

If $|n\rangle$ is the n th eigenstate of the harmonic potential

$$|n\rangle = N_n e^{-\alpha x^2/2} H_n(\alpha x)$$

$\uparrow$ norm. factor $\uparrow$ Hermite polynomial

$$N_n = \left[\frac{\alpha}{\sqrt{\pi}} 2^n n! \right]^{-1/2}$$

$$\alpha = \left[\frac{mk}{\hbar^2} \right]^{1/4} = \left[\frac{m\omega}{\hbar} \right]^{1/2}$$

Evaluate

$$\textcircled{1} \quad \langle m|x|n\rangle = N_m^* N_n \int_{-\infty}^{\infty} dx e^{-\alpha x^2/2} H_m(\alpha x) x e^{-\alpha x^2/2} H_n(\alpha x)$$

~~Let~~ $z = \alpha x$

$$\textcircled{2} \quad \langle m|x|n\rangle = \frac{1}{\alpha} N_m^* N_n \int_{-\infty}^{\infty} dz e^{-z^2/2} H_m(z) z H_n(z)$$

From the recursion relation

$$H_{n+1}(z) = 2z H_n(z) - 2n H_{n-1}(z)$$

$$\textcircled{3} \quad z H_n(z) = \frac{1}{2} H_{n+1}(z) + n H_{n-1}(z)$$

$$\textcircled{3} \Rightarrow \textcircled{2} \Rightarrow$$

(2)

$$\langle m|x|n\rangle = \frac{1}{2} N_m^* N_n \int_{-\infty}^{\infty} dz e^{-z^2} H_m(z) \left[n H_{n-1}(z) + \frac{1}{2} H_{n+1}(z) \right]$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dz e^{-z^2} N_m^*(z) \left[n \left(\frac{N_n}{N_{n-1}} \right) H_n H_{n-1}(z) + \frac{1}{2} \left(\frac{N_n}{N_{n+1}} \right) H_{n+1} H_{n+1}(z) \right]$$

$$= \frac{1}{2} \left\{ n \frac{N_n}{N_{n-1}} S_{m,n-1} + \frac{1}{2} \frac{N_n}{N_{n+1}} S_{m,n+1} \right\}$$

$$= \frac{1}{2} \left\{ n \left(\frac{1}{2^n} \right)^{1/2} S_{m,n-1} + \frac{1}{2} \left(2^{n+1} \right)^{1/2} S_{m,n+1} \right\}$$

$$\langle m|x|n\rangle = \frac{1}{2} \left\{ \left(\frac{n}{2} \right)^{1/2} S_{m,n-1} + \left(\frac{n+1}{2} \right)^{1/2} S_{m,n+1} \right\}$$

