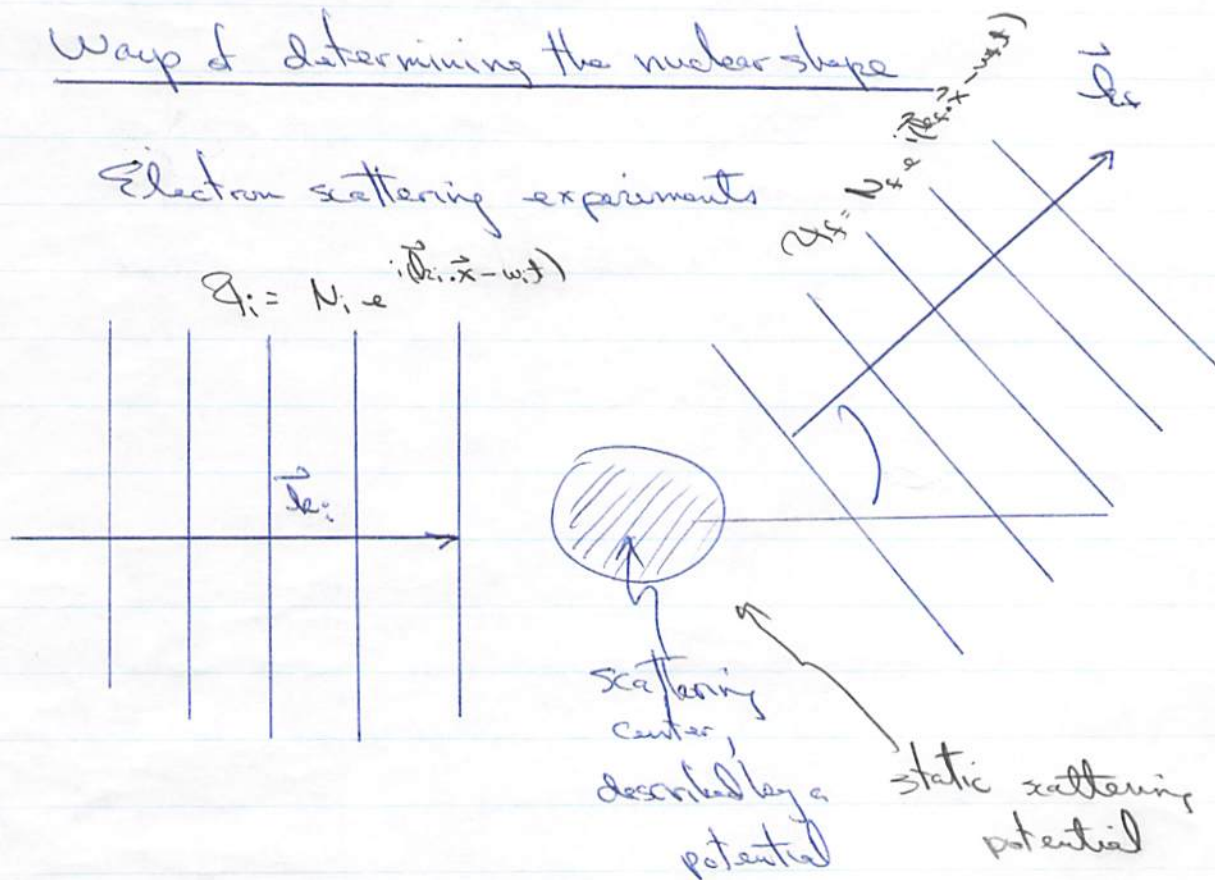


Supplemental notes to lecture 2

(1)

Way of determining the nuclear shape

Electron scattering experiments



Recall Fermi's Golden rule (K. 2.79) (K. 2.80)

$$\lambda = \frac{2\pi}{\hbar} \left| \int d\vec{x} \psi_f^*(\vec{x}) V_{\text{pert}}(\vec{x}) \psi_i(\vec{x}) \right|^2 \rho(E)$$

↑
density of final states

In this case we regard free plane waves as the basis solutions of the SE, and a localized potential, in this case the Coulomb potential of the nucleus, as the perturbation. It's crude, but it works. (This is called the "Born approximation", treating ψ_f, ψ_i as

aside ... distorted wave BA, partial wave analysis

(2)

free waves.)

