On the Determination of Functions From Their Integral Values Along Certain Manifolds

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Translated by P. C. Parks from the original German text

WHEN one integrates a function of two variables x, y—a *point function* $f(P)$ in the plane—subject to suitable regularity conditions along an arbitrary straight line g then one obtains in the integral values $F(g)$, a *line function*. In Part A of the present paper the problem which is solved is the inversion of this linear functional transformation, that is the following questions are answered: can every line function satisfying suitable regularity conditions be regarded as constructed in this way? If so, is f uniquely known from F and how can f be calculated?

In Part B a solution of the dual problem of calculating a line function $F(g)$ from its point mean values $f(P)$ is solved in a certain sense.

Finally in Part C certain generalizations are discussed, prompted by consideration of non-Euclidean manifolds as well as higher dimensional spaces.

The treatment of these problems, themselves of interest, gains enhanced importance through the numerous relationships that exist between this topic and the theory of logarithmic and Newtonian potentials. These will be mentioned at appropriate places in the text.

A. Determination of a Point Function in the Plane From Its Straight Line Integral Values

1) For all real points $P = [x, y]$ let $f(x, y)$ be a real function satisfying the following regularity conditions:

a_1) $f(x, y)$ is continuous;

b_1) the double integral,

$$\iint \frac{|f(x, y)|}{\sqrt{x^2 + y^2}}\, dx\, dy$$

extending over the whole plane, converges;

c_1) for an arbitrary point $P = [x, y]$ and each $r \geqq 0$ let

$$\bar{f}_P(r) = \frac{1}{2\pi} \int_0^{2\pi} f(x + r \cos \phi, y + r \sin \phi)\, d\phi$$

so that for every point P

$$\lim_{r \to \infty} \bar{f}_P(r) = 0.$$

Then the following theorems hold good.

Theorem I: The straight line integral value of f along the line g having the equation $x \cos \phi + y \sin \phi = p$ is given by

$$F(p, \phi) = F(-p, \phi + \pi) = \int_{-\infty}^{\infty} f(p \cos \phi - s \sin \phi, p \sin \phi + s \cos \phi)\, ds \quad \text{(I)}$$

and exists almost everywhere: this means that on every circle the set of tangency points of all tangents for which F does not exist has a linear measure of zero. □

Theorem II: Constructing the mean value of $F(p, \phi)$ for the tangents of the circle with centre $P = [x, y]$ and radius q as

$$\bar{F}_P(q) = \frac{1}{2\pi} \int_0^{2\pi} F(x \cos \phi + y \sin \phi + q, \phi)\, d\phi \quad \text{(II)}$$

then this integral converges absolutely for all P, q. □

Theorem III: The value of f is uniquely determined from F and can be calculated as

$$f(P) = -\frac{1}{\pi} \int_0^{\infty} \frac{d\bar{F}_P(q)}{q}. \quad \text{(III)}$$

This integral is understood in the sense of a Stieltjes integral and can also be defined by the formula

$$f(P) = \frac{1}{\pi} \lim_{\epsilon \to 0} \left(\frac{\bar{F}_P(\epsilon)}{\epsilon} - \int_{\epsilon}^{\infty} \frac{\bar{F}_P(q)}{q^2}\, dq \right). \quad \text{(III')}$$

□

Before proceeding with the proof we remark that the conditions a_1)–c_1) are invariant in the face of coordinate shifts in the plane. We can therefore always consider the origin [0, 0] as representing any point of the plane. One recognises now the double integral

$$\iint_{x^2+y^2>q^2} \frac{f(x, y)}{\sqrt{x^2 + y^2 - q^2}}\, dx\, dy \quad (1)$$

as absolutely convergent. Through the transformation

$$x = q \cos \phi - s \sin \phi, \quad y = q \sin \phi + s \cos \phi$$

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it becomes

$$\int_0^{2\pi} d\phi \int_0^{\infty} f(q\cos\phi - s\sin\phi,\ q\sin\phi + s\cos\phi)\,ds$$

$$= \int_0^{2\pi} d\phi \int_{-\infty}^{0} f(q\cos\phi - s\sin\phi,$$

$$q\sin\phi + s\cos\phi)\,ds$$

so that one can express its value as

$$\tfrac{1}{2}\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} f(q\cos\phi - s\sin\phi,\ q\sin\phi + s\cos\phi)\,ds$$

$$= \tfrac{1}{2}\int_0^{2\pi} F(q,\phi)\,d\phi = \pi\bar{F}_0(q).$$

The assertions of Theorems I and II follow from known properties of absolutely convergent double integrals.

