

NERS 312

Elements of Nuclear Engineering and Radiological Sciences II

aka **Nuclear Physics for Nuclear Engineers**

Lecture Notes for Chapter 12: Nuclear Models

Supplement to (Krane II: Chapter 5)

*Note: The lecture number corresponds directly to the chapter number in the online book.
The section numbers, and equation numbers correspond directly to those in the online book.*

12.0: Introduction...

Many of the ideas and methods we learned in studying atoms and their quantum behavior, carry over to nuclear physics. However, in some important ways, they are quite different:

1. We don't really know what the nucleon-nucleon potential is, but we do know that it has a central, $V(r)$, and non-central part, $V(\vec{x})$. That is the first complication.
2. The force on one nucleon not only depends on the position of the other nucleons, but also on the distances between the other nucleons! These are called *many-body* forces. That is the second complication.

12.0: Introduction...

Let us illustrate this in Figure 12.1, where we show the internal forces governing a ^3He nucleus.

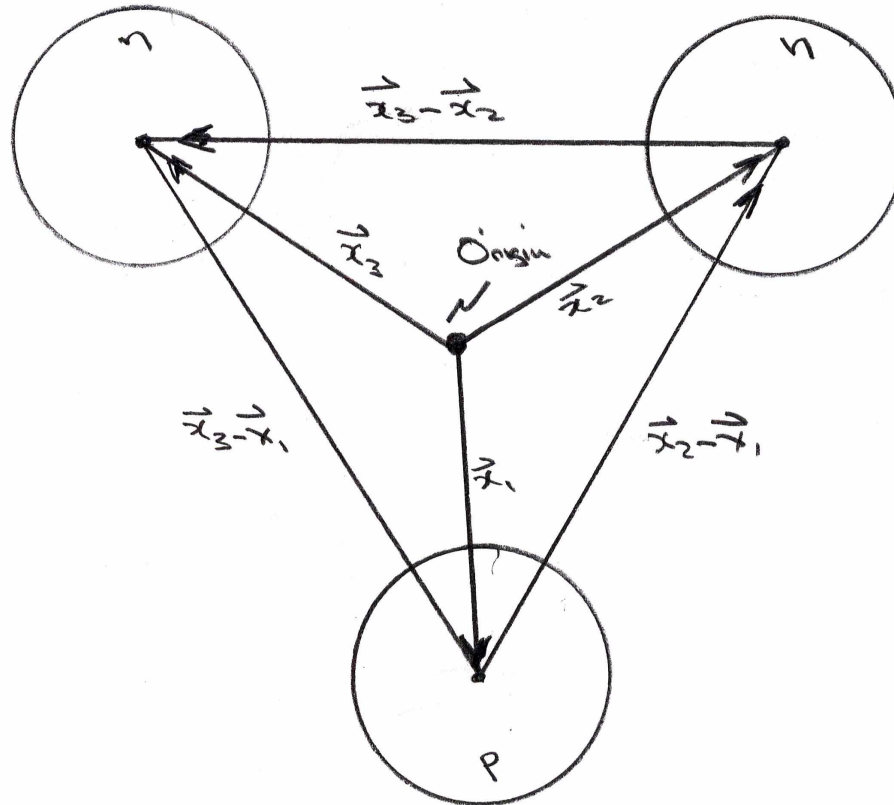


Figure 12.1: Theoretical sketch of a ^3He nucleus.

12.0: ...Introduction...

The potential on the proton at $\vec{x}_1$ is given by:

$$V_{nn}(\vec{x}_2 - \vec{x}_1) + V_{nn}(\vec{x}_3 - \vec{x}_1) + V_C(|\vec{x}_2 - \vec{x}_1|) + V_3(\vec{x}_1 - \vec{x}_2, \vec{x}_1 - \vec{x}_3, \vec{x}_2 - \vec{x}_3) , \quad (12.1)$$

where:

Potential term	Explanation
$V_{nn}(\vec{x}_2 - \vec{x}_1)$	2-body strong nuclear force between p at $\vec{x}_1$ and p at $\vec{x}_2$
$V_{nn}(\vec{x}_3 - \vec{x}_1)$	2-body strong nuclear force between p at $\vec{x}_1$ and n at $\vec{x}_3$
$V_C(\vec{x}_2 - \vec{x}_1)$	2-body Coulomb force between p at $\vec{x}_1$ and p at $\vec{x}_2$
$V_3(\dots)$	3-body force strong nuclear force (more explanation below)

The 2-body forces above follow from our discussion of the strong and Coulomb 2-body forces. However, the 3-body term is a fundamentally different thing.

12.0: ...Introduction...

Polarization effects are common in atomic physics as well.

You can think of V_3 as a “polarization” term—the presence of several influences, how 2 acts on 1 in the presence of 3, how 3 acts on 1 in the presence of 2, and how this is also affected by the distance between 2 and 3.

It may seem complicated, but it is familiar. People act this way! Person 1 may interact with person 2 in a different way if person 3 is present! These many-body forces are hard to get a grip on, in nuclear physics and in human social interaction. Nuclear theory is basically a phenomenological one based on measurement, and 3-body forces or higher order forces are hard to measure.

12.0: ...Introduction...

Polarization effects are common in atomic physics as well.

Figure 12.2, shows how an electron passing by, in the vicinity of two neutral atoms, polarizes the proximal atom, as well as more distant atoms.

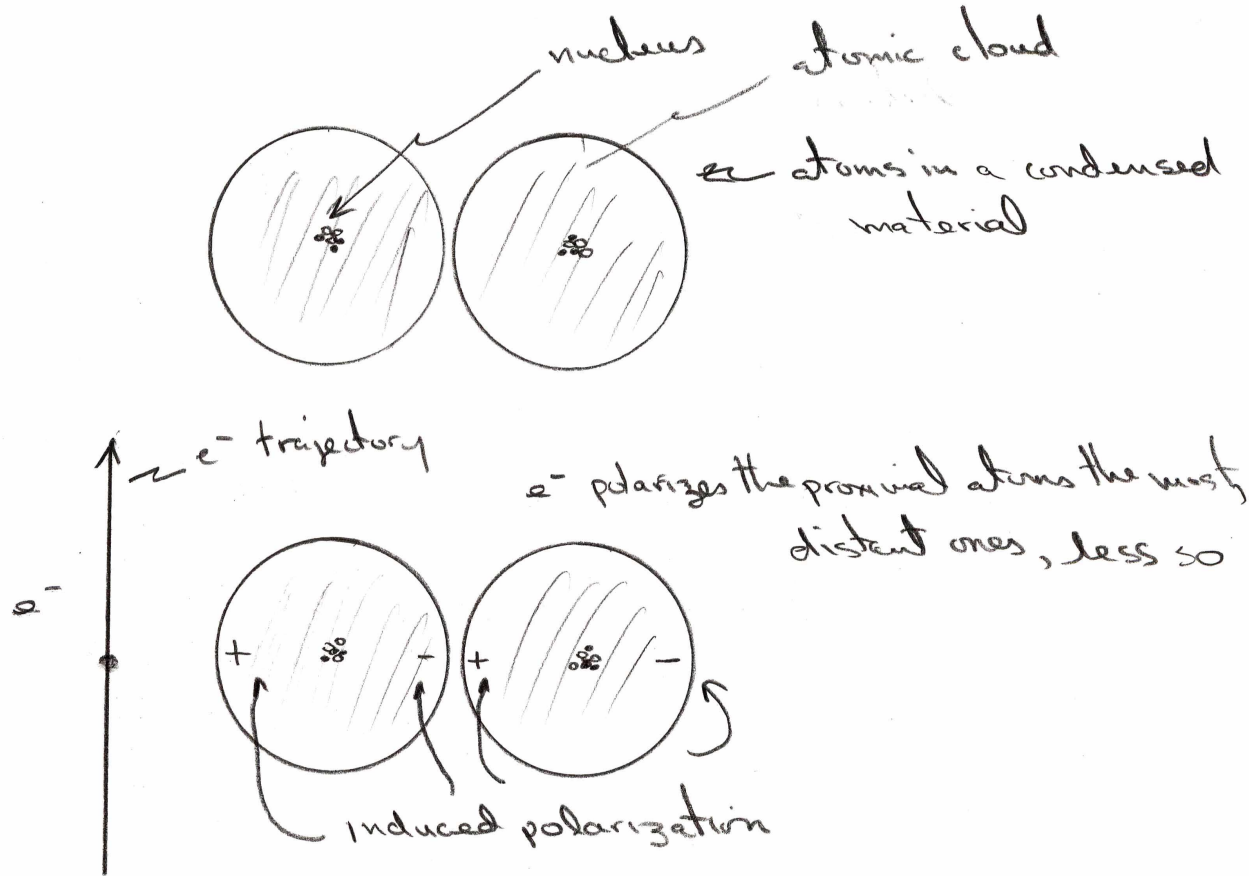


Figure 12.2: A depiction of polarization for an electron in condensed matter.

12.0: ...Introduction...

In nuclear physics, despite the complication of many-body forces, we shall persist with the development of simple models for nuclei.

These models organize the way we think about nuclei, based upon some intuitive guesses.

Should one of these guesses have predictive power, that is, it predicts some behavior we can measure, we have learned something—not the entire picture, but at least some aspect of it.

With no fundamental theory, this form of guesswork, phenomenology, is the best we can do.

12.1: The Shell Model...

Atomic systems show a very pronounced shell structure. See Figures 12.3 and 12.4.

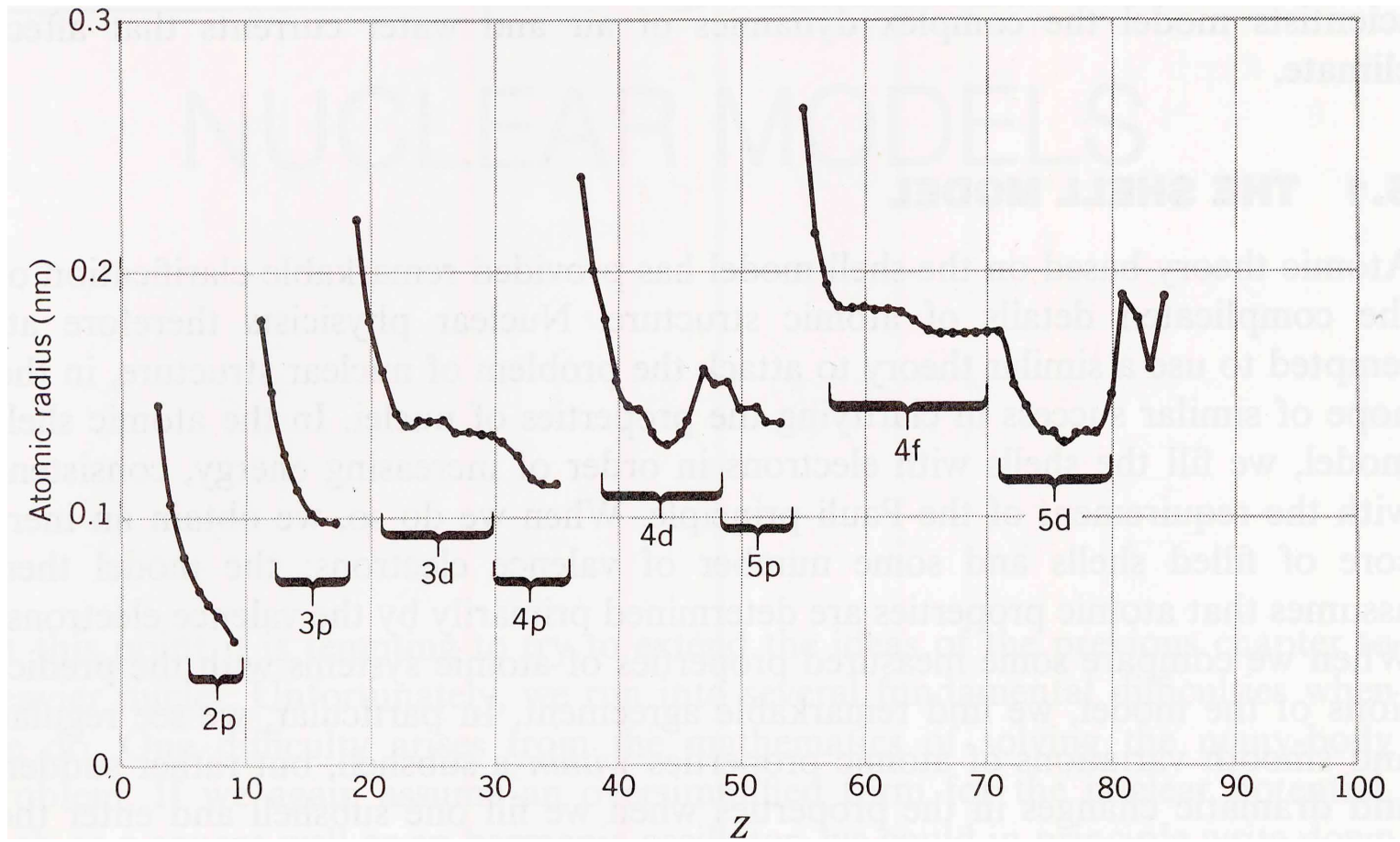


Figure 12.3: From Figure 5.1 in Krane's book, p. 118. This figure shows shell-induced regularities of the atomic radii of the elements, and the irregularities caused by shell transitions.

12.1: The Shell Model...

