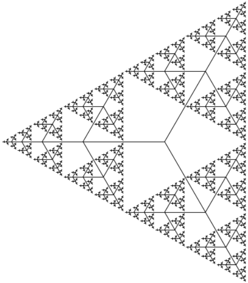
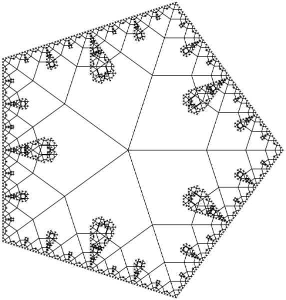
Free Groups and Infinite Trees



with

Noah Luntzlara

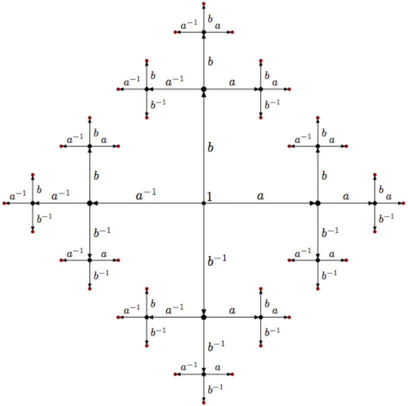
November 21, 2018



Honors 135.004
Fractals: Their Beauty and Topology
Connor Davis

Outline

- Group Theory
- Examples of Groups
- Free groups
- Cayley graphs



Group Theory

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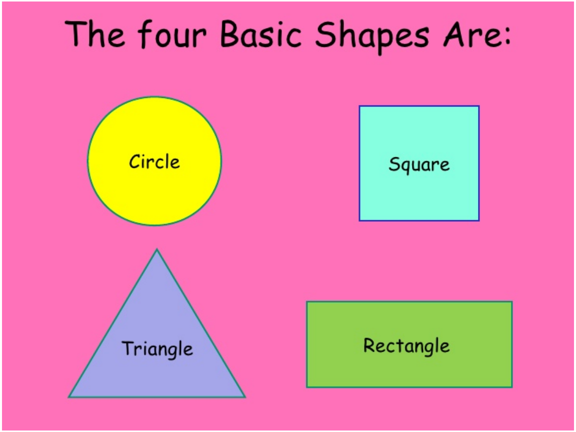
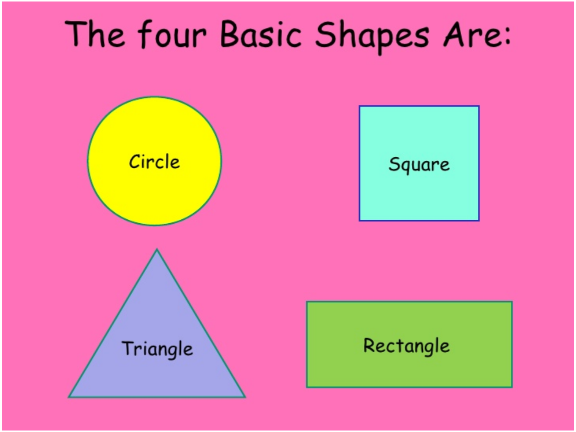
What is symmetry?

Symmetry


The four Basic Shapes Are:
Circle
Square
Triangle
Rectangle

Symmetry



Which is the most symmetric? Which is the least?

Symmetry

Definition

A *symmetry* is a transformation that leaves something unchanged.

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- Symmetries are always *reversible*.
- *Doing nothing* is always a symmetry.

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A *group* is a pair $(G, *)$, where G is a set and $*$ is a multiplication rule, which satisfies:

- **Associativity** $g * (h * k) = (g * h) * k$ always holds.
- There is an **identity** e in G such that for any g in G,
$$e * g = g \text{ and } g * e = g.$$
- Every g in G has an **inverse** g^{-1} in G such that
$$g * g^{-1} = e \text{ and } g^{-1} * g = e.$$

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  $GL_2(\mathbb{R}) = \{2 \cdot 2 \text{ matrices, entries in } \mathbb{R} \text{ and } \det = 1\}$.

N.B. For most pairs of matrices, $AB \neq BA$. For instance,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So generally $g * h = h * g$ isn't true in any group.

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Non-abelian groups.

- $GL_2(\mathbb{R})$.