To arrive at the formula (III) one can take the following course.

Substitution of polar coordinates in (1) above yields

$$\int_q^{\infty} r\,dr \int_0^{2\pi} \frac{f(r\cos\phi,\ r\sin\phi)}{\sqrt{r^2-q^2}}\,d\phi$$

or with the assistance of the mean value expression of c_1)

$$2\pi\int_q^{\infty} \frac{\bar{f}_0(r)\,r\,dr}{\sqrt{r^2-q^2}}.$$

Comparison with the earlier form of (1) gives

$$\bar{F}_0(q) = 2\int_q^{\infty} \frac{\bar{f}_0(r)\,r\,dr}{\sqrt{r^2-q^2}}. \qquad (2)$$

Substituting the variables $r^2 = v$, $q^2 = u$ into this integral equation of the first kind one can easily solve it through the known method of Abel to give the formula (III) for

$$\bar{f}_0(0) = f(0,0).$$

This approach appears difficult, however, without further conditions on f, so we prefer a direct verification. First, in order to demonstrate the equality of expressions (III) and (III′) one must prove that

$$\lim_{q\to\infty} \frac{\bar{F}_0(q)}{q} = 0.$$

From (2) above

$$\left|\frac{\bar{F}_0(q)}{q}\right| \leqq \frac{2}{q}\left|\int_q^{2q} \frac{\bar{f}_0(r)\,r\,dr}{\sqrt{r^2-q^2}}\right| + \frac{2}{q}\left|\int_{2q}^{\infty} \frac{\bar{f}_0(r)\,r\,dr}{\sqrt{r^2-q^2}}\right|$$

$$\leqq \frac{2}{q}\int_q^{2q} \frac{|\bar{f}_0(r)|\,r}{\sqrt{r^2-q^2}}\,dr + \frac{2}{q}\int_{2q}^{\infty} \frac{|\bar{f}_0(r)|\,r}{\sqrt{r^2-q^2}}\,dr$$

$$\leqq 2\sqrt{3}\,|\bar{f}_0(t)| + \frac{4}{\sqrt{3}q}\int_{2q}^{\infty} |\bar{f}_0(r)|\,dr \quad (q \leqq t \leqq 2q)$$

and this converges to zero as $q \to \infty$ on account of b_1) and c_1).

Through the introduction of (2) the right-hand side of (III′) now becomes

$$\frac{2}{\pi}\lim_{\epsilon\to 0}\left[\frac{1}{\epsilon}\int_{\epsilon}^{\infty} \frac{r\bar{f}_0(r)\,dr}{\sqrt{r^2-\epsilon^2}} - \int_{\epsilon}^{\infty} \frac{dq}{q^2}\int_q^{\infty} \frac{r\bar{f}_0(r)\,dr}{\sqrt{r^2-q^2}}\right].$$

By changing the order of integration in the second integral one can integrate with respect to q, recognizing the integral as an absolutely convergent double integral which justifies this change. One obtains for the whole expression above the formula

$$\frac{2}{\pi}\lim_{\epsilon\to 0} \epsilon\int_{\epsilon}^{\infty} \frac{\bar{f}_0(r)}{r\sqrt{r^2-\epsilon^2}}\,dr$$

which yields on taking the limit the value $\bar{f}_0(0) = f(0,0)$, as is not difficult to show.

2) Let $F(p,\phi) = F(-p,\phi+\pi)$ be a line function satisfying the following regularity conditions.

a_2) F and the derivatives F_p, F_{pp}, F_{ppp}, F_{ϕ}, $F_{p\phi}$, $F_{pp\phi}$, exist and are continuous for all $[p,\phi]$.

b_2) F, F_{ϕ}, pF_p, $pF_{p\phi}$, pF_{pp}, tend to zero as $p \to \infty$ uniformly in ϕ.

c_2) The integrals

$$\int_0^{\infty} F_{pp}\ln p\,dp,\quad \int_0^{\infty} F_{ppp}p\ln p\,dp,\quad \int_0^{\infty} F_{pp\phi}p\ln p\,dp$$

converge absolutely and uniformly in ϕ. Then we can prove [the following theorem].