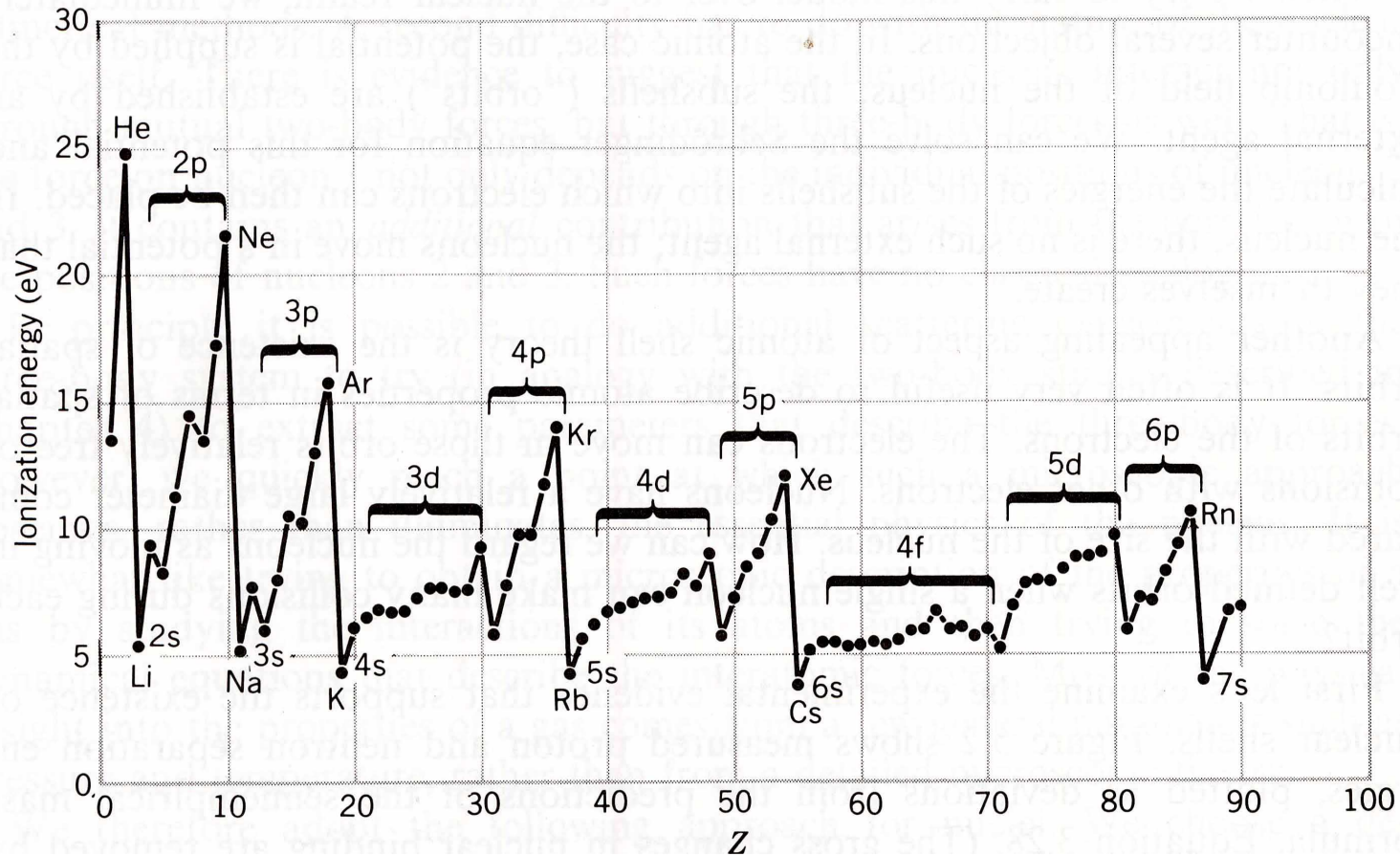
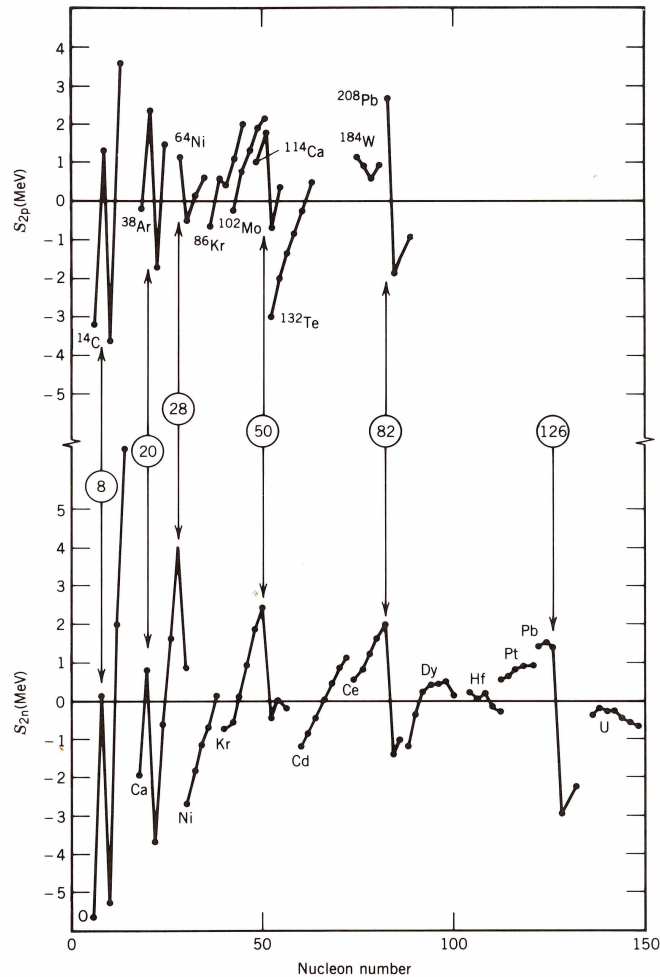


Figure 12.4: From Figure 5.1 in Krane's book, p. 118. This figure shows shell-induced regularities of the atomic radii of the elements, and the irregularities caused by shell transitions.

12.1: The Shell Model...



Nuclei, as well, show a “shell-like” structure, as seen in Figure 12.5.

The peaks of the separation energies (that is, those hardest to separate) occur when the Z or N correspond to major closed shells. The “magic” numbers, the closed major shells, occur at Z or N : 2, 8, 20, 28, 50, 82, & 126.

Figure 12.5: From Figure 5.2 in Krane’s book, p. 119. This figure shows shell-induced regularities of the $2p$ separation energies for sequences of isotones same N , and $2n$ separation energies for sequences of isotopes.

12.1: ...The Shell Model...

The stable magic nuclei

Isotopes	Explanation	Natural abundance (%)
${}^3_2\text{He}_1$	magic Z	1.38×10^{-4}
${}^4_2\text{He}_2$	doubly magic	99.99986
${}^{15}_7\text{N}_8$	magic N	0.366
${}^{16}_8\text{O}_8$	doubly magic	99.76
${}^{40}_{20}\text{Ca}_{20}$	doubly magic	96.94
${}^{42-48}_{20}\text{Ca}_{20}$	magic Z	
${}^{50}_{22}\text{Ti}_{28}$	magic N	5.2
${}^{52}_{24}\text{Cr}_{28}$	magic N	83.79
${}^{54}_{26}\text{Fe}_{28}$	magic N	5.8
${}^{86}\text{Kr}, {}^{87}\text{Rb}, {}^{88}\text{Sr}, {}^{89}\text{Y}, {}^{90}\text{Zr}, {}^{92}\text{Mo}$	magic $N = 50$	
$\vdots$	$\vdots$	$\vdots$
${}^{208}_{82}\text{Pb}_{126}$	doubly magic	52.3
${}^{209}_{83}\text{Bi}_{126}$	magic N	100, $t_{1/2} = 19 \pm 2 \times 10^{18} \text{ y}$

12.1: ...The Shell Model...

The Shell-Model idea

A nucleus is composed of a “core” that produces a potential that determines the properties of the “valence” nucleons. These properties determine the behavior of the nucleus much in the same way that the valence electrons in an atom determine its chemical properties.

The excitation levels of nuclei appears to be chaotic and inscrutable. However, there is order to the mess!

12.1: ...The Shell Model...

Figure 12.6 shows the energy levels predicted by the shell model using ever-increasing sophistication in the model of the “core” potential.

The harmonic oscillator potential as well as the infinite well potential predict the first few magic numbers. However, one must also include details of the profile of the nuclear skin, as well as introduce a spin-orbit coupling term, before the shells fall into place. In the next section we discuss the various components of the modern nuclear potential.

12.1: ...The Shell Model...

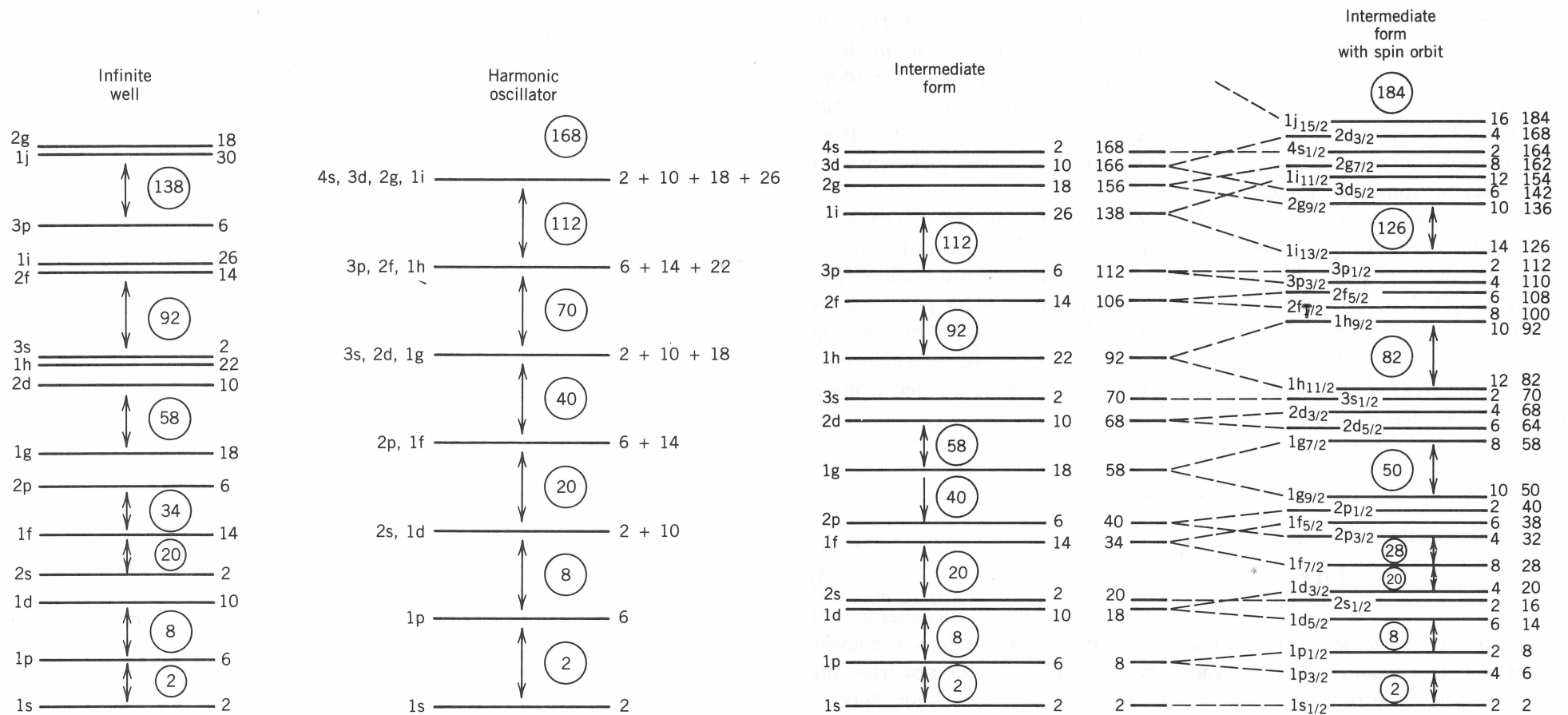


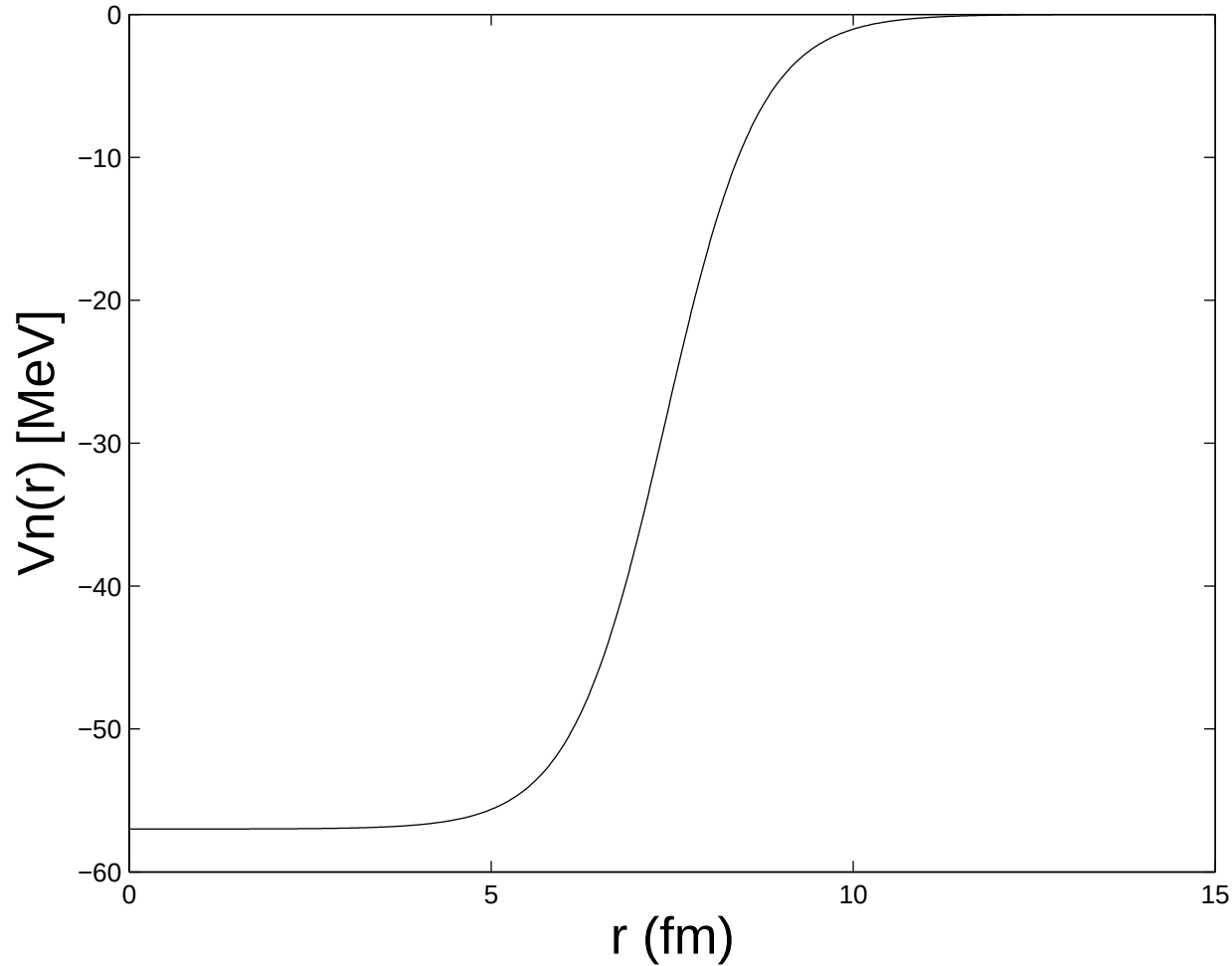
Figure 12.6: The shell model energy levels from Figures 5.4 (p. 121) and 5.6 p. 123 in Krane.

12.1: ...The Shell Model...

Details of the modern nuclear potential

A valence nucleon (p or n) feels the following central strong force from the core:

Shell model strong central potential for $(Z,A) = (82,208)$



12.1: ...The Shell Model...

This shape is modeled by the familiar Woods-Saxon form:

$$V_n(r) = \frac{-V_0}{1 + \exp\left(\frac{r-R_N}{t}\right)} \quad (12.2)$$

It is no coincidence that the form of this potential closely resembles the shape of the nucleus as determined by electron scattering experiments. The presence of the nucleons in the core, provides the force, and thus, the force is derived directly from the shape of the core.

12.1: ...The Shell Model...

In addition to the “bulk” attraction in (12.2), there is a symmetry term when there is an imbalance of neutrons and protons. This symmetry term is given by:

$$V_S = a_{\text{sym}} \left[\frac{(A - 2Z)^2}{A} \right], \quad (12.3)$$

The form of this potential can be derived from the parametric fit to the total binding energy of a nucleus given by Equation 10.38.

The parameters of the potential described above, are conventionally given as:

Parameter	Value	Interpretation
V_0	57 MeV	Potential depth of the core
R_N	$1.25A^{1/3}$	Nuclear radius
t	0.65 fm	Related to the nuclear skin depth
a_{sym}	34 MeV	Symmetry energy
a_{so}	1 fm	Spin-orbit coupling (discussed below)

12.1: ...The Shell Model...

If the valence nucleon is a proton, an additional central Coulomb repulsion must be applied:

$$V_C(r) = \frac{Ze^2}{4\pi\epsilon_0} \int d\vec{x}' \rho_p(r') \frac{1}{|\vec{x} - \vec{x}'|} = \frac{Ze^2}{4\pi\epsilon_0} \frac{2\pi}{r} \int dr' r' \rho_p(r') [(r + r') - |r - r'|] . \quad (12.4)$$

Recall that the proton density is normalized to unity by

$$1 \equiv \int d\vec{x}' \rho_p(r') = 4\pi \int dr' r'^2 \rho_p(r') .$$

Simple approximations to (12.4) treat the charge distribution as a uniform sphere with radius R_N . That is:

$$\rho_p(r) \approx \frac{3}{4\pi R_N^3} \Theta(R - r) .$$

12.1: ...The Shell Model...

