms a group!

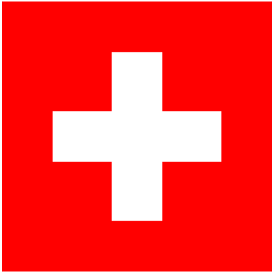
Proof. Recall:

- Symmetries are always *reversible*.
- *Doing nothing* is always a symmetry.

Thus we have **inverses**, **identity**. Furthermore, composition of transformations is **associative**.

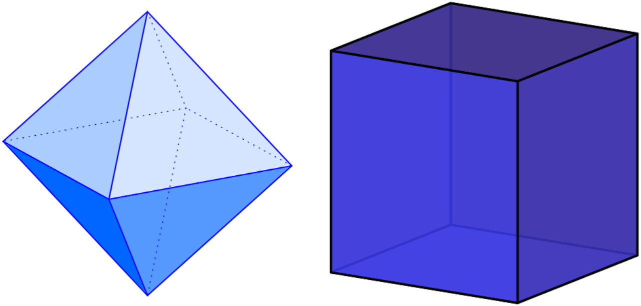
Symmetry Groups

What are the symmetry groups of these shapes?



Symmetry Groups

Amazing but true! These two objects have the same symmetry group:



Symmetry in Mathematics

The greatest developments in modern math have come from studying symmetries of mathematical structures.

Some objects with important symmetry groups:

- Vector spaces (Linear groups)
- Field extensions (Galois groups)
- Manifolds (Mapping class groups)
- L-functions (The modular group)
- Spacetime (Lorentz group and Poincaré group)

Free Groups

Words

Definition

Let S be a set of symbols. A *word* in S is an ordered list of elements in S, not necessarily distinct.

Example. $S = \{a, b, c, d, e, f, g\}$

$$w_1 = edcedc$$
$$w_2 = aaaaa = a^5$$
$$w_3 = cabbage = cab^2age$$
$$w_4 = f$$
$$w_5 = \emptyset$$

Words

Let S be a set of symbols. For every symbol x in S, associate a corresponding symbol x^{-1}. Call the set of such symbols S^{-1}. Consider words in $S \cup S^{-1}$.

Words may be *simplified* by deleting subwords of the form xx^{-1} or $x^{-1}x$. For instance, if $S = \{a, b, c\}$,

$$ab^3c^{-1}cb^{-1}c \to ab^3b^{-1}c \to ab^2c.$$

Definition

1. A word in $S \cup S^{-1}$ is *reduced* p.t. it can't be simplified.
2. Two words are *equivalent* p.t. they can be simplified to the same reduced word.

Concatenation

Definition

Concatenation is the operation $\circ$ on words which places the symbols of one word after the other.

Example. $w_1 = iam, w_2 = sam$

$$w_1 \circ w_2 = iamsam$$
$$w_2 \circ w_1 = samiam$$

N.B. concatenation is not commutative!

Free Group

Definition

The *free group* $(F_S, \circ)$ on *generating set* S is the group of non-equivalent words in $S \cup S^{-1}$, where $\circ$ is the concatenation operation.

Example. $S = \{a\}$

$$F_S = \{\ldots, a^{-2}, a^{-1}, \emptyset, a, a^2, \ldots\}$$

$$a^m \circ a^n = a^{m+n}$$

$$(F_S, \circ) \cong (\mathbb{Z}, +)$$

Free Group

Definition

The *free group* $(F_S, \circ)$ on *generating set* S is the group of non-equivalent words in $S \cup S^{-1}$, where $\circ$ is the concatenation operation.

Example. $S = \{a, b\}$

$$
\begin{aligned}
aba^2ba^{-1} \circ ab^{-1}a^{-1}ba^{-1}b^2 &= aba^2b\color{red}{a^{-1}a}\color{black}{b^{-1}a^{-1}ba^{-1}b^2} \\
&= aba^2\color{red}{bb^{-1}}\color{black}{a^{-1}ba^{-1}b^2} \\
&= aba^{\color{red}{2}}\color{red}{a^{-1}}\color{black}{ba^{-1}b^2} \\
&= \color{blue}{ababa^{-1}b^2}.
\end{aligned}
$$

Free Group

Definition

The *free group* $(F_S, \circ)$ on *generating set* S is the group of non-equivalent words in $S \cup S^{-1}$, where $\circ$ is the concatenation operation.