Theorem IV: Construct $f(P)$ from the formulae (III) or (III′), thus satisfying the conditions a_1), b_1), c_1) and yielding as the straight line integral value the above function $F(p,\phi)$. As a consequence of Theorem III it is the unique function of this kind. □

Substituting polar coordinates in (III) gives

$$f(\rho\cos\psi,\ \rho\sin\psi) = -\frac{1}{2\pi^2}\int_0^{\infty} \frac{dp}{p}\int_0^{2\pi}$$

$$F_p(\rho\cos\omega + p,\ \omega+\psi)\,d\omega$$

$$= \frac{1}{2\pi^2}\int_0^{\infty} \ln p\,dp \int_0^{2\pi}$$

$$F_{pp}(p + \rho\cos\omega,\ \omega+\psi)\,d\omega.$$

Now

$$\int_0^{2\pi} F_p(\rho\cos\omega + p,\ \omega+\psi)\,d\omega$$

$$= \int_0^{2\pi} F_p(\rho\cos\omega,\ \omega+\psi)\,d\omega$$

$$+ \int_0^{2\pi} d\omega \int_0^{p} F_{pp}(\rho\cos\omega + t,\ \omega+\psi)\,dt$$

and the first component is zero as $F(p,\phi) = F(-p,\phi-\pi)$. For this reason the product of the integral with $\ln p$ tends to zero as $p \to 0$. On account of the same property of F it follows also that

$$f(\rho \cos \psi, \rho \sin \psi) = \frac{1}{2\pi^2} \int_0^\pi d\omega \int_{-\infty}^{\infty} F_{pp}(p, \omega + \psi) \cdot \ln |p - \rho \cos \omega| \, dp. \tag{3}$$

It is sufficient that we show

$$\int_{-\infty}^{\infty} f(\rho, 0) \, d\rho = F(0, \pi/2) \tag{4}$$

as the conditions $a_2)$–$c_2)$ are invariant in the face of coordinate changes.

We put

$$F(p, \phi) = F(p, \pi/2) + \cos \phi \, G(p, \phi).$$

G satisfies obvious regularity conditions. It is now necessary, on account of this decomposition, to split $f(\rho, 0)$ into two parts $f_1(\rho)$ and $f_2(\rho)$ which are investigated separately. Because

$$\int_0^\pi \ln |p - \rho \cos \omega| \, d\omega = \begin{cases} \pi \ln \dfrac{|p| + \sqrt{p^2 - \rho^2}}{2}, & |p| > |\rho| \\ \pi \ln \dfrac{|\rho|}{2}, & |p| \leqq |\rho| \end{cases}$$

one obtains

$$\begin{aligned} f_1(\rho) &= \frac{1}{2\pi^2} \int_0^\pi d\omega \int_{-\infty}^{\infty} F_{pp}(p, \pi/2) \ln |p - \rho \cos \omega| \, dp \\ &= \frac{1}{2\pi} \int_{|\rho|}^{\infty} F_{pp}(p, \pi/2) \ln \frac{|p| + \sqrt{p^2 - \rho^2}}{|\rho|} \, dp \\ &\quad + \frac{1}{2\pi} \int_{-\infty}^{-|\rho|} F_{pp}(p, \pi/2) \ln \frac{|p| + \sqrt{p^2 - \rho^2}}{|\rho|} \, dp. \end{aligned}$$

This is now absolutely integrable with respect to ρ from $-\infty$ to $+\infty$ as will be obvious through exchanging the order of integration. One evaluates the integral as

$$\begin{aligned} \int_{-\infty}^{\infty} f_1(\rho) \, d\rho &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{pp}(p, \pi/2) \cdot d_p \int_{-|p|}^{|p|} \ln \frac{|p| + \sqrt{p^2 - \rho^2}}{|\rho|} \, d\rho \\ &= \frac{1}{2} \int_{-\infty}^{\infty} F_{pp}(p, \pi/2) |p| \, dp = F(0, \pi/2). \end{aligned}$$

As far as $f_2(\rho)$ is concerned, as we shall show, it is also absolutely integrable and when integrated from $-\infty$ to $+\infty$, gives zero.