However, a more sophisticated approach would be to use the nuclear shape suggested by (12.2), that is:

$$\rho_p(r) = \frac{\rho_0}{1 + \exp\left(\frac{r - R_N}{t}\right)},$$

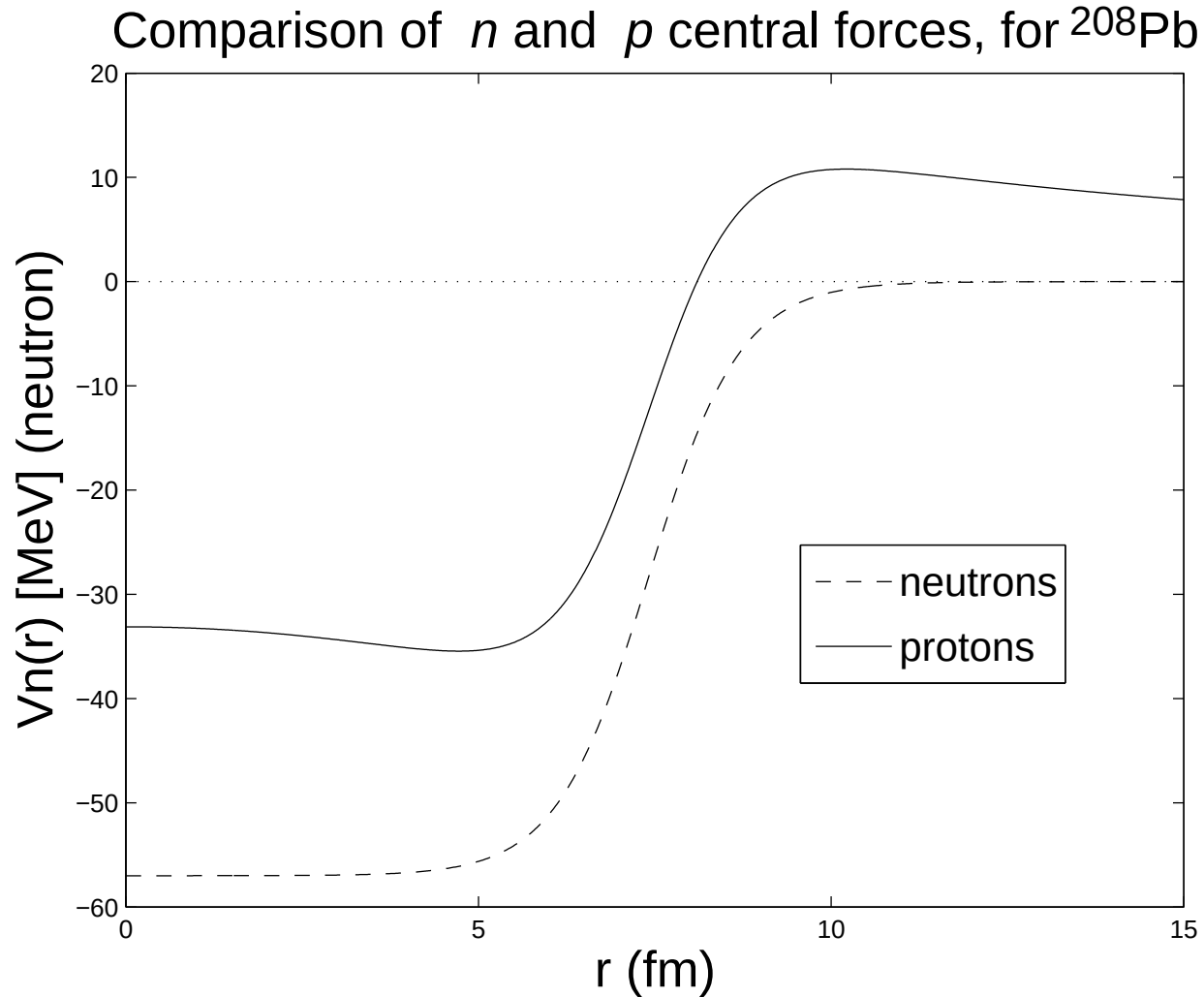
and determining ρ_0 from the normalization condition above.

Herein resides a beautifully constructed extra-credit problem!

Every opportunity is disguised as a seemingly insurmountable problem!

12.1: ...The Shell Model...

Contrasting the central part of the potentials felt by a single valence neutron and proton...



12.1: ...The Shell Model...

The spin-orbit potential

The spin-orbit potential has the form:

$$V_{so}(\vec{x}) = -\frac{a_{so}^2}{r} \frac{dV_n(r)}{dr} \langle \vec{l} \cdot \vec{s} \rangle . \quad (12.5)$$

The radial derivative in the above equation is only meant to be applied where the nuclear density is changing rapidly.

From <https://www.aps.org/publications/apsnews/200808/physicshistory.cfm>

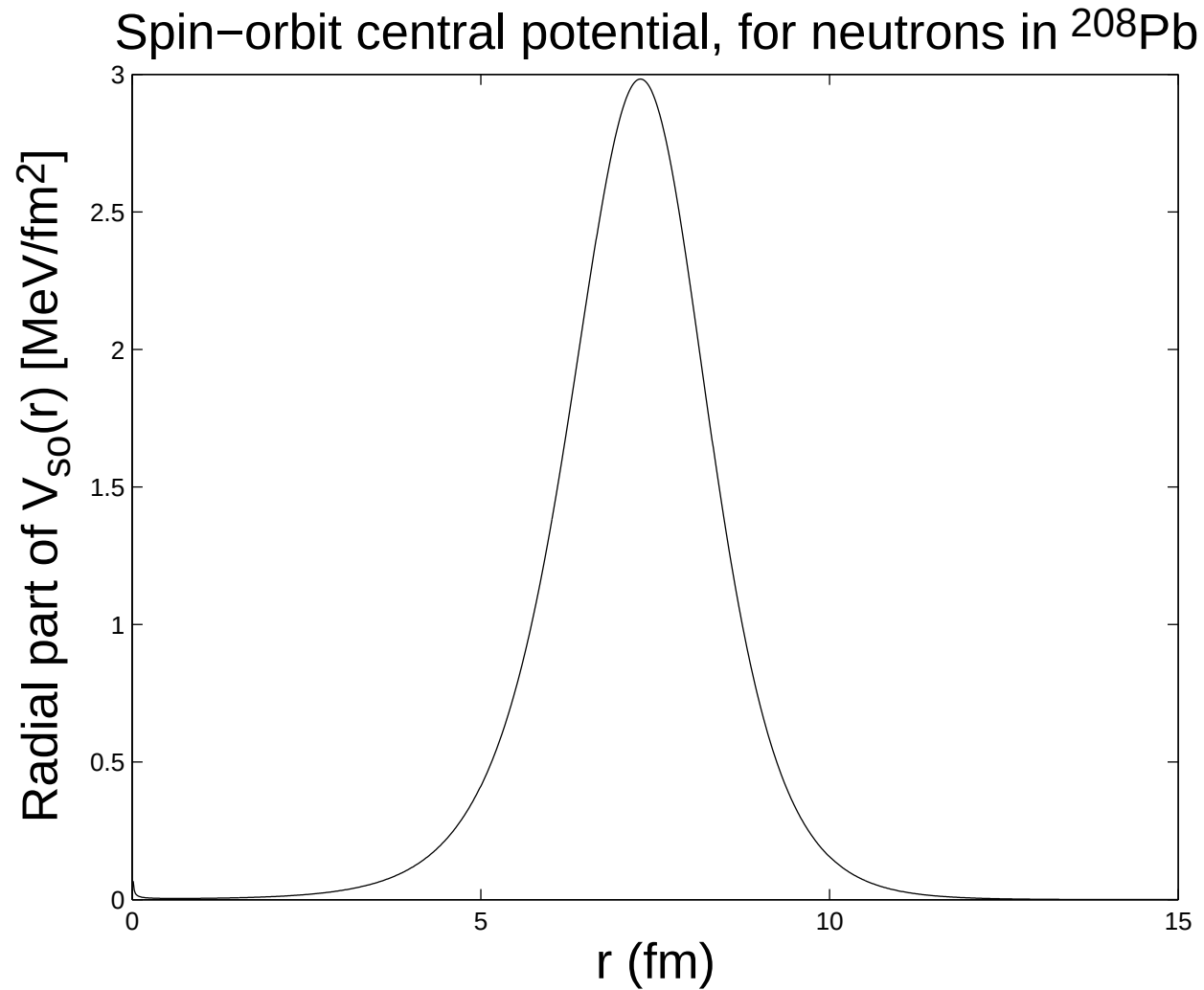
Maria Goeppert Mayer, who made important discoveries about nuclear structure, is one of only two women to have won the Nobel Prize in physics. ... The key insight came to Goeppert Mayer when Enrico Fermi happened to ask her if there was any evidence of spin-orbit coupling. She immediately realized this was the answer. Goeppert Mayer was now able to calculate energy levels and magic numbers.

The idea here is that it is the valence nucleons that determine the properties of the nucleus, and $dV_n(r)/dr$ is a measure of where they are. The core nucleons spins all average to zero, and, presumably, do not participate in the spin-orbit term.

Here's another extra credit opportunity! I have not been able to find a refereed paper, that states, unambiguously, why this works so well, except that, "Hey It works!"

12.1: ...The Shell Model...

Shape of $\frac{dV_n(r)}{rdr}$...



12.1: ...The Shell Model...

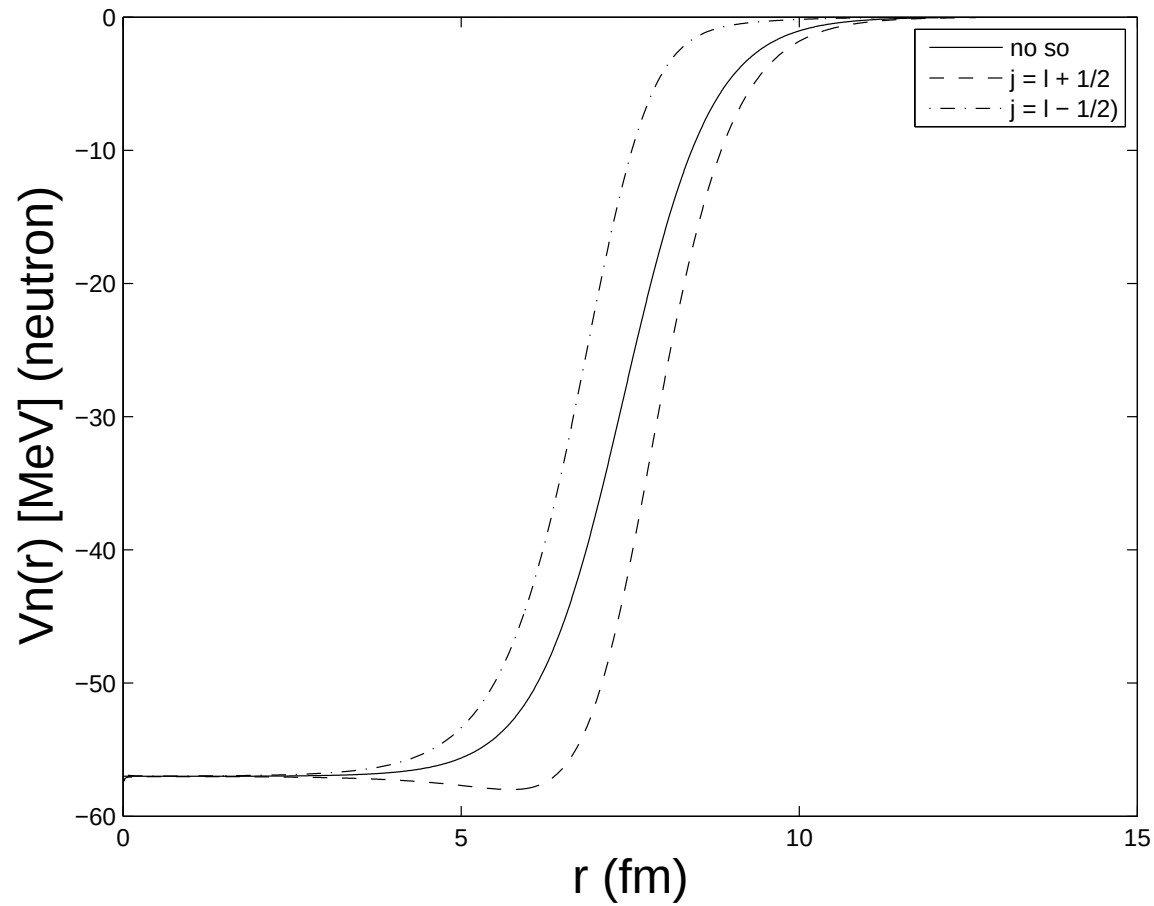


Figure 12.7: The potential of a ^{208}Pb nucleus as seen by a single valence neutron.

12.1: ...The Shell Model...

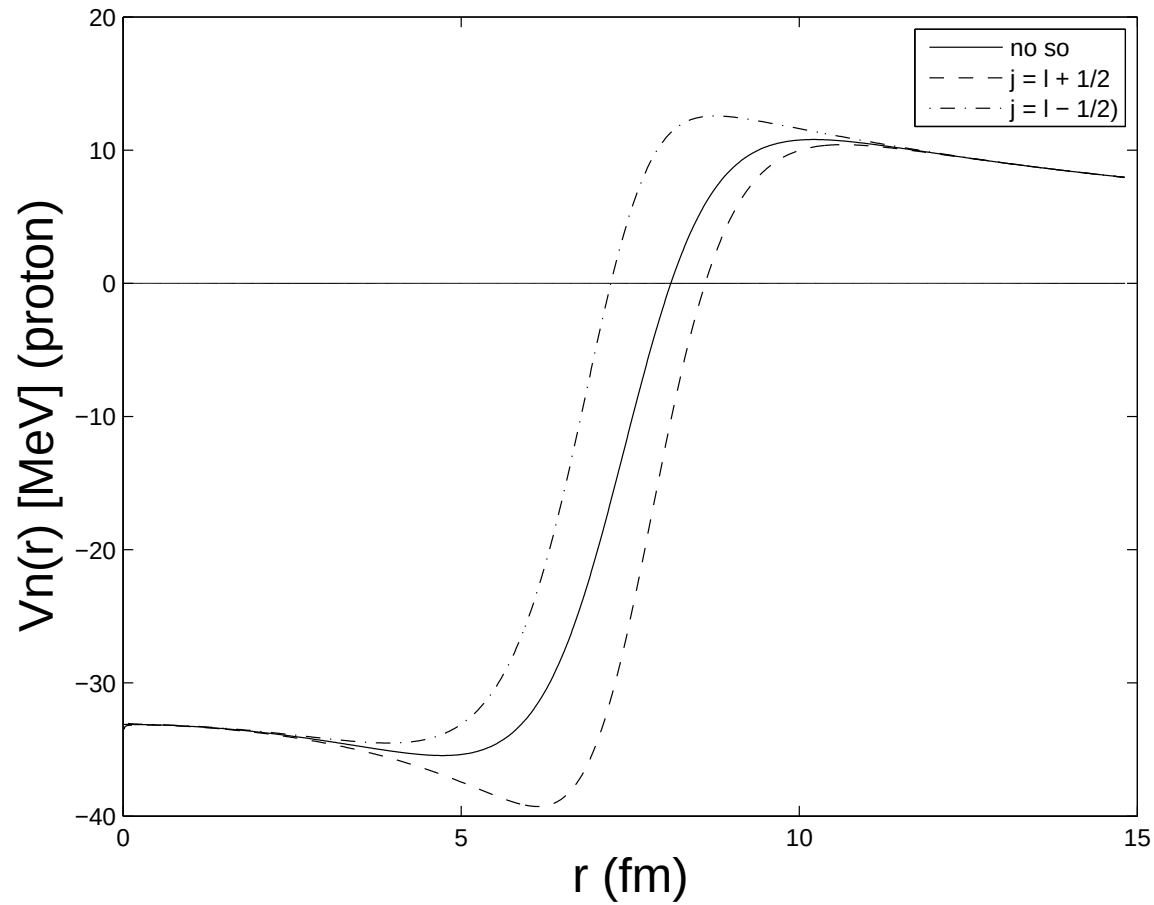


Figure 12.8: The potential of a ^{208}Pb nucleus as seen by a single valence proton. Note the effect of the Coulomb potential on the the potential near the origin (parabolic shape there), as well as the presence of the Coulomb barrier.