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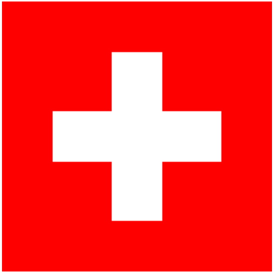
Thus we have **inverses**, **identity**. Furthermore, composition of transformations is **associative**.

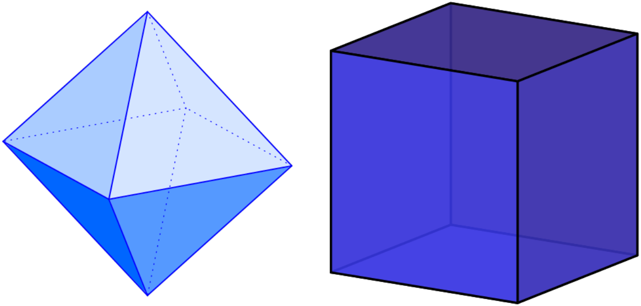
Symmetry Groups

What are the symmetry groups of these shapes?



Symmetry Groups

Amazing but true! These two objects have the same symmetry group:



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- Spacetime (Lorentz group and Poincaré group)

Free Groups

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Words may be *simplified* by deleting subwords of the form xx^{-1} or $x^{-1}x$. For instance, if $S = \{a, b, c\}$,

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Definition

1. A word in $S \cup S^{-1}$ is *reduced* p.t. it can't be simplified.
2. Two words are *equivalent* p.t. they can be simplified to the same reduced word.

Concatenation

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Concatenation is the operation $\circ$ on words which places the symbols of one word after the other.

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N.B. concatenation is not commutative!

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$$(F_S, \circ) \cong (\mathbb{Z}, +)$$

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Thus the (finitely generated) free groups are $F_1, F_2, F_3, \ldots$

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Associativity. Word concatenation is associative:

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$$a_1 \cdots a_k \circ a_k^{-1} \cdots a_1^{-1} = a_k^{-1} \cdots a_1^{-1} \circ a_1 \cdots a_k = e.$$

Relations

We impose *relations* $r_1, r_2, \ldots, r_k$ in F_S on a free group by declaring

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Example. The *free abelian group* $\mathbb{Z}_S$ from F_S by setting

$$aba^{-1}b^{-1} = e$$

for every pair of generators a, b.

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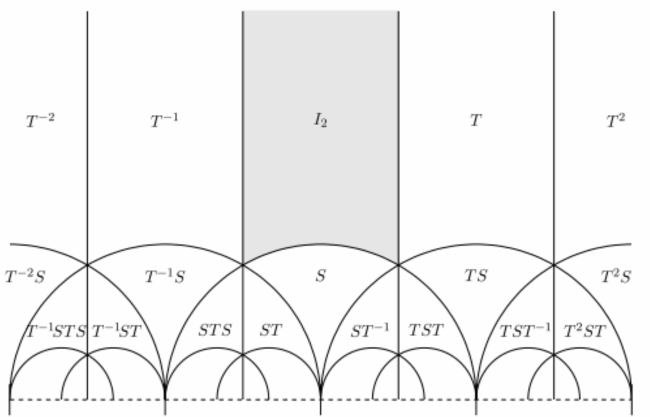

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Every group has a presentation!

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Given a presentation of a group, and two words w_1, w_2, how can you tell if they are equal in the group?

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Theorem

There is no algorithm which can solve the word problem.

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Given a group G and a presentation with generators S, its *Cayley graph* is the graph with the elements of G as vertices and edges $g - (s * g)$ if s is a generator.

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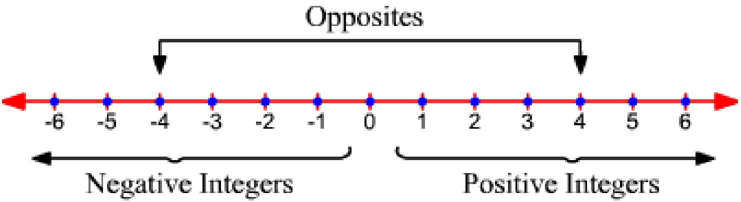
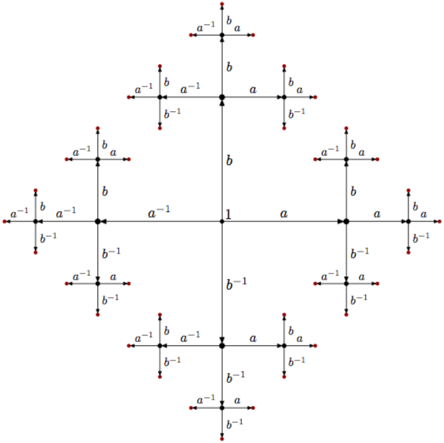


Cayley Graphs

Example. The free group F_2 on 2 generators.