We can of course write $f_2(\rho)$ in the following way:

$$\begin{aligned} f_2(\rho) &= \frac{1}{2\pi^2} \int_0^\pi d\omega \int_{-\infty}^{\infty} G_{pp}(p, \omega) \ln |p - \rho \cos \omega| \cdot \cos \omega \, dp \\ &= \frac{1}{2\pi^2} \int_0^\pi d\omega \int_{-\infty}^{\infty} G_{pp}(p, \omega) \left[\ln \left| \frac{p - \rho \cos \omega}{\rho \cos \omega} \right| \cos \omega + \frac{\rho \, p \cos^2 \omega}{1 + \rho^2 \cos^2 \omega} \right] dp \end{aligned}$$

in that the added terms integrate to give zero and in this form the integration with respect to ρ results in an *absolutely* convergent triple integral. It is

$$\int_{-\infty}^{\infty} \left\| \left[\ln \left| \frac{p - \rho \cos \omega}{\rho \cos \omega} \right| \cos \omega + \frac{\rho \, p \cos^2 \omega}{1 + \rho^2 \cos^2 \omega} \right] \right\| d\rho = |p| \int_{-\infty}^{+\infty} \left\| \left[\ln \left| 1 - \frac{1}{\tau} \right| + \frac{p^2 \tau}{1 + p^2 \tau^2} \right] \right\| d\tau = \lambda(p)$$

(putting $|p| \tau = \rho \cos \omega$)

with

$$\lim_{|p| \to \infty} \frac{\lambda(p)}{|p| \ln |p|} = 2.$$

The integration with respect to ρ yields as the value of the integral

$$\int_{-\infty}^{+\infty} f_2(\rho) \, d\rho = 0$$

whereby (4) is proved.

We have now to show that f satisfies the conditions $a_1)$–$c_1)$. The continuity follows from the representation (3) on account of assumptions $a_2)$–$c_2)$. Condition $b_1)$ is equally satisfied because

$$\int_{-\infty}^{+\infty} |f(\rho \cos \psi, \rho \sin \psi)| \, d\rho$$

is integrable with respect to ψ, as one easily sees.

To prove $c_1)$ we set

$$\begin{aligned} \overline{f_0}(\rho) &= \frac{1}{2\pi} \int_0^{2\pi} f(\rho \cos \psi, \rho \sin \psi) \, d\psi \\ &= \frac{1}{4\pi^2} \int_0^\pi d\omega \int_0^{2\pi} d\psi \int_{-\infty}^{\infty} F_{pp}(\rho, \psi) \ln |p - \rho \cos \omega| \, dp \\ &= \frac{1}{4\pi^2} \int_0^{2\pi} d\psi \left[\int_{-\infty}^{-|\rho|} F_{pp}(\rho, \psi) \ln \frac{|p| + \sqrt{p^2 - \rho^2}}{2} \, dp \right. \\ &\quad + \int_{|\rho|}^{+\infty} F_{pp}(\rho, \psi) \ln \frac{|p| + \sqrt{p^2 - \rho^2}}{2} \, dp \\ &\quad \left. + F_p(\rho, \psi) \ln \frac{\rho}{2} - F_p(\rho, \psi) \ln \frac{\rho}{2} \right] \end{aligned}$$

from which one can recognize the correctness of $c_1)$. Thus, Theorem IV is proved.

B. Determination of a Line Function From Its Point Mean Values

3) Let $F(p, \phi) = F(-p, \phi + \pi)$ be a line function that satisfies the following regularity conditions.

a₃) F, F_ϕ, F_p are continuous with $|F_\phi| < M$ for all p, ϕ;

b₃) $F_p \ln |p|$ converges to zero uniformly in ϕ as $p \to \infty$;

c₃) $\int_{-\infty}^{\infty} |F_p| \cdot \ln |p|\, dp$ converges uniformly in ϕ.

These conditions are moreover invariant with respect to coordinate shifts.

We construct the point mean value of $F(p, \phi)$ for $P = [x, y]$:

$$f(x, y) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} F(x \cos \phi + y \sin \phi, \phi)\, d\phi. \quad (5)$$

Then we have [the following theorem].