12.1: ...The Shell Model...

The shape of this potential was shown, for a valence neutron in Figure 12.7, and for a valence proton in Figure 12.8. For this demonstration, the core nucleus was ^{208}Pb .

The l in the figures, to highlight the spin-orbit coupling, was chosen to be $l = 10$.

12.1: ...The Shell Model...

Evaluating the spin-orbit term

Recall, $\vec{j} = \vec{l} + \vec{s}$. Hence, $j^2 = l^2 + 2\vec{l} \cdot \vec{s} + s^2$. Thus, $\vec{l} \cdot \vec{s} = (1/2)(j^2 - l^2 - s^2)$, and $\langle \vec{l} \cdot \vec{s} \rangle = (1/2)[j(j+1) - l(l+1) - s(s+1)]$.

The valence nucleon has spin-1/2. To determine the splitting of a given l into $j = l \pm \frac{1}{2}$ levels, we calculate, therefore:

$$\begin{aligned}\langle \vec{l} \cdot \vec{s} \rangle_{j=l+\frac{1}{2}} &= [(l + 1/2)(l + 3/2) - l(l + 1) - 3/4]/2 \\ &= l/2 \\ \langle \vec{l} \cdot \vec{s} \rangle_{j=l-\frac{1}{2}} &= [(l - 1/2)(l + 1/2) - l(l + 1) - 3/4]/2 \\ &= -(l + 1)/2 \\ \langle \vec{l} \cdot \vec{s} \rangle_{j=l+\frac{1}{2}} - \langle \vec{l} \cdot \vec{s} \rangle_{j=l-\frac{1}{2}} &= (2l + 1)/2\end{aligned}\tag{12.6}$$

$V_{so}(r)$ is negative, and so, the higher $j = l + \frac{1}{2}$ (orbit and spin angular momenta are aligned) is more tightly bound.

12.1: ...The Shell Model...

Determining the ground state I^π in the shell model

The spin and parity assignment may be determined by considering the nuclear potential described so far, plus one additional idea, the “Extreme Independent Particle Model” (EIPM). The EIPM is an addendum to the shell model idea, and it is expressed as follows.

All the characteristics of a given nucleus are determined by the unpaired valence nucleons. All pairs of like nucleons cancel one another's spins and parities.

The EIPM is based upon the spin-spin force that provides an incentive for the pairing to occur. When it comes to magnets, with orbital or spin ones, opposites really do attract.

Applying EIPM for the example of two closely related nuclei is demonstrated in Figure 12.9.

12.1: ...The Shell Model...

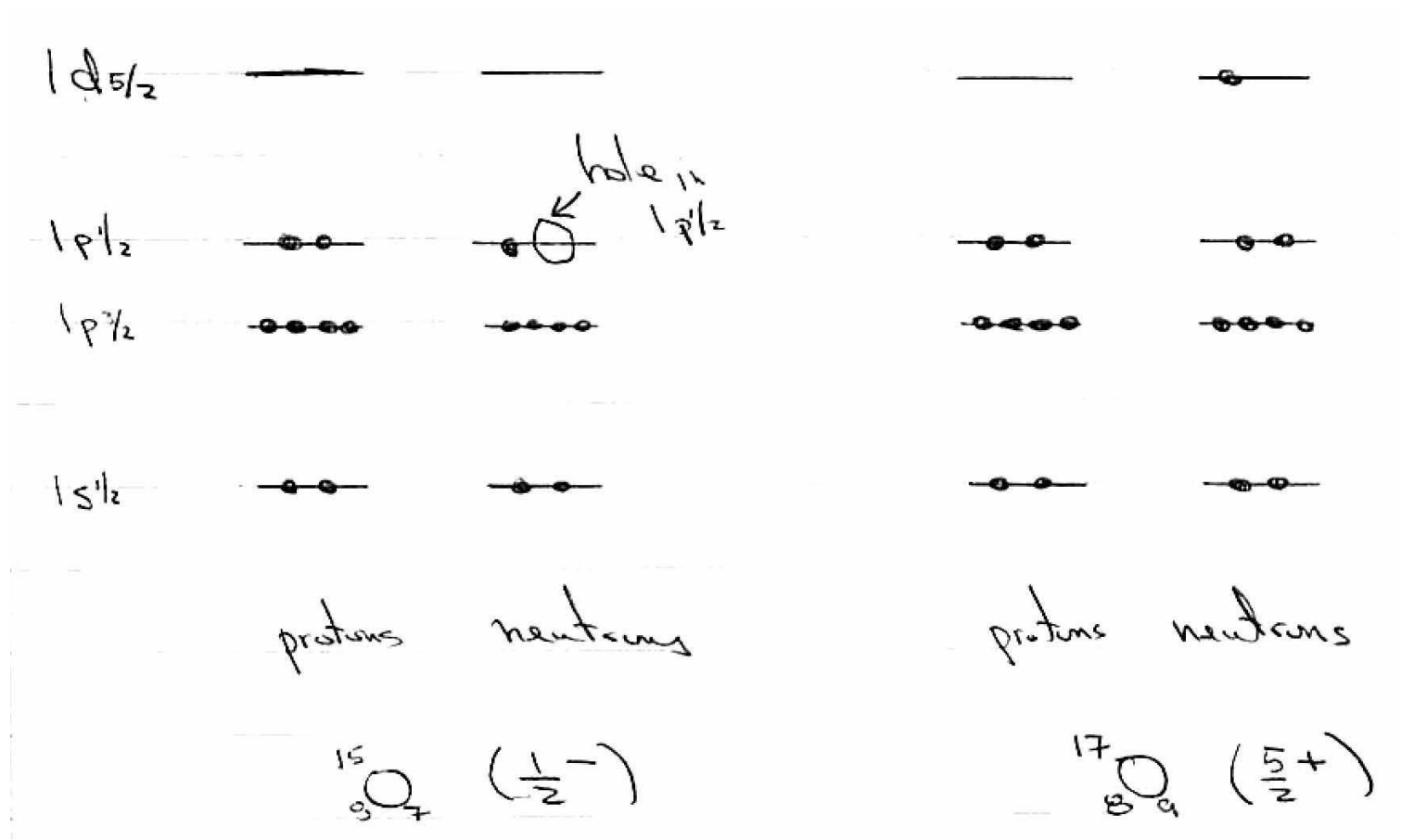


Figure 12.9: A demonstration of the spin and parity assignment for ^{15}O and ^{17}O . $I^\pi(^{15}\text{O}) = \frac{1}{2}^-$, while $I^\pi(^{17}\text{O}) = \frac{5}{2}^+$.

This figure awaits the attention of a student seeking extra credit, and who would like to learn how to make drawings in $\text{\LaTeX}$.

12.1: ...The Shell Model...

Another demonstration of the success of the EIPM model is to consider the isotopes of Oxygen.

Isotope of O	I^π , measured	I^π , EIPM prediction	decay mode	$t_{1/2}$ /abundance
^{12}O	0^+ (est)	0^+	$2p$	$\approx 10^{-21}$ s
^{13}O	$\frac{3}{2}^-$	$\frac{3}{2}^-$	β^+, p	8.6 ms
^{14}O	0^+	0^+	γ, β^+	70.60 s
^{15}O	$\frac{1}{2}^-$	$\frac{1}{2}^-$	ε, β^+	2.037 m
^{16}O	0^+	0^+		99.757%
^{17}O	$\frac{5}{2}^+$	$\frac{5}{2}^+$		0.038%
^{18}O	0^+	0^+		0.205%
^{19}O	$\frac{5}{2}^+$	$\frac{5}{2}^+$	β^-, γ	26.9 s
^{20}O	0^+	0^+	β^-, γ	13.5 s
^{21}O	?	$\frac{5}{2}^+$	β^-, γ	3.4 s
^{22}O	0^+ (est)	0^+	β^-, γ	2.2 s
^{23}O	?	$\frac{1}{2}^+$	β^-, n	0.08 s
^{24}O	0^+ (est)	0^+	β^-, γ, n	65 ms

12.1: ...The Shell Model...

Other successes ...

Isotope	I^π
${}^{13}_5\text{Be}_8$	$\frac{3}{2}^-$
${}^{14}_6\text{C}_8$	0^+
${}^{15}_7\text{N}_8$	$\frac{1}{2}^-$
${}^{16}_8\text{O}_8$	0^+
${}^{17}_9\text{F}_8$	$\frac{5}{2}^+$
${}^{18}_{10}\text{Ne}_8$	0^+

12.1: ...The Shell Model...

EIPM prediction of the magnetic moment of the nucleus

The shell model, and its EIPM interpretation, can be tested by measuring and calculating the magnetic moment of a nucleus. Thus, the last unpaired nucleon determines the magnetic moment of the entire nucleus. Recall from Chapter 10, the definition of magnetic moment, μ , of a nucleus:

$$\mu = \mu_N(g_l l_z + g_s s_z) , \quad (12.7)$$

where

Symbol	Meaning	Value
μ_N	Nuclear magnetron	$5.05078324(13) \times 10^{-27} \text{ J/T}$
g_l	Orbital gyromagnetic ratio	0 (neutron), 1 (proton)
g_s	Spin gyromagnetic ratio	$-3.82608545(46)$ (neutron) $5.585694713(90)$ (proton)
l_z	Maximum value of m_l	$l_z = \max(m_l)$
s_z	Maximum value of m_s	$s_z = \max(m_s) = 1/2$

12.1: ...The Shell Model...

However, neither $\vec{l}$ nor $\vec{s}$ is precisely defined for nuclei (recall the Deuteron) due to the strong spin-orbit coupling. Consequently, l_z and s_z can not be known precisely. However, total angular momentum, $\vec{j}$ and its maximum z -projection, j_z are precisely defined, and thus measurable.

Since $j_z = l_z + s_z$, we may rewrite (12.7) as:

$$\mu = [g_l j_z + (g_s - g_l) s_z] \mu_N / \hbar . \quad (12.8)$$

Computing the expectation value (*i.e.* the measured value) of μ gives:

$$\langle \mu \rangle = [g_l j + (g_s - g_l) \langle s_z \rangle] \mu_N / \hbar . \quad (12.9)$$

Since $\vec{j}$ is the only measurable vector in the nucleus, we can determine $\langle s_z \rangle$ from its projection along $\vec{j}$.

Thus, using projection vector language...

12.1: ...The Shell Model...

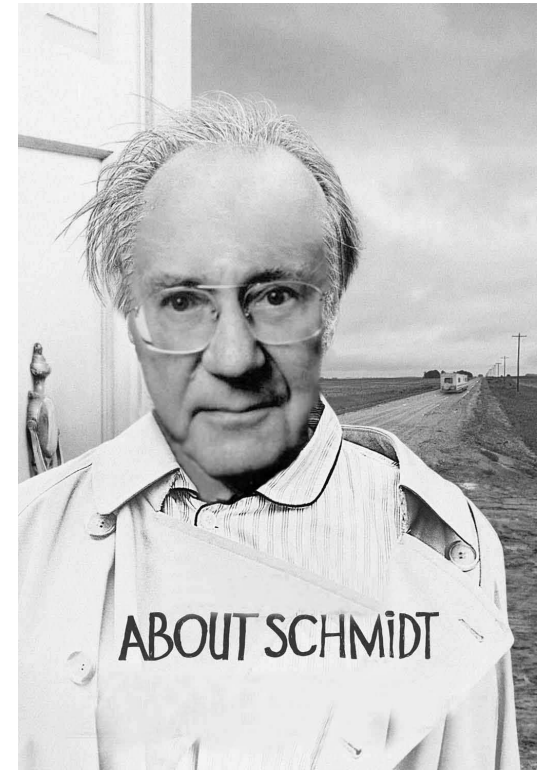
$$\begin{aligned}\vec{s}_j &= \vec{j} \frac{(\vec{s} \cdot \vec{j})}{\vec{j} \cdot \vec{j}} , \\ s_z = \hat{z} \cdot \vec{s}_j &= j_z \frac{\vec{s} \cdot \vec{j}}{\vec{j} \cdot \vec{j}} , \\ \langle s_z \rangle &= j \frac{\langle \vec{s} \cdot \vec{j} \rangle}{j(j+1)} = \frac{\langle \vec{s} \cdot \vec{j} \rangle}{(j+1)} = \frac{\langle \vec{j} \cdot \vec{j} \rangle - \langle \vec{l} \cdot \vec{l} \rangle + \langle \vec{s} \cdot \vec{s} \rangle}{2(j+1)} , \\ \langle s_z \rangle &= \frac{j(j+1) - l(l+1) + s(s+1)}{2(j+1)} , \\ \langle s_z \rangle_{j=l+1/2} &= 1/2 , \\ \langle s_z \rangle_{j=l-1/2} &= -\frac{j}{2(j+1)} .\end{aligned}\tag{12.10}$$

Substituting the results of (12.10) into (12.9) gives:

$$\begin{aligned}\langle \mu \rangle_{j=l+1/2} &= \mu_N [g_l(j - \tfrac{1}{2}) + \tfrac{1}{2}g_s] \\ \langle \mu \rangle_{j=l-1/2} &= \mu_N \left[g_l \frac{j(j + \frac{3}{2})}{(j+1)} - \frac{g_s}{2} \frac{j}{(j+1)} \right]\end{aligned}\tag{12.11}$$

12.1: ...The Shell Model...

Comparisons of measurements with theory are given in Figure 12.10, for odd-neutron and odd-proton nuclei. These nuclei are expected to give the best agreement with the EIPM. The theoretical lines are known as Schmidt lines, honoring the first person who developed the theory. Generally, the trends in the data are followed by the Schmidt lines, though the measured data is significantly lower. The reason for this is probably a “polarization effect”, where the intrinsic spin of the odd nucleon is shielded by the other nucleons in the nucleus as well as the virtual exchange mesons. This is very similar to a charged particle entering a condensed medium and polarizing the surrounding atoms, thereby reducing the effect of its charge. This can be interpreted as a reduction in charge by the surrounding medium. (The typical size of this reduction is only about 1–2%. However, in a nucleus, the forces are much stronger, and hence, so is the polarization. The typical reduction factor applied to the nucleons are g_s (in nucleus) $\approx 0.6g_s$ (free).



12.1: ...The Shell Model...