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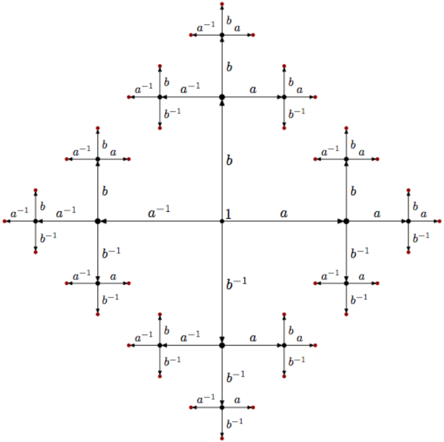
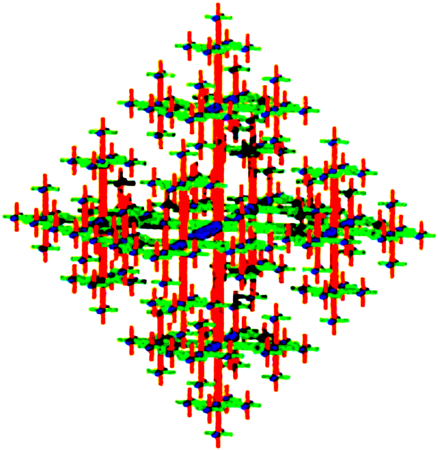


This graph is used in the proof of the Banach-Tarski theorem.

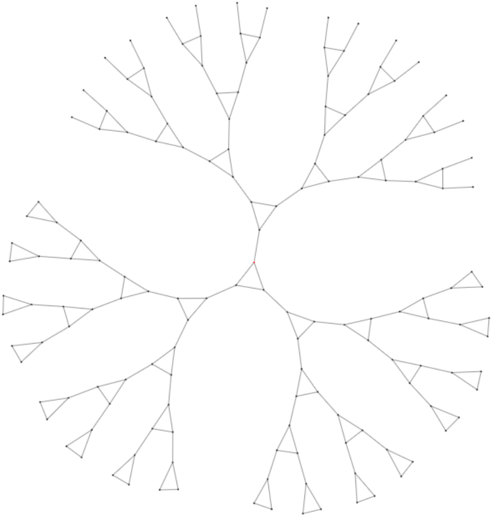
Cayley Graphs

Example. The free group F_3 on 3 generators.



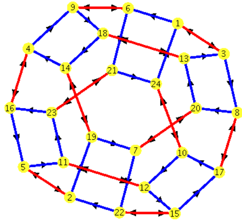
Cayley Graphs

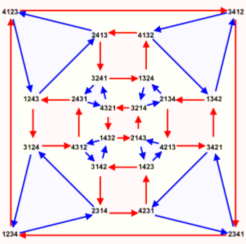
Example. PSL$(2, \mathbb{Z})$ with generators $\{s, t\}$.

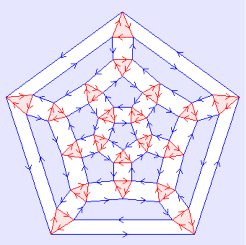


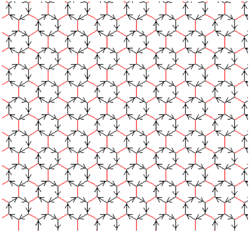
Cayley Graphs

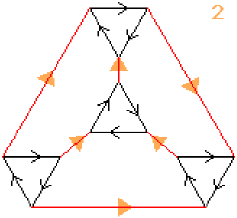
Some random Cayley Graphs

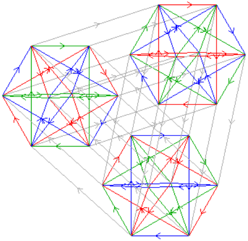












The End