Theorem V: Through the expression for f, F is uniquely determined and is given by

$$F\left(0, \frac{\pi}{2}\right) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dx}{x} \int_{-\infty}^{\infty} f_x(x, y)\, dy$$

where the integral with respect to x is understood in the sense of the Cauchy principal value and the value of F for any other straight line can be derived from the given formula through an appropriate coordinate transformation. □

For the proof, we next derive from (5):

$$\int_{-A}^{B} f_x(x, y)\, dy = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} d\phi \int_{-A}^{B} F_p(x \cos \phi + y \sin \phi, \phi) \cos \phi\, dy \quad (6)$$

where A and B are positive constants. We now put, as was done analogously earlier on:

$$F(p, \phi) = F(p, 0) + \sin \phi\, G(p, \phi)$$

where $G(p, \phi)$ remains restricted to the integration region and takes the limiting value of zero as $p \to \infty$. From

$$\int_{-A}^{B} G_p(x \cos \phi + y \sin \phi, \phi) \cos \phi \cdot \sin \phi\, dy$$
$$= [G(x \cos \phi + B \sin \phi, \phi) - G(x \cos \phi - A \sin \phi, \phi)] \cos \phi$$

it follows that the second component of (6) takes the limiting value of zero for $A \to \infty$, $B \to \infty$, so that it remains only to investigate the first component. Through the analogous integration one recognizes that in this first component the integral with respect to ϕ likewise will be zero for $A \to \infty$, $B \to \infty$ over any interval *not* containing $\phi = 0$; it remains therefore to examine

$$\lim_{\substack{A \to \infty \\ B \to \infty}} \int_{-\epsilon}^{+\epsilon} d\phi \int_{-A}^{B} F_p(x \cos \phi + y \sin \phi, 0) \cos \phi\, dy, \quad 0 < \epsilon \leq \pi/2.$$

One can write this integral as

$$\frac{1}{\pi} \int_{-\epsilon}^{+\epsilon} d\phi \int_{x\cos\phi - A\sin\phi}^{x\cos\phi + B\sin\phi} F_p(p, 0) \cot \phi\, dp$$

and obtain from it, when A and B take sufficiently large values, through exchange of the order of integration and after some calculation, the value

$$\frac{1}{\pi} \int_{x\cos\epsilon - B\sin\epsilon}^{x\cos\epsilon + B\sin\epsilon} \ln \frac{(B^2 + x^2) \sin \epsilon}{|Bp - x\sqrt{B^2 + x^2 - p^2}|} F_p(p, 0)\, dp$$
$$+ \frac{1}{\pi} \int_{x\cos\epsilon - A\sin\epsilon}^{x\cos\epsilon + A\sin\epsilon} \ln \frac{(A^2 + x^2) \sin \epsilon}{|Ap - x\sqrt{A^2 + x^2 - p^2}|} F_p(p, 0)\, dp.$$

It is sufficient to determine the limiting value of the second integral for $A \to \infty$. We write it thus:

$$\frac{1}{\pi} \ln (A \sin \epsilon) [F(x \cos \epsilon + A \sin \epsilon, 0) - F(x \cos \epsilon - A \sin \epsilon, 0)]$$
$$+ \frac{1}{\pi} \int_{x\cos\epsilon - A\sin\epsilon}^{x\cos\epsilon + A\sin\epsilon} \ln \frac{1}{|p - x|} F_p(p, 0)\, dp$$
$$+ \frac{1}{\pi} \int_{x\cos\epsilon - A\sin\epsilon}^{x\cos\epsilon + A\sin\epsilon} \ln \frac{|Ap + x\sqrt{A^2 + x^2 - p^2}|}{A|p + x|} F_p(p, 0)\, dp.$$

Since the logarithm in the last integral tends *uniformly* to zero for $A \to \infty$ then the limiting value follows as

$$-\frac{1}{\pi} \int_{-\infty}^{+\infty} F_p(p, 0) \ln |p - x|\, dp$$

whereby the limiting value of (6) is obtained as

$$\int_{-\infty}^{+\infty} f_x(x, y)\, dy = -\frac{2}{\pi} \int_{-\infty}^{+\infty} F_p(p, 0) \ln |p - x|\, dp.$$

It should be noted here that the latter expression represents the limiting values of the imaginary part of an analytic function which is regular in the upper plane and for which the limiting values of the real part have the value $2F(x, 0)$. We now form in the sense of the formula in Theorem V:

$$-\int_{-\infty}^{+\infty} \frac{dx}{x} \int_{-\infty}^{+\infty} f_x(x, y)\, dy = \frac{2}{\pi} \int_{0}^{\infty} \frac{dx}{x} \int_{-\infty}^{+\infty} F_p(p, 0) \ln \left|\frac{p - x}{p + x}\right| dx$$

so that this double integral is absolutely convergent and because

$$\int_{0}^{\infty} \ln \left|\frac{p - x}{p + x}\right| \frac{dx}{x} = -\frac{\pi^2}{2} \operatorname{sgn} p$$

it leads precisely to the formula in Theorem V.