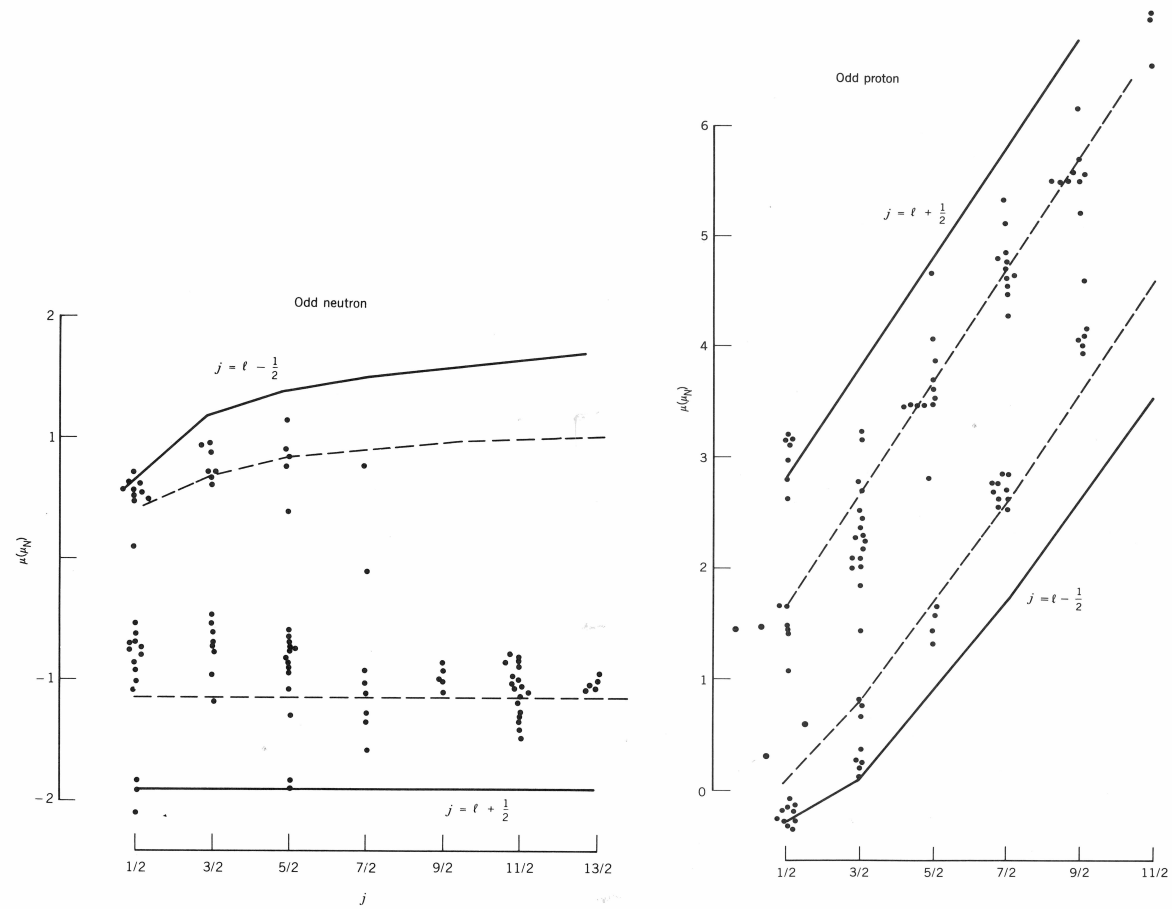


Figure 12.10: From Krane's Figure 5.9, p. 127

12.1: ...The Shell Model...

Shell model and EIPM prediction of the quadrupole moment of the nucleus

Recall the definition of the quadrupole moment of a nucleus, given in Equation 10.53 namely:

$$\langle Q \rangle = \int d\vec{x} \psi_N^*(\vec{x})(3z^2 - r^2)\psi_N(\vec{x}) .$$

where the maximum projection of I along the z -axis is used in the measurement.

When there is a single proton in the valence shell of an odd- A nucleus, the above equation may be written:

$$\begin{aligned} \langle Q_{\text{sp}} \rangle &= \langle R_{nl} Y_{l,m_l=l} | 3z^2 - r^2 | R_{nl} Y_{l,m_l=l} \rangle \\ &= \langle r^2 \rangle \langle Y_{ll} | 3 \cos^2 \theta - 1 | Y_{ll} \rangle \\ &= \langle r^2 \rangle \langle Y_{ll} | 2 - 3 \sin^2 \theta | Y_{ll} \rangle \end{aligned}$$

Using

$$Y_{ll} = \sqrt{\frac{(2l_1)!}{4\pi}} \frac{1}{2^l l!} e^{il\phi} \sin^l \theta$$

it is possible to show that:

$$\langle Q_{\text{sp}} \rangle = -\frac{2j-1}{2(j+1)} \langle r^2 \rangle . \quad (12.12)$$

12.1: ...The Shell Model...

Single holes in shells

When a shell is missing only one proton to be closed, we expect that the single-hole, and single-proton quadrupole moments to be related by:

$$\langle Q_{\text{sh}} \rangle = -\langle Q_{\text{sh}} \rangle$$

12.1: ...The Shell Model...

Krane's Table 5.1 (p. 129) bears this out, despite the generally poor agreement in the absolute value of the quadrupole moment as predicted by theory.

Table 5.1 Shell-Model Quadrupole Moments

Shell-Model State	Calculated Q (single proton)	Measured Q			
		Single Particle		Single Hole	
		p	n	p	n
$1p_{3/2}$	-0.013	-0.0366(^7Li)		+0.0407(^{11}B)	+0.053(^9Be)
$1d_{5/2}$	-0.036	-0.12(^{19}F)	-0.026(^{17}O)	+0.140(^{27}Al)	+0.201(^{25}Mg)
$1d_{3/2}$	-0.037	-0.08249(^{35}Cl)	-0.064(^{33}S)	+0.056(^{39}K)	+0.45(^{35}S)
$1f_{7/2}$	-0.071	-0.26(^{43}Sc)	-0.080(^{41}Ca)	+0.40(^{59}Co)	+0.24(^{49}Ti)
$2p_{3/2}$	-0.055	-0.209(^{63}Cu)	-0.0285(^{53}Cr)	+0.195(^{67}Ga)	+0.20(^{57}Fe)
$1f_{5/2}$	-0.086		-0.20(^{61}Ni)	+0.274(^{85}Rb)	+0.15(^{67}Zn)
$1g_{9/2}$	-0.13	-0.32(^{93}Nb)	-0.17(^{73}Ge)	+0.86(^{115}In)	+0.45(^{85}Kr)
$1g_{7/2}$	-0.14	-0.49(^{123}Sb)		+0.20(^{139}La)	
$2d_{5/2}$	-0.12	-0.36(^{121}Sb)	-0.236(^{91}Zr)		+0.44(^{111}Cd)

Data for this table are derived primarily from the compilation of V. S. Shirley in the *Table of Isotopes*, 7th ed. (New York: Wiley, 1978). The uncertainties in the values are typically a few parts in the last quoted significant digit.

12.1: ...The Shell Model...

Krane's Figure 5.10 (p. 130) ...

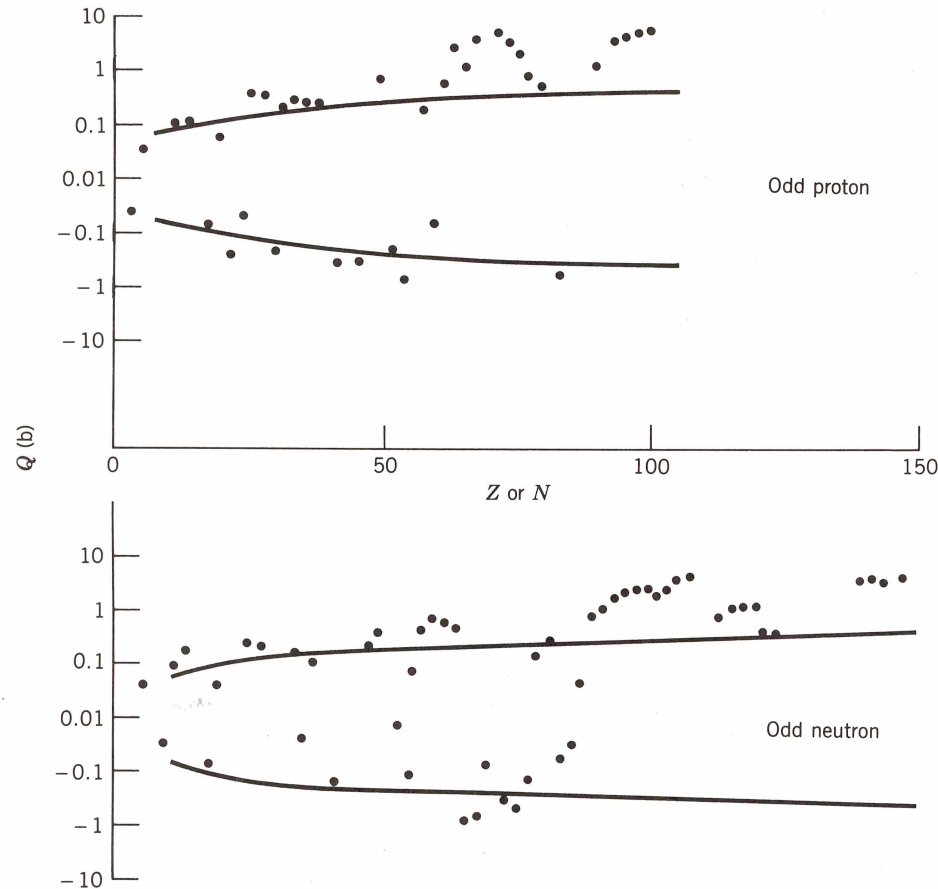


Figure 5.10 Experimental values of electric quadrupole moments of odd-neutron and odd-proton nuclei. The solid lines show the limits $Q \sim \langle r^2 \rangle$ expected for shell-model nuclei. The data are within the limits, except for the regions $60 < Z < 80$, $Z > 90$, $90 < N < 120$, and $N > 140$, where the experimental values are more than an order of magnitude larger than predicted by the shell model.

12.1: ...The Shell Model...

It is possible that the large discrepancy between measurement and theory for these data is due to the non-central spin-orbit force. Like the deuteron, it is possible that two or more orbital spin levels contribute to the quadrupole moment.

Even more astonishing is the measured quadrupole moment for single neutron, single-neutron hole data. There is no theory for this! Neutrons are not charged, and therefore, if Q were determined by the “last unpaired nucleon in” idea, Q would be zero for these states. It might be lesser in magnitude, but it is definitely not zero!

There is much more going on than the EIPM or shell models can predict. These are collective effects, whereby the odd neutron perturb the shape of the nuclear core, resulting in a measurable quadrupole moment. EIPM and the shell model can not address this physics. It is also known that the shell model prediction of quadrupole moments fails catastrophically for $60 < Z < 80$, $Z > 90$ $90 < N < 120$ and $N > 140$, where the measured moments are an order of magnitude greater. This is due to collective effects, either multiple particle behavior or a collective effect involving the entire core. We shall investigate these in due course.

12.1: ...The Shell Model...

Q for multiple valence nucleons

For multiple nucleons in the valence shell, we may state a few things quantitatively.

Returning to the definition of the quadrupole moment:

$$\langle Q \rangle = \int d\vec{x} \psi_N^*(\vec{x})(3z^2 - r^2)\psi_N(\vec{x}) .$$

The core nucleons are made up of closed shells, for which their contribution to $\langle Q \rangle = 0$. The composite wavefunction, comprised of the valence nucleons, has quantum numbers, I and m_I , as well as parity. We can write, using $3 \cos^2 \theta - 1 = \sqrt{(5/16\pi)} Y_{2,0}$ (see Krane, p. 27 for Y s) the value for $\langle Q \rangle$ as:

$$\langle Q \rangle = \langle r^2 \rangle \langle I, m_I = I | 3 \cos^2 \theta - 1 | I, m_I = I \rangle = \langle r^2 \rangle \langle II | \sqrt{(5/16\pi)} Y_{2,0} | II \rangle ,$$

and infer that all $I = 0, 1/2$ nuclei have $\langle Q \rangle = 0$. This is also an experimentally observed fact, and strong evidence of the strength of the pairing force.

12.1: ...The Shell Model...

Shell model predictions of excited states

If the EIPM were true, we could measure the shell model energy levels by observing the decays of excited states.

Recall the shell model energy diagram, and let us focus on the lighter nuclei.

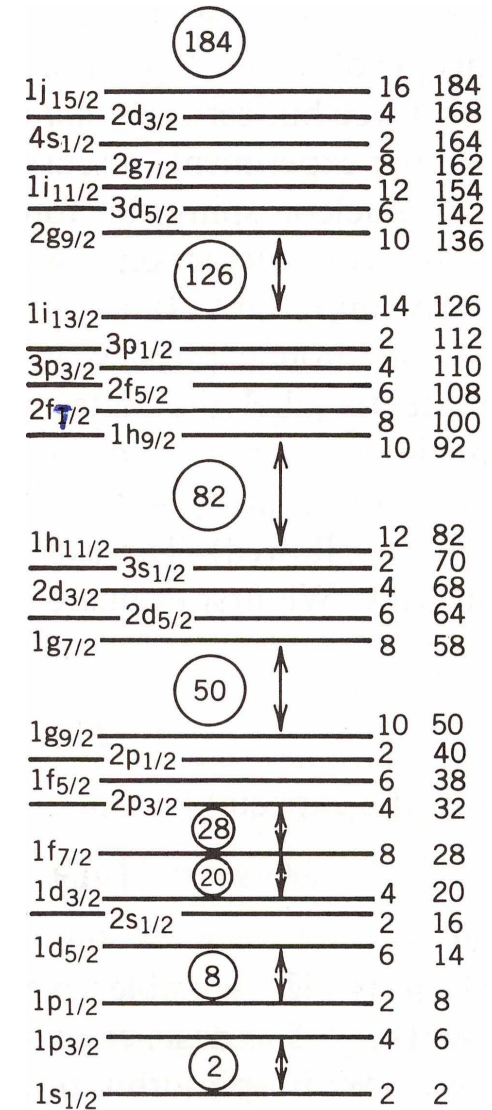


Figure 12.11: The shell model

12.1: ...The Shell Model...

Let us see if we can predict and compare the excited states of two related light nuclei:

$${}^{17}_8\text{O}_9 = [{}^{16}_8\text{O}_8] + 1n, \text{ and } {}^{17}_9\text{F}_8 = [{}^{16}_8\text{O}_8] + 1p.$$

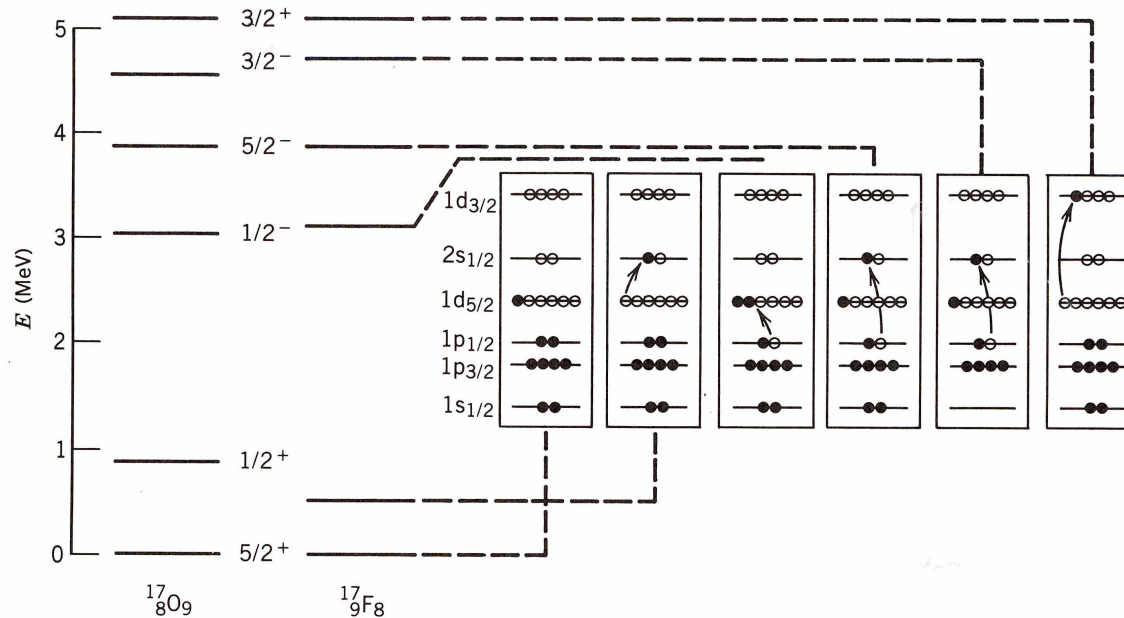


Figure 5.11 Shell-model interpretation of the levels of ${}^{17}\text{O}$ and ${}^{17}\text{F}$. All levels below about 5 MeV are shown, and the similarity between the levels of the two nuclei suggests they have common structures, determined by the valence nucleons. The even-parity states are easily explained by the excitation of the single odd nucleon from the $d_{5/2}$ ground state to $2s_{1/2}$ or $1d_{3/2}$. The odd-parity states have more complicated structures; one possible configuration is shown, but others are also important.

Figure 12.12: The low-lying excited states of ${}^{17}_8\text{O}_9$ and ${}^{17}_9\text{F}_8$. Krane's Figure 5.11, p. 131

12.1: ...The Shell Model...