Generalization of 1D plane wave to 3D

In 1D $\Psi_{\text{free}}(x) = N e^{i(kx - \omega t)}$

norm factor

$k = \sqrt{\frac{2mE}{\hbar^2}}$; $\hbar\omega = E$

In 3D $\Psi_{\text{free}}(\vec{x}) = N e^{i(\vec{k} \cdot \vec{x} - \omega t)}$

Let's calculate this matrix element in detail

$\Psi_i(\vec{x}) = N_i e^{i(\vec{k}_i \cdot \vec{x} - \omega_i t)}$

$\Psi_f(\vec{x}) = N_f e^{i(\vec{k}_f \cdot \vec{x} - \omega_f t)}$

$V_{fi} = \int d\vec{x} \Psi_f^*(\vec{x}) V_p(\vec{x}) \Psi_i(\vec{x})$

$= \int d\vec{x} N_f^* e^{-i(\vec{k}_f \cdot \vec{x} - \omega_f t)} V_p(\vec{x}) N_i e^{i(\vec{k}_i \cdot \vec{x} - \omega_i t)}$

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$V_{fi} = N_f^* N_i \int d\vec{x} e^{i(\vec{k}_i - \vec{k}_f) \cdot \vec{x}} V_p(\vec{x}) e^{-i(\omega_i - \omega_f)t}$

just as in Rutherford scattering, we expect $E_f = E_i$, elastic scattering, if the target is left unchanged.

$\omega_i = \omega_f$

$\omega_i = \omega_f$

$|\vec{k}_i| = |\vec{k}_f|$

$|\vec{k}_i| = |\vec{k}_f|$

$$\vec{q} = \vec{r}_i - \vec{r}_e$$

(3)

$$V_i = N_e N_i \underbrace{\int d\vec{x} e^{i\vec{q} \cdot \vec{x}} V_p(\vec{x})}$$

note: This is just the 3D Fourier integral. By Fourier inversion, if I measure $V_i(\vec{q})$, then I can obtain $V_p(\vec{x})$!

What is $V_p(\vec{x})$?

If we have a charge distribution $g(\vec{x}')$, arising from Z protons

$$V_p(\vec{x}) = \underbrace{-}_{\substack{\uparrow \\ \text{attractive}}} \underbrace{(Ze)(e)}_{\substack{\text{proton charge} \\ \text{electron charge}}} \underbrace{\int d\vec{x}' \frac{g(\vec{x}')}{|\vec{x} - \vec{x}'|}}_{\substack{\text{from electrostatics}}}$$

note: $Q = (Ze) \int d\vec{x}' g(\vec{x}') = Ze$

So, $\int d\vec{x}' g(\vec{x}') = 1$ proton probability distribution



Continuing

$$V_{fi} = N_f^* N_i \int d\vec{x} e^{i\vec{q} \cdot \vec{x}} \int d\vec{x}' \left(\frac{-Ze^2}{4\pi\epsilon_0} \right) \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$$= -\frac{Ze^2}{4\pi\epsilon_0} \left(\int d\vec{x}' \rho(\vec{x}') \right) \int d\vec{x} \frac{e^{i\vec{q} \cdot \vec{x}}}{|\vec{x} - \vec{x}'|}$$

$$e^{i\vec{q} \cdot \vec{x}} = e^{i\vec{q} \cdot (\vec{x} - \vec{x}') + i\vec{q} \cdot \vec{x}'} = e^{i\vec{q} \cdot (\vec{x} - \vec{x}')} e^{i\vec{q} \cdot \vec{x}'}$$

$$= -\frac{Ze^2}{4\pi\epsilon_0} \int d\vec{x}' e^{i\vec{q} \cdot \vec{x}'} \rho(\vec{x}') N_f^* N_i \int d\vec{x} \frac{e^{i\vec{q} \cdot (\vec{x} - \vec{x}')}}{|\vec{x} - \vec{x}'|}$$

$\uparrow$ integrated over the charge distribution $\uparrow$ integrated over all space

$$V_{fi} = \underbrace{\int d\vec{x}' e^{i\vec{q} \cdot \vec{x}'} \rho(\vec{x}')}_{\text{Fourier integral of the charge distribution}} \left[\underbrace{-\frac{Ze^2}{4\pi\epsilon_0} N_f^* N_i \int d\vec{u} \frac{e^{i\vec{q} \cdot \vec{u}}}{|\vec{u}|}}_{\text{This is a function of } q \text{ only, and is related to the intensity of the } V_{fi} \text{ vs. angle}} \right]$$