4) Now let f be a point function with the following regularity properties.

a₄) f with its derivatives of up to the second order inclusive are continuous;

b₄) the expressions $f(x, y)$, $\sqrt{x^2 + y^2} \cdot \ln (x^2 + y^2)$

$\cdot f_x(x, y)$, $\sqrt{x^2 + y^2} \cdot \ln (x^2 + y^2) \cdot f_y(x, y)$ have the limiting value zero for $x^2 + y^2 \to \infty$;

c_4) the integrals

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} D_1 f \cdot \frac{\ln (x^2 + y^2)}{\sqrt{x^2 + y^2}} \, dx \, dy$$

$$\text{and} \quad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} D_2 f \cdot \ln (x^2 + y^2) \, dx \, dy$$

where $D_1 f$ represents every first derivative of f and $D_2 f$ every second derivative, all converge absolutely.

These conditions are again invariant with coordinate changes.

Then we have [the following theorem].

Theorem VI: The straight line function calculated from f according to the formula in Theorem V takes the point mean values $f(x, y)$. □

It is sufficient to furnish the proof for the origin. For an arbitrary straight line through it, Theorem V yields, following a partial integration

$$F(0, \phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [f_{xx} \cos^2 \phi + 2 f_{xy} \sin \phi \cos \phi + f_{yy} \sin^2 \phi] \ln |x \cos \phi + y \sin \phi| \, dx \, dy$$

or, introducing polar coordinates ρ, ψ:

$$F(0, \phi) = \frac{1}{2\pi} \int_0^{\infty} \rho \, d\rho \int_0^{2\pi} \left[\frac{\partial^2 f}{\partial \rho^2} \cos^2 (\phi - \psi) + 2 \frac{\partial^2 f}{\partial \rho \partial \psi} \frac{\sin (\phi - \psi) \cos (\phi - \psi)}{\rho} + \frac{\partial^2 f}{\partial \psi^2} \frac{\sin^2 (\phi - \psi)}{\rho^2} + \frac{\partial f}{\partial \rho} \frac{\sin^2 (\phi - \psi)}{\rho} - 2 \frac{\partial f}{\partial \psi} \frac{\sin (\phi - \psi) \cos (\phi - \psi)}{\rho^2} \right] \ln |\rho \cos (\phi - \psi)| \, d\psi.$$

In order to construct the point mean value for [0, 0] one can carry out the integration with respect to ψ in the double integral from 0 to 2π and then divide by 2π. The term with $\partial^2 f / \partial \psi^2$ that appears drops out in the integration with respect to ψ and so there remains

$$\frac{1}{2\pi} \int_0^{2\pi} d\psi \int_0^{\infty} \left[\frac{1}{4} \left(\rho \frac{\partial^2 f}{\partial \rho^2} - \frac{\partial f}{\partial \rho} \right) + \frac{1}{2} \ln \frac{\rho}{2} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) \right] d\rho$$

which reduces in fact to $f(0, 0)$.

In order to demonstrate the uniqueness of F, we must prove that Conditions a_3)–c_3) are satisfied, this clearly requires further conditions on f.

5) It is desirable to find a place for the following remark, for which I am indebted to Herr W. Blanschke, concerning in general the status of Problem B: both problems considered here are closely related to the theory of Newtonian potentials. We consider particularly the conversion of point function f to its straight line mean values F as a linear functional transformation

$$F = Rf$$

and likewise the conversion of line function F to its point mean value v:

$$v = BF$$

suggesting that, compounded as

$$v = Hf = B[Rf] = BRf$$

the transformation $H = BR$ should be considered.

One now sees immediately that Hf is none other than the Newtonian potential of the plane covered with the mass density $(1/\pi) f$ calculated at points of the plane itself. It follows that one can obtain the inversion of the transformation H from a remark of G. Herglotz; this results in

$$f(P) = H^{-1} v = -\frac{1}{2} \int_0^{\infty} \frac{d\bar{v}_P(r)}{r} = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Delta v(x', y')}{r_{PP'}} \, dx' \, dy'$$

where $\bar{v}_P$ is a mean value expression analogous to that introduced earlier and Δ denotes the Laplace operator.