The first excited state of $^{17}_8\text{O}_9$ and $^{17}_9\text{F}_8$ has $I^\pi = 1/2^+$. This is explained by the EIPM interpretation. The “last in” unpaired nucleon at the $1d_{5/2}$ level is promoted to the $2s_{1/2}$ level, vacating the $1d$ shell. The second excited state with $I^\pi = 1/2^-$ does not follow the EIPM model. Instead, it appears that a core nucleon is raised from the $1p_{1/2}$ level to the $1d_{5/2}$ level, joining another nucleon there and canceling spins. The $I^\pi = 1/2^-$ is determined by the unpaired nucleon left behind. Nor do the third and fourth excited states follow the EIPM prescription. The third and fourth excited states seem to be formed by a core nucleon raised from the $1p_{1/2}$ level to the $2s_{1/2}$ level, leaving three unpaired nucleons. Since I is formed from the coupling of j 's of $1/2$, $1/2$ and $5/2$, we expect $3/2 \leq I \leq 7/2$. $3/2$ is the lowest followed by $5/2$. Not shown, but expected to appear higher up would be the $7/2$. The parity is negative, because parity is multiplicative. Symbolically, $(-1)^p \times (-1)^d \times (-1)^s = -1$. Finally, the fifth excited state does follow the EIPM prescription, raising the “last in” unbound nucleon to $d_{3/2}$ resulting in an $I^\pi = 3/2^+$.

12.1: ...The Shell Model...

Hints of collective structure

Krane's discussion on this topic is quite good.

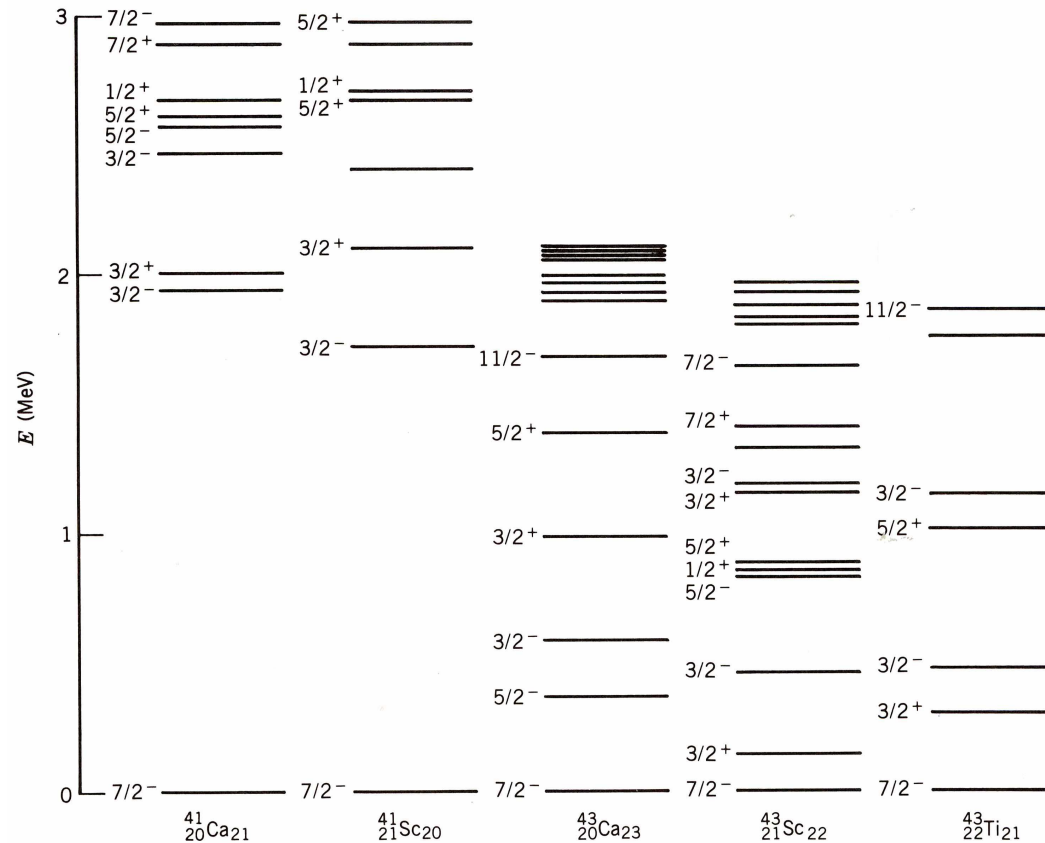


Figure 5.12 Energy levels of nuclei with odd particles in the $1f_{7/2}$ shell.

Figure 12.13: The low-lying excited states of $^{41}_{20}\text{Ca}_{21}$, $^{41}_{21}\text{Sc}_{20}$, $^{43}_{20}\text{Ca}_{23}$, $^{43}_{21}\text{Sc}_{22}$, $^{43}_{22}\text{Ti}_{21}$. Krane's figure 5.12, p. 132

12.1: ...The Shell Model...

Verification of the shell model

Krane has a very interesting discussion on a demonstration of the validity of the shell model by investigating the behavior of s states in heavy nuclei. In this demonstration, the difference in the proton charge distribution (measured by electrons), is compared for Thallium, $^{205}_{81}\text{Tl}_{124}$ and Lead $^{206}_{82}\text{Pb}_{124}$.

$$\rho_{\text{p}}^{^{205}_{81}\text{Tl}_{124}}(r) - \rho_{\text{p}}^{^{206}_{82}\text{Pb}_{124}}(r)$$

^{206}Pb has a magic number of protons and 124 neutrons while ^{205}Tl has the same number of neutrons and 1 less proton. That proton is in an $s_{1/2}$ orbital. So, the measurement of the charge density is a direct investigation of the effect of an unpaired proton coursing though the tight nuclear core, whilst on its s -state meanderings.

12.1: ...The Shell Model

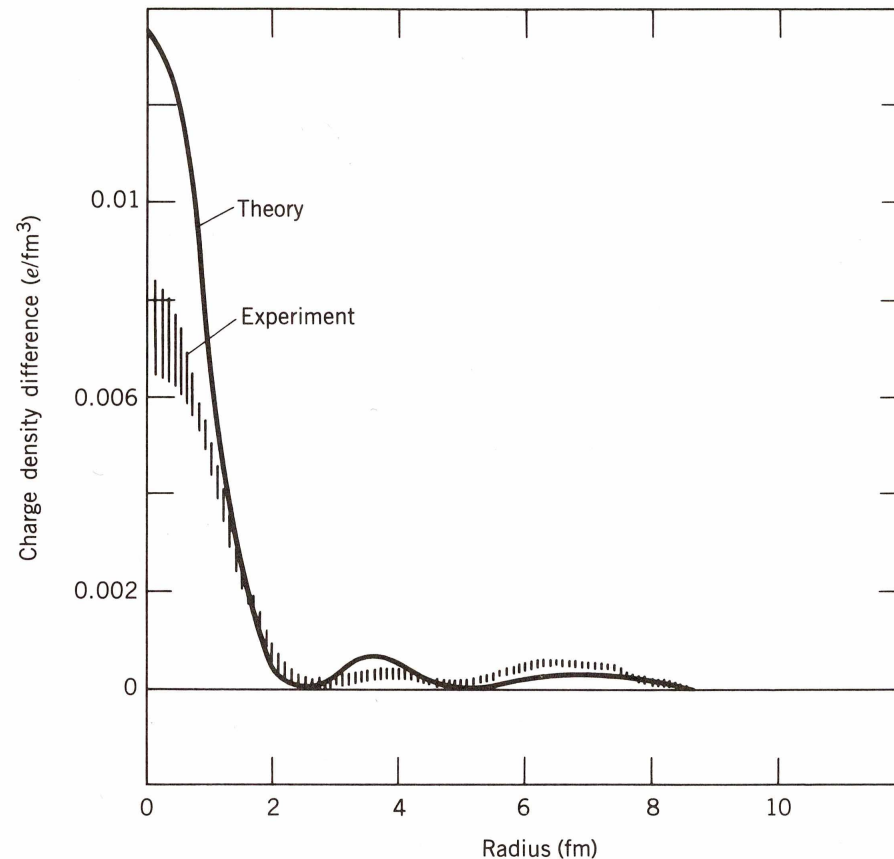


Figure 5.13 The difference in charge density between ^{205}Tl and ^{206}Pb , as determined by electron scattering. The curve marked “theory” is just the square of a harmonic oscillator 3s wave function. The theory reproduces the variations in the charge density extremely well. Experimental data are from J. M. Cavedon et al., *Phys. Rev. Lett.* **49**, 978 (1982).

12.2: Even- Z , even- N Nuclei and Collective Structure...

All even/even nuclei are $I^\pi = 0^+$, a clear demonstration of the effect of the pairing force. All even/even nuclei have an anomalously small 1st excited state at 2^+ that can not be explained by the shell model (EIPM or not). Read Krane pp. 134–138.

In the figure on the next page, observe that, except near closed shells, there is a smooth downward trend in $E(2^+)$, the binding energy of the lowest 2^+ states. Regions $150 < A < 190$ and $A > 220$ seem very small and consistent.

Q_2 is small for $A < 150$. Q_2 is large and negative for $150 < A < 190$ suggesting an *oblate* deformation

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

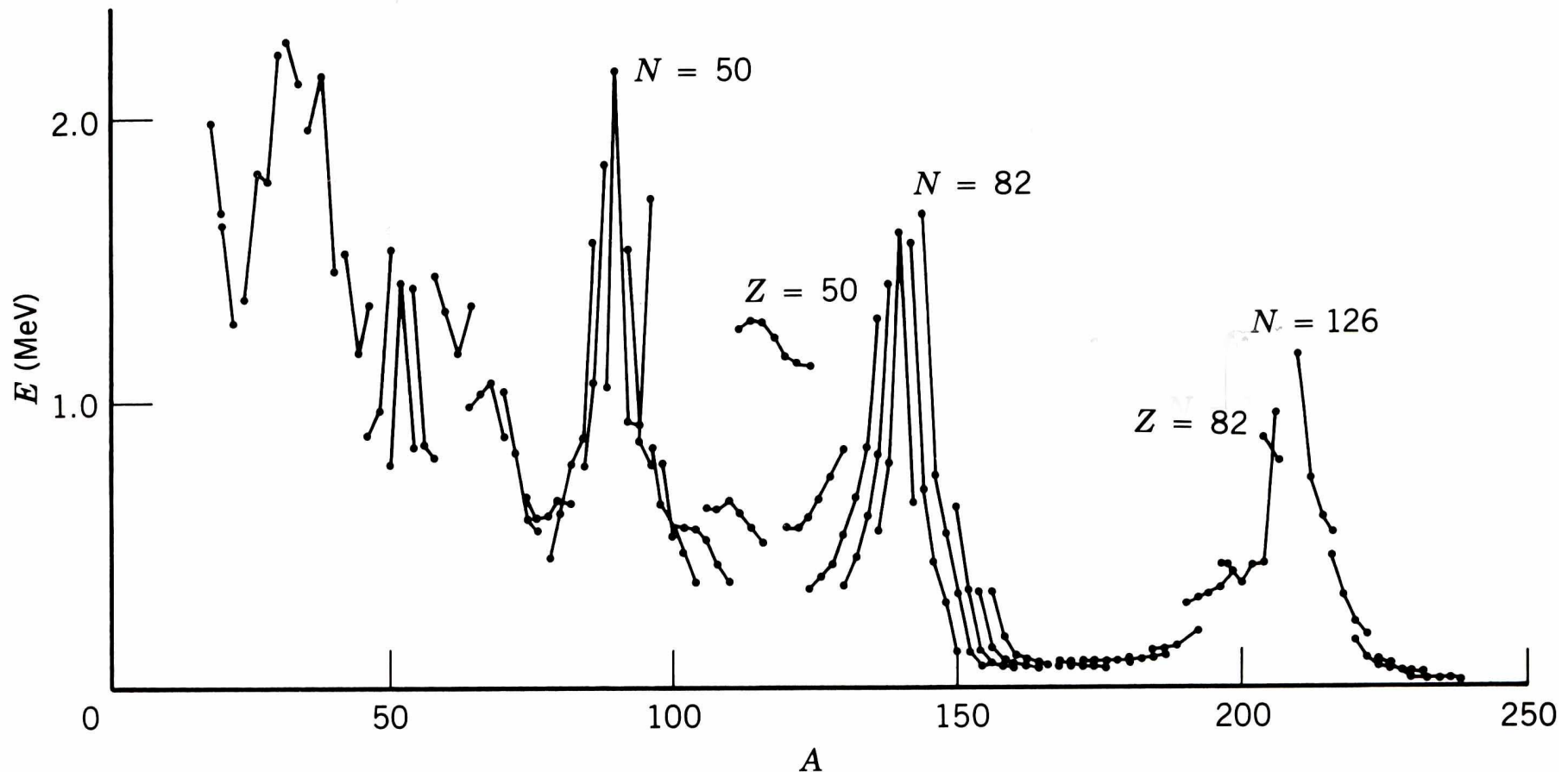
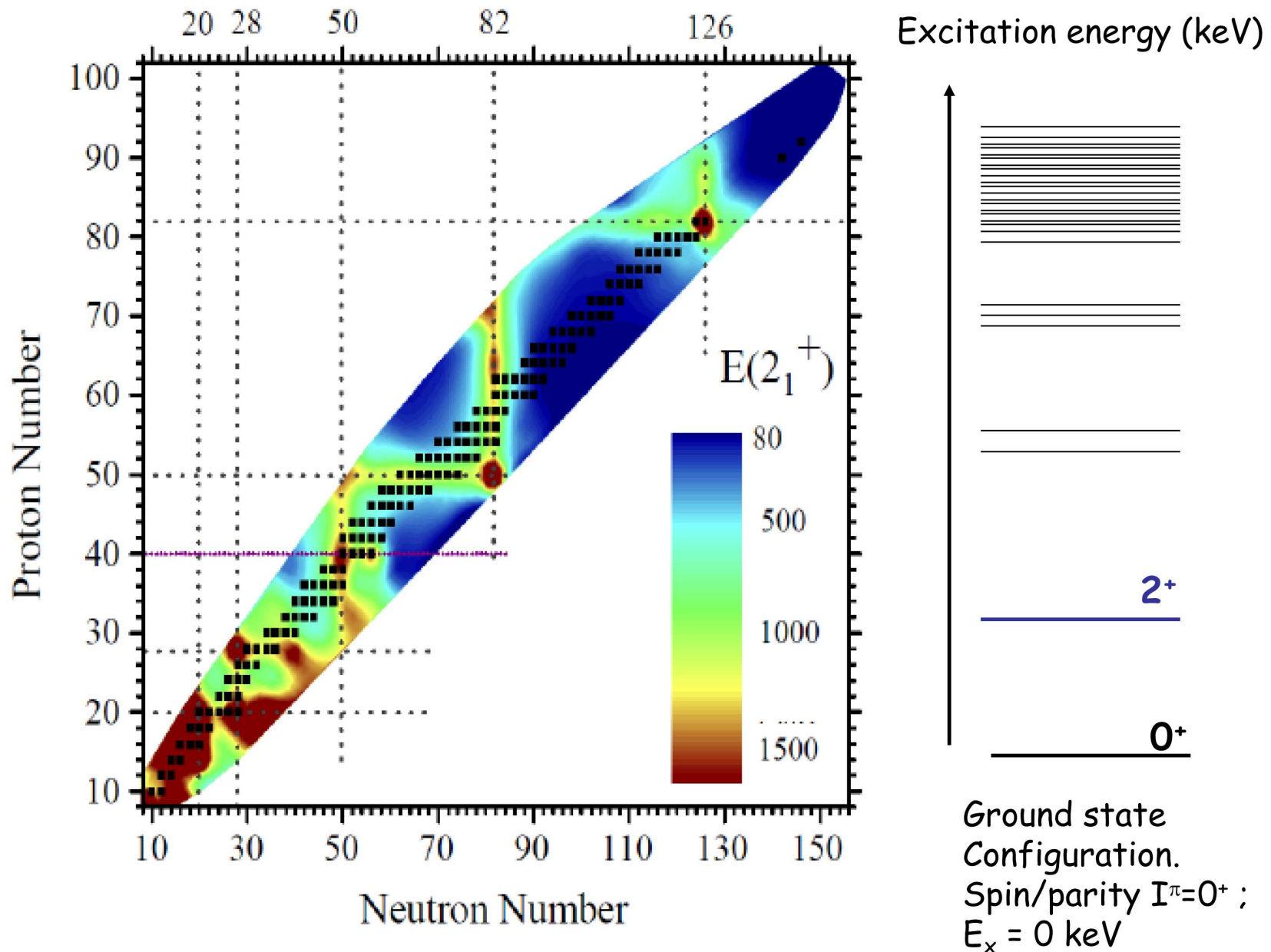


Figure 5.15a Energies of lowest 2^+ states of even- Z , even- N nuclei. The lines connect sequences of isotopes.