→ called the "nuclear form factor"

Contains all the info about the shape

shape factor

$$V_{fi} = F(\vec{q}) f(\vec{q})$$

experimentally measure

central
eqn
used to
infer
 $g(\vec{x})$ from
exp't

$$\left\{ \begin{array}{l} \text{measured} \\ \left[\frac{2\pi}{x} \frac{(V_{fi} V_{fi}^*)}{f^*(\vec{q}) f(\vec{q})} g_E \right] = F^*(\vec{q}) F(\vec{q}) \end{array} \right. \rightarrow \begin{array}{l} \text{charge distribution} \\ \downarrow \\ \text{infer } g(\vec{x}) \end{array}$$

from theory

Let's carry this a little further

$$F(\vec{q}) = \int d\vec{x}' e^{i\vec{q} \cdot \vec{x}'} g(\vec{x}')$$

For spherical nuclei $g(\vec{x}') = g(r')$

$$F(\vec{q}) = \int_0^{2\pi} d\phi' \int_0^\pi \sin\theta' d\theta' \int_0^\infty r'^2 dr' e^{i\vec{q} \cdot \vec{x}'} g(r')$$

without loss of generality (since spherical)
we can take $\vec{q} = q\hat{z}$, then

(6)

$$\begin{aligned}
 F(q) &= \int_0^{2\pi} d\phi' \int_0^\pi \sin\theta' d\theta' \int_0^\infty r'^2 dr' g(r') e^{i\mathbf{q} \cdot \mathbf{r}'} \\
 &= (2\pi) \int_0^\infty r'^2 dr' g(r') \int_{-\infty}^{\infty} d\mu e^{i\mathbf{q} \cdot \mathbf{r}'} \\
 &= 2\pi \int_0^\infty r'^2 dr' g(r') \frac{1}{iqr'} \underbrace{\left[e^{iqr'} - e^{-iqr'} \right]}_{2i \sin qr'} \\
 &= \frac{4\pi}{q} \int_0^\infty r dr' g(r') \sin(qr')
 \end{aligned}$$

$$F(q) = \frac{4\pi}{q} \int_0^\infty r dr' \sin(qr') g(r')$$

$$= \frac{4\pi R^3}{3} [\sin qR - qR \cos qR]$$

for a solid sphere with radius R
uniformly charged

is the special case of
spherical nuclei

What is $|\vec{q}|$?

$$\vec{q} = \vec{k}_i - \vec{k}_f$$

$$q^2 = (\vec{k}_i - \vec{k}_f) \cdot (\vec{k}_i - \vec{k}_f)$$

$$= k_i^2 - 2\vec{k}_i \cdot \vec{k}_f + k_f^2$$

$$q^2 = k_i^2 - 2k_i k_f \cos \theta + k_f^2$$

But $k_i = k_f$ (elastic) so

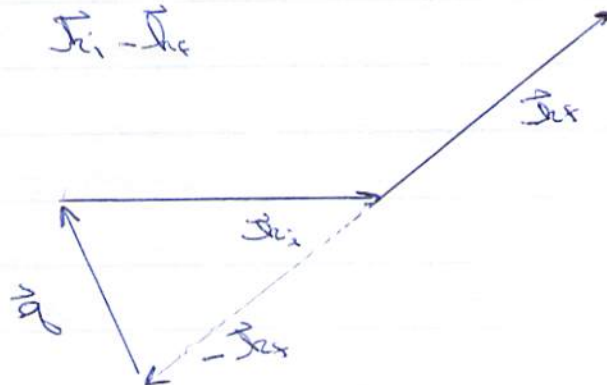
$$\begin{aligned} q^2 &= k^2 - 2k^2 \cos \theta + k^2 \\ &= 2k^2(1 - \cos \theta) \\ &= 2k^2(1 - \cos(\frac{\theta}{2} + \frac{\theta}{2})) \\ &= 2k^2(1 - (\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2})) \\ &= 2k^2(1 - \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}) \\ &= 2k^2(\sin^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}) \\ &= 4k^2 \sin^2 \frac{\theta}{2} \end{aligned}$$

$$\text{So, } q = 2k \sin \frac{\theta}{2}$$

$$\text{Since } \hbar k = p$$

$$q = \frac{2p \sin \frac{\theta}{2}}{\hbar}$$

$$\vec{q} = \vec{k}_i - \vec{k}_f$$



$\hbar \vec{q}$ is the momentum transfer

$\hbar q$ is its magnitude

What about this other factor $f(\vec{q})$ used in the analysis?