Now it is obvious that the inversions directly introduced in 1)–3) are achieved through the expressions

$$R^{-1} = H^{-1} B \quad \text{or} \quad B^{-1} = R H^{-1}.$$

Actually I first discovered the inversion formula (IV) in this way, but a rigorous development of this conception appeared to be more difficult than the direct verification and it also failed in the equally important non-Euclidean cases.

Finally, it should be noted that the basic regularity conditions in Sections A and B are obviously far from the most general, as may be shown by simple examples.

C. Generalizations

6) A far-reaching generalization of the problem treated in A can be formulated as follows: let a surface S be given on which an arc differential ds is somehow defined as well as a double infinite sheaf of curves C on S. It is required to determine a point function of the surface from its integral values $\int f ds$ along the curves C.

One obtains the nearest specialization when one takes a non-Euclidean plane as S, the appropriate arc element for ds and straight lines for the curves C. In the elliptical cases one can bring into play the results of spherical geometry; in a known manner one interprets a diametrical point-pair of the sphere as a point of the elliptical plane obtaining the result that a direction function on the sphere—that means equal valued at diametric points—can be determined from its great circle integral values. Minkowski [1] has been the first to consider this result in principle and to have solved it through the use of the spherical functions; P. Funk [2] later realized the Minkowski solution and showed how one can find the solution with the help of Abelian integral equations. I also am indebted to this method for the solution of Problem A. Funk's solution is altogether analogous to (III), only substituting in the denominator the sine of the spherical radius and adding to the integral the value of F at the pole of the great circle in question divided by π. The stated result also has an analogous solution to (III) in the hyperbolic plane

$$f(P) = -\frac{1}{\pi}\int_0^\infty \frac{d\bar{F}_P(q)}{\sinh q}$$

(here the curvature measure is taken as -1) as one can show totally conforms to the derivation of III indicated in I.

In both cases, one can also for example pose the problem analogous to B. In the elliptical geometry one obtains nothing new by virtue of the absolute polarity: in the hyperbolic case a solution analogous to Theorem V appears not to exist.

A second specialization follows when one takes as the curves C circles with constant radius (in Euclidean or non-Euclidean geometries). Here one can make use of the Minkowski treatment using spherical functions on the sphere and solve the problem to a certain extent. It is interesting in this case that the uniqueness of the solution can be lost; there are certain radii ρ determined by the zeros of the Legendre polynomials of even order for which line functions on the sphere integrate to give zero along every circle of spherical radius ρ without vanishing identically. In the Euclidean case the integral theorem of Bessel functions takes the place of the spherical function sequence; here there are always functions that integrate on all circles of constant radius giving zero and still do not vanish identically; if the radius is in unity then (in polar coordinates ρ, ϕ) these functions are

$$J_n(x_\upsilon\rho)\cos n\phi, \quad J_n(x_\upsilon\rho)\sin n\phi$$

and their linear combinations where x_υ, is a zero of J_0. In hyperbolic cases the so-called spherical functions take the place of the Bessel functions for which the integral theorem of Weyl [3] is appropriate. The results are analogous to the Euclidean cases.

7) In another direction the results of A and B can be generalized by transposition to higher dimensions. In a Euclidean R^n one can seek to determine a point function $f(P) = f(x_1, x_2, \cdots, x_n)$ from its integral values $F(\alpha_1, \alpha_2, \cdots \alpha_n, p)$ on all hyperplanes $\alpha_1 x_1 + \cdots + \alpha_n x_n = p$ $(\alpha_1^2 + \cdots + \alpha_n^2 = 1)$. Analogous to the procedure followed in 1 we construct the mean value $\bar{F}_0(q)$ of F on the tangent planes of the sphere of centre of $(0, 0, \cdots 0)$ and radius q. It is given by the $n - 1$ fold integral

$$\bar{F}_0(q) = \frac{1}{\Omega_n}\int F(\alpha, q)\, d\omega$$

where $d\omega$ is the surface element, $\Omega_n = [2\pi^{n/2}]/[\Gamma(n/2)]$ is the surface area of the n-dimensional sphere $x_1^2 + \cdots + x_n^2 = 1$.