12.2: ...Even- Z , even- N Nuclei and Collective Structure...



This figure thanks to Prof. Patrick (Paddy) Regan of the University of Surrey - Guildford

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Quadrupole moment systematics

Observe the figure below. The regions between $150 < A < 190$ and $A > 220$ are markedly different. The ratio of $E(4^+)/E(2^+)$. One also notes something “special” about the regions: $150 < A < 190$ and $A > 220$.

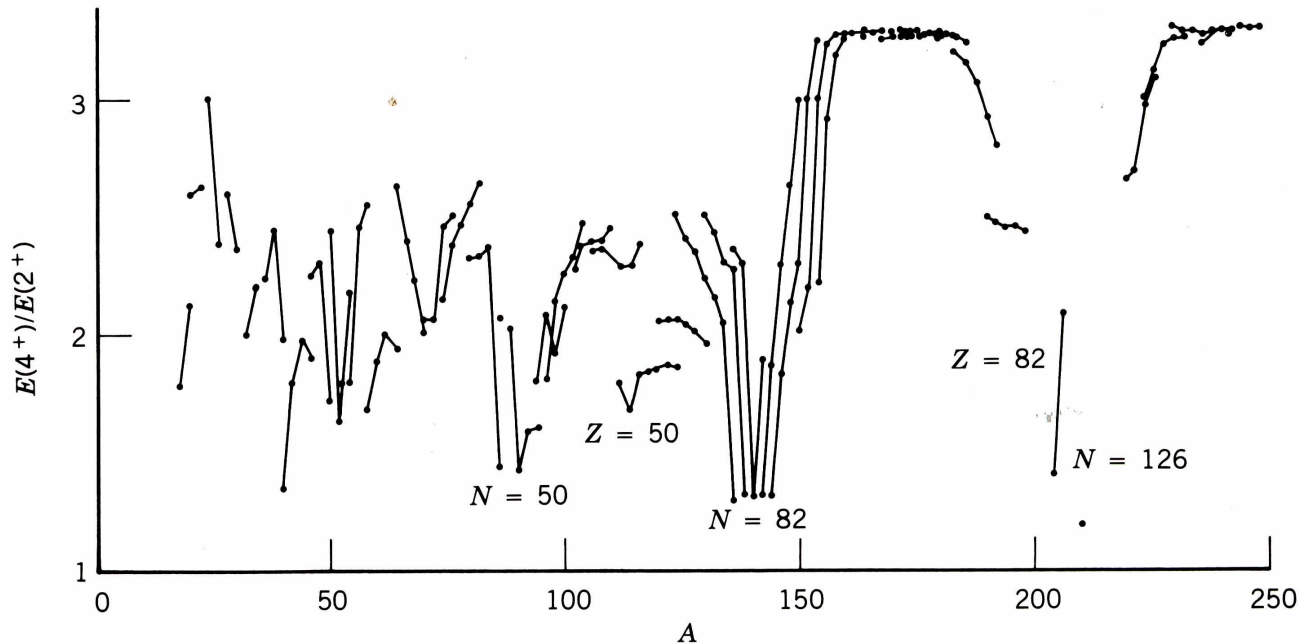
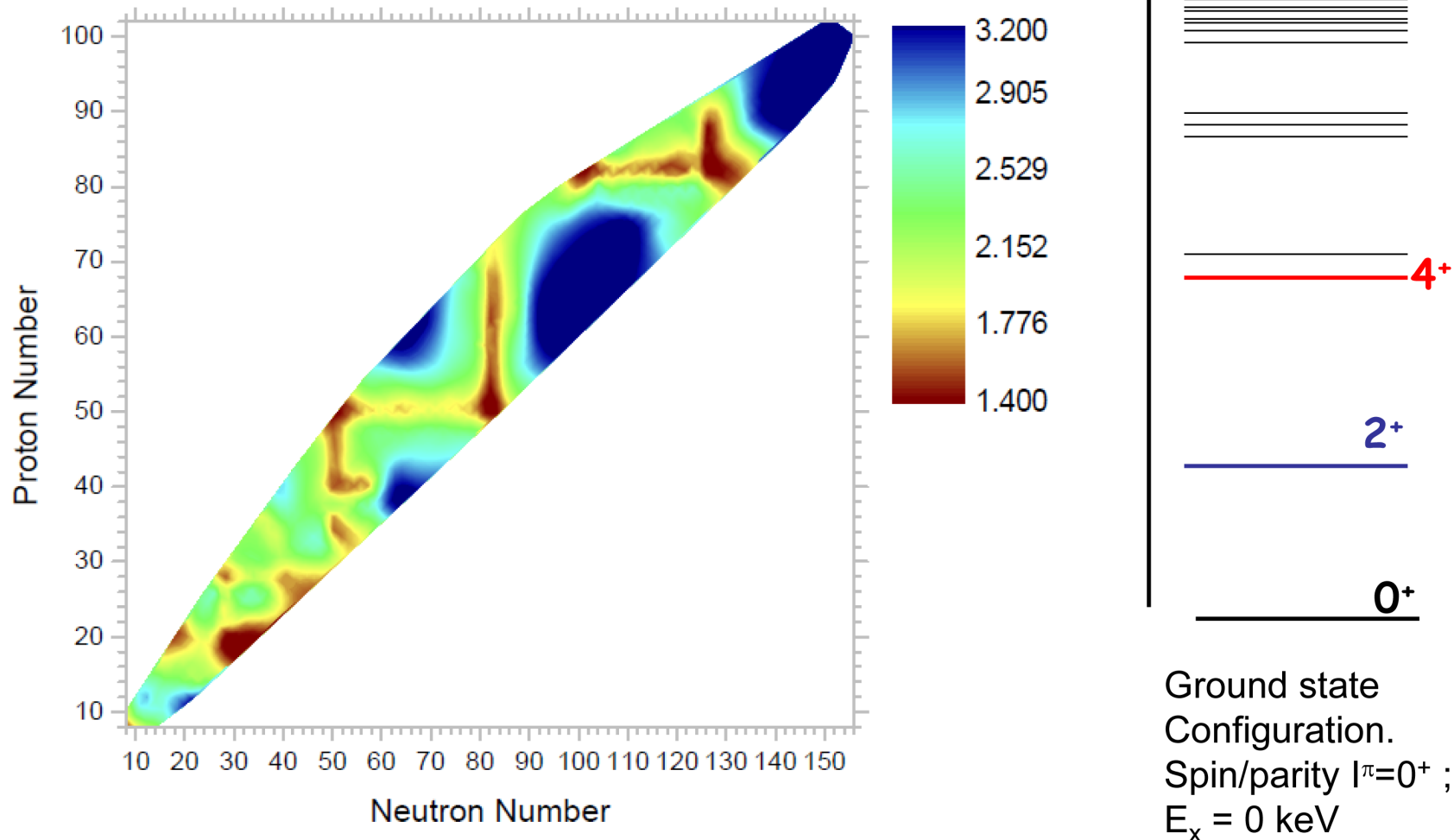


Figure 5.15b The ratio $E(4^+)/E(2^+)$ for the lowest 2^+ and 4^+ states of even- Z , even- N nuclei. The lines connect sequences of isotopes.

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

4+/2+ energy ratio:
mirrors 2+ systematics.



This figure thanks

to Prof. Patrick (Paddy) Regan of the University of Surrey - Guildford

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

All this evidence suggests a form of “collective behavior” that is described by the *Liquid Drop Model* (LDM) of the nucleus.

The Liquid Drop Model of the Nucleus

In the the Liquid Drop Model is familiar to us from the semi-empirical mass formula (SEMF). When we justified the first few terms in the SEMF, we argued that the bulk term and the surface term were characteristics of a cohesive, attractive mass of nucleons, all in contact with each other, all in motion, much like that of a fluid, like water. Adding a nucleon liberates a certain amount of energy, identical for each added nucleon. The gives rise to the bulk term. The bulk binding is offset somewhat by the deficit of attraction of a nucleon at or near the surface. That nucleon has fewer neighbors to provide full attraction. Even the Coulomb repulsion term can be considered to be a consequence of this model, adding in the extra physics of electrostatic repulsion. Now we consider that this “liquid drop” may have collective (many or all nucleons participating) excited states, in the quantum mechanical sense.

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

These excitations are known to have two distinct forms:

- Vibrational excitations, about a spherical or ellipsoidal shape. All nucleons participate in this behavior. (This is also known as phonon excitation.)
- Rotational excitation, associated with rotations of the entire nucleus, or possibly only the valence nucleons participating, with perhaps some “drag” on a non-rotating spherical core. (This is also known as roton excitation.)

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Nuclear Vibrations (Phonons)

Here we characterize the nuclear radius as have a temporal variation in polar angles in the form:

$$R(\theta, \phi, t) = R_{\text{avg}} + \sum_{\lambda=1}^{\Lambda} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu}(t) Y_{\lambda\mu}(\theta, \phi) , \quad (12.13)$$

Here, R_{avg} is the “average” radius of the nucleus, and $\alpha_{\lambda\mu}(t)$ are temporal deformation parameters. Reflection symmetry requires that $\alpha_{\lambda,-\mu}(t) = \alpha_{\lambda\mu}(t)$. Equation (12.13) describes the surface in terms of sums total angular momentum components $\vec{\lambda}\hbar$ and their z -components, $\mu\hbar$. The upper bound on λ is some upper bound Λ . Beyond that, presumably, the nucleus can not longer be bound, and flies apart. If we insist that the nucleus is an incompressible fluid, we have the further constraints:

$$V_N = \frac{4\pi}{3} R_{\text{avg}}^3$$
$$0 = \sum_{\lambda=1}^{\Lambda} |\alpha_{\lambda,0}(t)|^2 + 2 \sum_{\lambda=1}^{\Lambda} \sum_{\mu=1}^{\lambda} |\alpha_{\lambda\mu}(t)|^2 \quad (12.14)$$

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

The λ deformations are shown in Figure 12.14 for $\lambda = 1, 2, 3$.

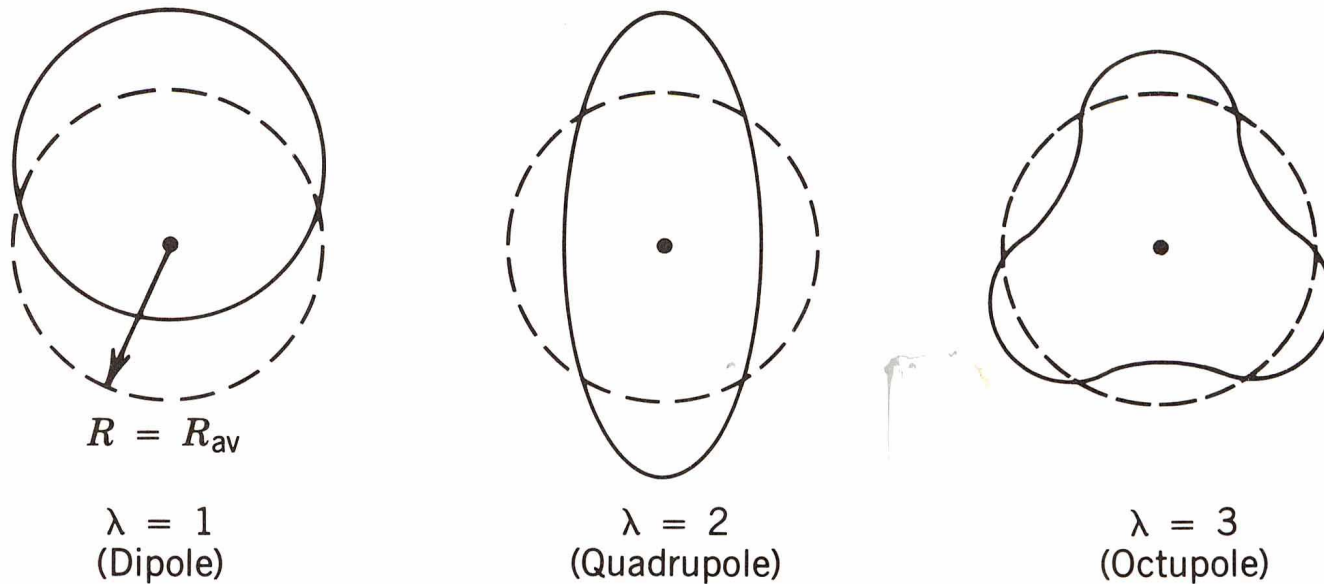


Figure 5.18 The lowest three vibrational modes of a nucleus. The drawings represent a slice through the midplane. The dashed lines show the spherical equilibrium shape and the solid lines show an instantaneous view of the vibrating surface.

Figure 12.14: In this figure, nuclear surface deformations are shown for $\lambda = 1, 2, 3$

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Dipole phonon excitation

The $\lambda = 1$ formation is a *dipole* excitation. Nuclear deformation dipole states are not observed in nature, because a dipole excitation is tantamount to a oscillation of the center of mass.

Quadrupole phonon excitation

The $\lambda = 2$ excitation is called a *quadrupole excitation* or a *quadrupole-phonon excitation*, the latter being more common. Since $\pi = (-1)^\lambda$, the parity of the quadrupole phonon excitation is always positive, and it's $I^\pi = 2^+$.

Octopole-phonon excitation

The $\lambda = 3$ excitation is called an *octopole excitation* or an *octopole-phonon excitation*, the latter being more common. Since $\pi = (-1)^\lambda$, the parity of the octopole-phonon excitation is always negative, and it's $I^\pi = 3^-$.

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Two-quadrupole phonon excitation

Now is gets interesting! Let us consider a two-quadrupole phonon excitation. We know, from 311, that $\vec{\lambda} = \vec{\lambda}_1 + \vec{\lambda}_2 = \vec{2} + \vec{2} = \{\vec{4}, \vec{3}, \vec{2}, \vec{1}, \vec{0}\}$, since quadrupole spins add in the quantum mechanical way. Let us enumerate all the apparently possible combinations of $|\mu_1\rangle$ and $|\mu_2\rangle$ for a two-phonon excitation:

$\mu = \mu_1 + \mu_2$	Possible combinations $ \mu_1\rangle \mu_2\rangle$	d	$\mu_{\lambda=4}$	$\mu_{\lambda=3}$	$\mu_{\lambda=2}$	$\mu_{\lambda=1}$	$\mu_{\lambda=0}$
4	$ 2\rangle 2\rangle$	1	y				
3	$ 2\rangle 1\rangle, 1\rangle 2\rangle$	2	y	y			
2	$ 2\rangle 0\rangle, 1\rangle 1\rangle, 0\rangle 2\rangle$	3	y	y	y		
1	$ 2\rangle -1\rangle, 1\rangle 0\rangle, 0\rangle 1\rangle, -1\rangle 2\rangle$	4	y	y	y	y	
0	$ 2\rangle -2\rangle, 1\rangle -1\rangle, 0\rangle 0\rangle, -1\rangle 1\rangle, -2\rangle 2\rangle$	5	y	y	y	y	y
-1	$ 1\rangle -2\rangle, 0\rangle -1\rangle, -1\rangle 0\rangle, -2\rangle 1\rangle$	4	y	y	y	y	
-2	$ 0\rangle -2\rangle, -1\rangle -1\rangle, -2\rangle 0\rangle$	3	y	y	y		
-3	$ -1\rangle -2\rangle, -2\rangle -1\rangle$	2	y	y			
-4	$ -2\rangle -2\rangle$	1	y				
		$\sum d = 25$	9	7	5	3	1

It would appear that we could make two-quadrupole phonon states with $I^\pi = 4^+, 3^-, 2^+, 1^-, 0^+$. However, phonons are unit spin excitations, and follow Bose-Einstein statistics!