N.B. F_S depends only on the number of generators $|S|$.

Thus the (finitely generated) free groups are $F_1, F_2, F_3, \ldots$

Free Group

Definition

The *free group* $(F_S, \circ)$ on *generating set* S is the group of non-equivalent words in $S \cup S^{-1}$, where $\circ$ is the concatenation operation.

We check that the free group is a group.

Associativity. Word concatenation is associative:

$$w_1 \circ (w_2 \circ w_2) = a_1 \cdots a_j b_1 \cdots b_k c_1 \cdots c_\ell = (w_1 \circ w_2) \circ w_2$$

Identity. $e = \emptyset$: $\emptyset \circ w = w \circ \emptyset = w$.
Inverses. $g = a_1 \cdots a_k$, $g^{-1} = a_k^{-1} \cdots a_1^{-1}$

$$a_1 \cdots a_k \circ a_k^{-1} \cdots a_1^{-1} = a_k^{-1} \cdots a_1^{-1} \circ a_1 \cdots a_k = e.$$

Relations

We impose *relations* $r_1, r_2, \ldots, r_k$ in F_S on a free group by declaring

$$r_1 = r_2 = \cdots = r_k = e,$$

and following through all the implications.

Example. The *free abelian group* $\mathbb{Z}_S$ from F_S by setting

$$aba^{-1}b^{-1} = e$$

for every pair of generators a, b.

Relations

We impose *relations* $r_1, r_2, \ldots, r_k$ in F_S on a free group by declaring

$$r_1 = r_2 = \cdots = r_k = e,$$

and following through all the implications.

Example. The *projective special linear group* $\mathrm{PSL}(2, \mathbb{Z})$ from $F_{\{s,t\}}$ by

$$s^2 = (st)^3 = e.$$

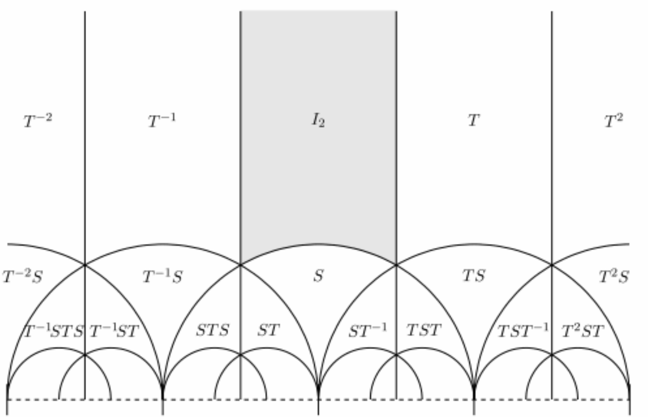
$\mathrm{PSL}(2, \mathbb{Z})$ can be realized as symmetries on the upper half plane $\mathbb{H}^2 = \{x + iy \in \mathbb{C} : x > 0\}$ formed by

$$S : z \mapsto -1/z, \qquad T : z \mapsto z + 1$$

$\mathbf{PSL}(2, \mathbb{Z})$

$\mathrm{PSL}(2, \mathbb{Z})$ can be realized as symmetries on the upper half plane $\mathbb{H}^2 = \{x + iy \in \mathbb{C} : x > 0\}$ formed by

$$S : z \mapsto -1/z, \qquad T : z \mapsto z + 1$$



Presentations

Definition

A *presentation* of a group is a way of writing it as a free group with relations.

Theorem

Every group has a presentation!

The Word Problem

Given a presentation of a group, and two words w_1, w_2, how can you tell if they are equal in the group?

The Word Problem

The Word Problem

Given a presentation of a group G with finitely many relations $r_1, \ldots, r_k$ and two words w_1, w_2 written in the generators, how can you tell if they are equal in G?

Example. In $\mathrm{PSL}(2, \mathbb{Z})$ [$= F_{\{s,t\}}$ with $s^2 = (st)^3 = e$],

$$sts = t^{-1}st^{-1}$$

Theorem

There is no algorithm which can solve the word problem.