Recall,

$$f(\vec{q}) = \frac{-Ze^2}{4\pi\epsilon_0} N_f N_i \int d\vec{u} \frac{e^{i\vec{q}\cdot\vec{u}}}{|\vec{u}|}$$

$$\int d\vec{u} \frac{e^{i\vec{q}\cdot\vec{u}}}{|\vec{u}|} \underset{\substack{\uparrow \\ \text{all space}}}{=} \int_0^{2\pi} d\phi \int_{-1}^1 dp \int_0^\infty u^2 du \frac{e^{i q u p}}{u}$$

why assume $\vec{q} = q \hat{z}$

$$\begin{aligned} &= (2\pi) \int_0^\infty u du \int_{-1}^1 dp \underbrace{e^{i q u p}}_{\frac{1}{i q u} (e^{i q u} - e^{-i q u})} \\ &= \frac{2 \sin q u}{q u} \end{aligned}$$

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$$= \frac{4\pi}{q} \lim_{\Lambda \rightarrow \infty} \int_0^{\Lambda} du \sin qu$$

$$= \frac{4\pi}{q} - \frac{1}{q} \lim_{\Lambda \rightarrow \infty} \cos qu \Big|_0^{\Lambda}$$

$$= \frac{4\pi}{q^2} \left(1 - \lim_{\Lambda \rightarrow \infty} \cos q\Lambda \right)$$

set to 0, why?

really wave packets, and the stuff at ∞ always dies out

$$f(q) = \frac{-Ze^2}{4\pi\epsilon_0} N_f N_i \frac{4\pi}{q^2}$$

$$f^*(q) f(q) = \left(\frac{Ze^2}{\epsilon_0} \right)^2 \frac{1}{q^4} \underbrace{|N_f|^2}_{=1} \underbrace{|N_i|^2}_{=1}$$

$$f^*(q) f(q) = \frac{(Ze^2/\epsilon_0)^2}{q^4}$$

$$f^*(q)f(q) = \left(\frac{Ze^2}{\epsilon_0} \right)^2 \left(\frac{\hbar}{2p \sin \frac{\theta}{2}} \right)^4$$

$$= \left(\frac{Ze^2 \hbar^2}{4\epsilon_0} \right)^2 \frac{1}{p^4 \sin^4 \frac{\theta}{2}}$$

Note: If the incoming charged particle was non-relativistic $p^2 = 2mK$

$$f^*(q)f(q) \stackrel{NR}{=} \left(\frac{Ze^2 \hbar^2}{16\epsilon_0 m} \right)^2 \frac{1}{K^2} \frac{1}{\sin^4 \frac{\theta}{2}}$$

Rutherford scattering

So,

$$|V_{fi}|^2 = \underbrace{F(\vec{r})F(\vec{r})}_{\uparrow \text{ modification of RS due to finite size of the nucleus}} \underbrace{f^*(q)f(q)}_{\text{Rutherford scattering (RS)}}$$

Recall,

$$F(\vec{q}) = \int d\vec{x} e^{i\vec{q}\cdot\vec{x}} g(\vec{x})$$

If $1q$

$$\text{if } \frac{1}{q} = \frac{h}{2p\sin\frac{\theta}{2}} \gg R_{\text{nucleus}}$$

i.e. wavelength of incoming particle
 $\gg R_{\text{nucleus}}$.

$$e^{i\vec{q}\cdot\vec{x}} \approx 1$$

$$F(\vec{q}) \rightarrow 1$$

$$\therefore |V_{fi}|^2 = \text{RS formula}$$

As $p \uparrow$ wavelength approaches the nuclear size, and its structure can be resolved

As $p \uparrow \uparrow$ the internal structure of the nucleus can be resolved (quarks)

Show figure on p. 58