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Therefore, only symmetric combinations can occur. Accounting for this, as we have done following, leads us to conclude that the only possibilities are: $I^\pi = 4^+, 2^+, 0^+$.

$\mu = \mu_1 + \mu_2$	Symmetric combinations	d	$\mu_{\lambda=4}$	$\mu_{\lambda=2}$	$\mu_{\lambda=0}$
4	$ 2\rangle 2\rangle$	1	y		
3	$(2\rangle 1\rangle + 1\rangle 2\rangle)$	1	y		
2	$(2\rangle 0\rangle + 0\rangle 2\rangle), 1\rangle 1\rangle$	2	y	y	
1	$(2\rangle -1\rangle + -1\rangle 2\rangle), (1\rangle 0\rangle + 0\rangle 1\rangle)$	2	y	y	
0	$(2\rangle -2\rangle + -2\rangle 2\rangle), (1\rangle -1\rangle + -1\rangle 1\rangle), 0\rangle 0\rangle$	3	y	y	y
-1	$(1\rangle -2\rangle + -2\rangle 1\rangle), (0\rangle -1\rangle + -1\rangle 0\rangle)$	2	y	y	
-2	$(0\rangle -2\rangle + -2\rangle 0\rangle), -1\rangle -1\rangle$	2	y	y	
-3	$(-1\rangle -2\rangle + -2\rangle -1\rangle)$	1	y		
-4	$ -2\rangle -2\rangle$	1	y		
		$\sum d = 15$	9	5	1

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Three-quadrupole phonon excitations

For three-quadrupole phonon excitations $\vec{\lambda} = \vec{\lambda}_1 + \vec{\lambda}_2 + \vec{\lambda}_3$ one can show easily (hah!) that the Bose-Einstein combinations that survive are $I^\pi = 6^+, 4^+, 3^+, 2^+, 0^+$.

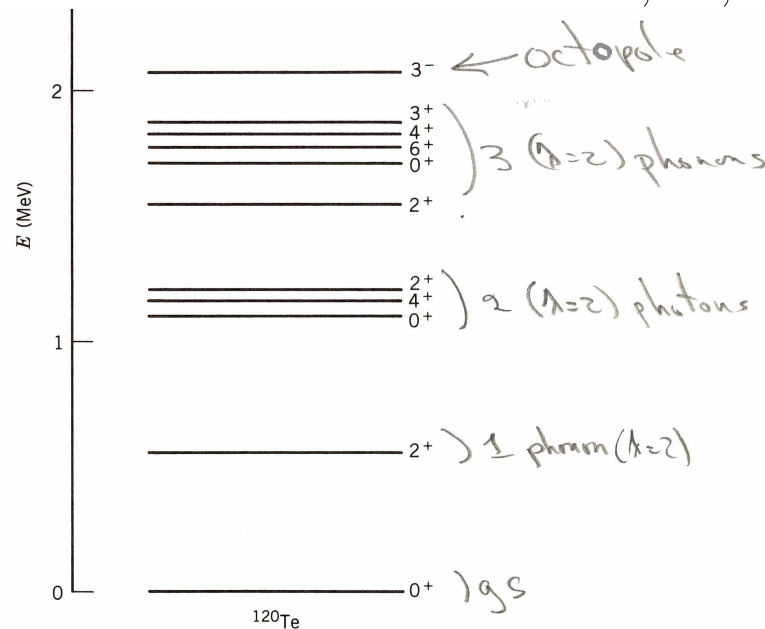


Figure 5.19 The low-lying levels of ^{120}Te . The single quadrupole phonon state (first 2^+), the two-phonon triplet, and the three-phonon quintuplet are obviously seen. The 3^- state presumably is due to the octupole vibration. Above 2 MeV the structure becomes quite complicated, and no vibrational patterns can be seen.

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Nuclear Rotations (Rotons)

Nuclei in the mass range $150 < A < 190$ and $A > 200$ have permanent non-spherical deformations. The quadrupole moments of these nuclei are larger by about an order of magnitude over their non-deformed counterparts.

This permanent deformation is usually modeled as follows:

$$R_N(\theta) = R_{\text{avg}}[1 + \beta Y_{20}(\theta)] . \quad (12.15)$$

(12.15) describes (approximately) an ellipse, if β , the deformation parameter β is small, that is $\beta \ll 1$. β is related to the “eccentricity” of an ellipse as follows,

$$\beta = \frac{4}{3} \sqrt{\frac{\pi}{5}} \frac{\Delta R}{R_{\text{avg}}} , \quad (12.16)$$

where ΔR is the difference between the semimajor and semiminor axes of the ellipse. When $\beta > 0$, the nucleus is a *prolate* ellipsoid (football shape) and *oblate* (curling stone) when $\beta < 0$.

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

When $\beta < 0$, the nucleus is an *oblate* ellipsoid (shaped like a curling stone). Or, if you like, if you start with a spherical blob of putty and roll it between your hands, it becomes prolate. If instead, you press it between your hands, it becomes *prolate*, like a football.

The relationship between β and the quadrupole moment of the nucleus is:

$$Q = \frac{3}{\sqrt{5\pi}} R_{\text{avg}}^2 Z \beta \left[1 + \frac{2}{7} \left(\frac{5}{\pi} \right)^{1/2} \beta + \frac{9}{28\pi} \beta^2 \right] . \quad (12.17)$$

Note: This is a correction to Krane's equation (5.16). The β -term has a coefficient of 0.16, rather than 0.36 as implied by (12.17). Typically, this correction is about 10%. The additional term provided in (12.17) provides about another 1% correction.

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Energy of rotation

Classically, the energy of rotation, E_{rot} is given by:

$$E_{\text{rot}} = \frac{1}{2} \mathcal{I} \omega^2, \quad (12.18)$$

where $\mathcal{I}$ is the *moment of inertia* and ω is the rotational frequency. The transition to Quantum Mechanics is done as follows:

$$E_{\text{rot}}^{\text{QM}} = \frac{1}{2} \frac{\vec{\mathcal{I}} \cdot \vec{\mathcal{I}}}{\mathcal{I}} \omega^2 = \frac{1}{2} \frac{(\vec{\mathcal{I}} \omega) \cdot (\vec{\mathcal{I}} \omega)}{\mathcal{I}} = \frac{1}{2} \frac{\langle (\vec{I} \hbar) \cdot (\vec{I} \hbar) \rangle}{\mathcal{I}} = \frac{\hbar^2}{2\mathcal{I}} \langle \vec{I} \cdot \vec{I} \rangle = \frac{\hbar^2}{2\mathcal{I}} I(I+1) \quad (12.19)$$

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Technical aside: Moment of Inertia?



Figure 12.15: A rigid body in rotation.

Imagine that an object is spinning around the z -axis, which cuts through its center of mass, as shown in Figure 12.15. We place the origin of our coordinate system at the object's center of mass. The angular frequency of rotation is ω .

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

The element of mass, dm at $\vec{x}$ is $\rho(\vec{x})d\vec{x}$, where $\rho(\vec{x})$ is the mass density. [$M = \int d\vec{x} \rho(\vec{x})$]. The speed of that mass element, $|v(\vec{x})|$ is $\omega r \sin \theta$. Hence, the energy of rotation, of that element of mass is:

$$dE_{\text{rot}} = \frac{1}{2}dm |v(\vec{x})|^2 = \frac{1}{2}d\vec{x} [\rho(\vec{x})r^2 \sin^2 \theta] \omega^2 . \quad (12.20)$$

Integrating over the entire body gives:

$$E_{\text{rot}} = \frac{1}{2}\mathcal{I}\omega^2 , \quad (12.21)$$

which defines the moment of inertia to be:

$$\mathcal{I} = \int d\vec{x} \rho(\vec{x})r^2 \sin^2 \theta . \quad (12.22)$$

The moment of inertia is an intrinsic property of the object in question.

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Example 1: Moment of inertia for a spherical nucleus

Here,

$$\rho(\vec{x}) = M \frac{3}{4\pi R_N^3} \Theta(R_N - r) .$$

Hence,

$$\begin{aligned} \mathcal{I}_{\text{sph}} &= M \frac{3}{4\pi R_N^3} \int_{|\vec{x}| \leq R_N} d\vec{x} r^2 \sin^2 \theta \\ &= \frac{3M}{2R_N^3} \int_0^{R_N} dr r^4 \int_0^\pi \sin \theta d\theta \sin^2 \theta \\ &= \frac{3MR_N^2}{10} \int_{-1}^1 d\mu (1 - \mu^2) \\ \mathcal{I}_{\text{sph}} &= \frac{2}{5} M R_N^2 \end{aligned} \tag{12.23}$$

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Example 2: Moment of inertia for an elliptical nucleus

Here, the mass density is a constant, but within a varying radius given by (12.15), namely

$$R_N(\theta) = R_{\text{avg}}[1 + \beta Y_{20}(\theta)] .$$

The volume of this nucleus is given by:

$$\begin{aligned} V &= \int_{|\vec{x}| \leq R_{\text{avg}}[1 + \beta Y_{20}(\mu)]} d\vec{x} \\ &= 2\pi \int_{-1}^1 d\mu \int_0^{R_{\text{avg}}[1 + \beta Y_{20}(\mu)]} r^2 dr \\ &= \frac{2\pi R_{\text{avg}}^3}{3} \int_{-1}^1 d\mu [1 + \beta Y_{20}(\mu)]^3 \end{aligned} \tag{12.24}$$

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

$$\begin{aligned}\mathcal{I}_\ell &= \int d\vec{x} \rho(\vec{x}) r^2 \sin^2 \theta \\ &= \frac{M}{V} (2\pi) \int_{-1}^1 d\mu (1 - \mu^2) \int_0^{R_{\text{avg}}[1+\beta Y_{20}(\mu)]} dr r^4 \\ &= \frac{M R_{\text{avg}}^5}{V} \frac{2\pi}{5} \int_{-1}^1 d\mu (1 - \mu^2) [1 + \beta Y_{20}(\mu)]^5 \\ \mathcal{I}_\ell &= M R_{\text{avg}}^2 \left(\frac{3}{5} \right) \left[\int_{-1}^1 d\mu (1 - \mu^2) [1 + \beta Y_{20}(\mu)]^5 \right] \bigg/ \left[\int_{-1}^1 d\mu [1 + \beta Y_{20}(\mu)]^3 \right] \quad (12.25)\end{aligned}$$

(12.25) is a ratio a 5th-order polynomial in β , to a 3rd-order polynomial in β . However, it can be shown that it is sufficient to keep only $O(\beta^2)$. With,

$$Y_{20}(\mu) = \sqrt{\frac{5}{16\pi}} (3\mu^2 - 1)$$

(12.25) becomes:

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

$$\begin{aligned}\mathcal{I}_\ell &= \left(\frac{2}{5}\right) MR_{\text{avg}}^2 \left[1 - \frac{1}{2} \sqrt{\frac{5}{\pi}} \beta + \frac{71}{28\pi} \beta^2 + O(\beta^3) \right] \\ &= \left(\frac{2}{5}\right) MR_{\text{avg}}^2 \left[1 - 0.63\beta + 0.81\beta^2 + (< 1\%) \right] .\end{aligned}\tag{12.26}$$

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Rotational bands

$E_{\text{rot}}(I^\pi)$	Value	Interpretation
$E(0^+)$	0	ground state
$E(2^+)$	$6(\hbar^2/2\mathcal{I})$	1st rotational state
$E(4^+)$	$20(\hbar^2/2\mathcal{I})$	2nd rotational state
$E(6^+)$	$42(\hbar^2/2\mathcal{I})$	3rd rotational state
$E(8^+)$	$72(\hbar^2/2\mathcal{I})$	4th rotational state
$\vdots$	$\vdots$	$\vdots$

Using $\mathcal{I}_{\text{rigid}}$, assuming a rigid body, gives a spacing that is low by a factor of about off by about 2–3. Using

$$\mathcal{I}_{\text{fluid}} = \frac{9}{8\pi} M_N R_{\text{avg}}^2 \beta$$

for a fluid body in rotation, gives a spacing that is high by a factor of about off by about 2–3. Thus the truth for a nucleus, is somewhere in between:

$$\mathcal{I}_{\text{fluid}} < \mathcal{I}_N < \mathcal{I}_{\text{rigid}}$$

12.2: ...Even- Z , even- N Nuclei and Collective Structure...

Actually, the moment of inertia of a fluid body is an ill-defined concept. There are two ways I can think of, whereby the moment of inertia may be reduced. One model could be that of a “static non-rotating core”. From (12.27), this would imply that:

$$\mathcal{I}_\ell = - \left(\frac{2}{5} \right) M R_{\text{avg}}^2 \left[\frac{1}{2} \sqrt{\frac{5}{\pi}} \beta - \frac{71}{28\pi} \beta^2 \right] \approx - \left(\frac{2}{5} \right) M R_{\text{avg}}^2 [\beta - 0.81\beta^2] .$$

Another model would be that of viscous drag, whereby the angular frequency becomes a function of r and θ .

For example, $\omega = \omega_0 (r \sin \theta / R_{\text{avg}})^n$. One can show that the reduction, R_n in $\mathcal{I}$ is of the form $R_{n+1} = \frac{2(n+2)}{7+2n} R_n$, where $R_0 \equiv 1$.

A “parabolic value”, $n = 2$, gives the correct amount of reduction, about a factor of 2. This also makes some sense, since rotating liquids obtain a parabolic shape.

Chapter 12: Nuclear Models...

Things to think about ...

Chapter 12.0: Introduction ...

1. What is a many-body force? Gives some examples? Why is it important in nuclear physics?

Chapter 12.1: The Shell Model ...

1. What are “magic” numbers? How are they obtained?
2. What is the major Shell-Model idea:
3. Describe the details of the modern nuclear potential. What is the Woods-Saxon form?
4. How does one model the spin-orbit potential?
5. Evaluate the $\langle \vec{l} \cdot \vec{s} \rangle$ term in the spin-orbit potential.
6. What is the EIPM idea?
7. Review the prediction of the magnetic moment of the nucleus in the EIPM. How good is it? What can you conclude?

Chapter 12: ...Nuclear Models

Chapter 12.1: The Shell Model ...

8. How do you determine the ground state I^π in the shell model? Review some examples.
9. How well does the Shell model and the EIPM predict of the quadrupole moment of the nucleus?
10. How do you predict excited states from the shell model and the EIPM?
11. What experimental data led to the conclusion of collective structure of the nucleus?
12. Justify the assertion that all $I = 0, 1/2$ nuclei have no quadrupole moments.
13. What conclusions do you draw from Figure 5.13 in Krane?

Chapter 12.2: Even- Z , even- N Nuclei and Collective Structure ...

1. What do Figures(5.15) and (5.16) in Krane suggest?
2. What is the “Liquid Drop Model” of the Nucleus
3. What is a “phonon”?
4. Why don't nuclei have a dipole phonon excitation?

Chapter 12: ...Nuclear Models

Chapter 12.2: Even- Z , even- N Nuclei and Collective Structure ...

5. What is a quadrupole phonon excitation?
6. What is an octopole phonon excitation?
7. How do you couple spins in a two-quadrupole phonon excitation?
8. How do you couple spins in three-quadrupole phonon excitations?
9. What is a roton?
10. Compare the energy spectra of phonons and rotons.
11. what is a “moment of Inertia”?
12. Calculate the moment of inertia for a spherical nucleus.
13. Calculate the moment of inertia for an elliptical nucleus.
14. What is the signature of rotational bands?