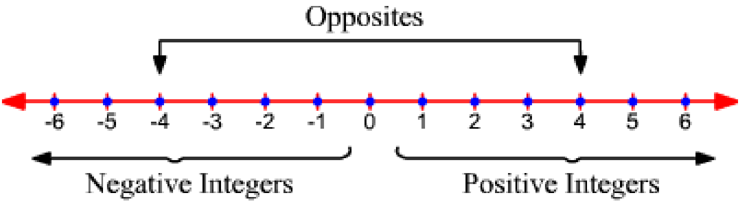
Cayley Graphs

Cayley Graphs

Definition

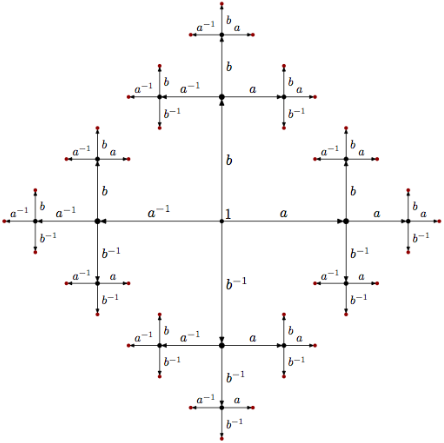
Given a group G and a presentation with generators S, its *Cayley graph* is the graph with the elements of G as vertices and edges $g - (s * g)$ if s is a generator.

Example. $G = (\mathbb{Z}, +)$, $S = \{1\}$.



Cayley Graphs

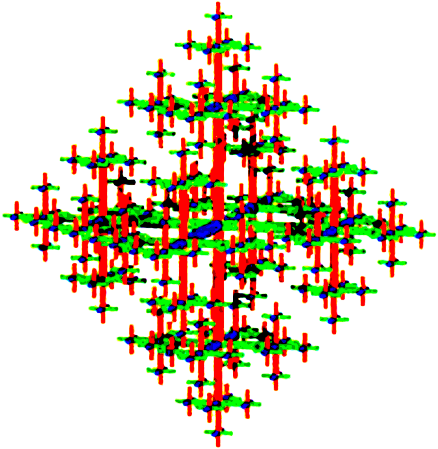
Example. The free group F_2 on 2 generators.



This graph is used in the proof of the Banach-Tarski theorem.

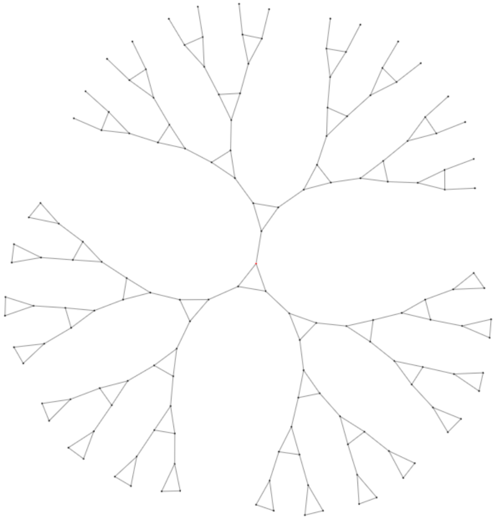
Cayley Graphs

Example. The free group F_3 on 3 generators.



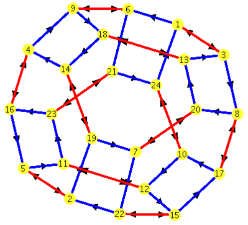
Cayley Graphs

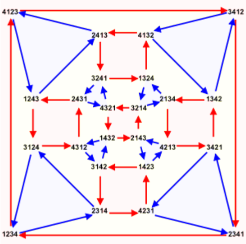
Example. $\mathrm{PSL}(2,\mathbb{Z})$ with generators $\{s, t\}$.

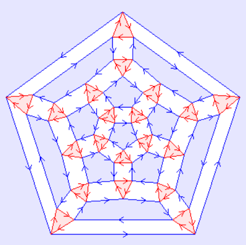


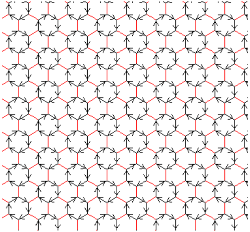
Cayley Graphs

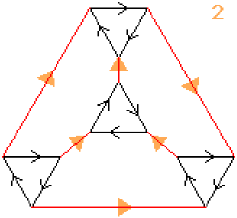
Some random Cayley Graphs

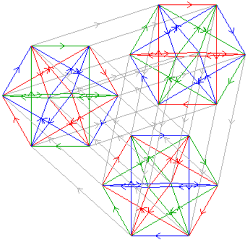












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