One can describe $\bar{F}_0$ through an n-fold integral in f and indeed this is

$$\bar{F}_0(q) = \frac{\Omega_{n-1}}{\Omega_n}\iint_{x_1^2+\cdots+x_n^2>q^2}\cdots\int f(x_1, \cdots x_n) \cdot \frac{(x_1^2 + \cdots + x_n^2 - q^2)(n-3)/2}{(x_1^2 + \cdots + x_n^2)(n-2)/2}\, dx_1 \cdots dx_n \tag{7}$$

or in a frequently used mean value expression

$$\bar{F}_0(q) = \Omega_{n-1}\int_q^\infty \bar{f}_0(r)(r^2 - q^2)^{(n-3)/2}\, r\, dr.$$

This is the formula analogous to (2) to which is connected the corresponding conclusions. The substitution $r^2 = v$, $q^2 = u$ leads to the integral equation

$$\Phi(u) = \frac{\Omega_{n-1}}{2}\int_u^\infty \phi(v)(v - u)^{(n-3)/2}\, dv.$$

If n is *even* $(n/2 - 1)$ differentiations with respect to u gives the the same equation as (2) and one can find from this

$$\phi(0) = f(0, 0, \cdots, 0).$$

Therefore, for the construction of f from a given F, both differentiation and an integral operation are necessary. For *odd n* this integral operation becomes redundant because $(n - 1)/2$-fold differentiation now gives

$$\phi(0) = \frac{2(-1)(n-1)/2}{\Omega_{n-1}\left(\frac{n-3}{2}\right)!}\,\Phi^{(n-3)/2}(0).$$

The three-dimensional case turns out to be particularly simple; one can treat this case also by a method that is analogous to 5) and which yields very elegant results. From (7) the point mean value of F for $q = 0$ emerges as

$$\bar{F}_0 = \frac{1}{2}\iiint \frac{f(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}\, dx\, dy\, dz$$

which is considered as the Newtonian potential of the space filled with a mass density $(1/2)f$. Hence, it follows that

$$f(x, y, z) = \frac{1}{2\pi}\Delta\bar{F}$$

where $\bar{F}$ indicates the point mean value of F.

Here one can also solve the problem which is analogous to B and obtain by the methods indicated in 5) for planar functions F of which the point mean values f are known

$$F(E) = -\frac{1}{2\pi} \iint \Delta f \, d\sigma$$

where $d\sigma$ is the surface element of the plane. Δ is the Laplace operator for three-dimensional space and the integration is extended over the whole plane E.

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References

[1] H. Minkowski, *Collected Works, Vol. 2*, p. 277 *et seq.*
[2] P. Funk, *Math. Ann.* vol. 74, pp. 283–288.
[3] H. Von Weyl, *Gött. Nachr.*, p. 454, 1910.

Translator's Notes

The original German text was published in *Berichte der Sächsischen Akadamie der Wissenschaft*, vol. 69, pp. 262–277, 1917. (Session of April 30, 1917.)

I have tried to keep closely to the original German text, but occasionally I have made a small deviation to improve the readability of the English version.

I have added the symbol □ to denote the end of the statements of the theorems. I have also changed the notation l in the original (which denotes logarithm) to the more familiar ln, and on the original page 275 the German fraktur sin q is changed to the more usual modern form for the hyperbolic sine of q which is sinh q.

The original contains a number of misprints in the mathematical expressions which I have corrected. These errors include the omission of bars over $\bar{F}_P(q)$, the omission of r in the numerator of (2) on p. 265 and of $r\,dr$ some 12 lines later, as well as $1/q$ in the second term on the right-hand side of the subsequent inequality. On page 266 an ϵ was missing in the expression just before the start of Section 2 of the paper. ϕ was printed instead of ψ at several places in the original paper.

The references have been numbered and collected at the end of the paper instead of appearing as footnotes.

I hope that by translating the present paper into English I shall have made this pioneering work of Johann Radon widely available to the many research workers in the medical image processing field who make use of what are now called "Radon transforms."

At the time of making this translation I was unaware of the English translation of this paper by R. Lohner which appears as Appendix A in the following book: S. R. Dean, *The Radon Transform and Some of Its Applications* (New York: Wiley-Interscience, 1983). I am grateful to Prof. A. K. Louis (Universität Kaiserslautern) for this information: