

NERS 312

Elements of Nuclear Engineering and Radiological Sciences II

aka **Nuclear Physics for Nuclear Engineers**

Lecture Notes for Chapter 11: The Force Between Nucleons

Supplement to (Krane II: Chapter 4)

*Note: The lecture number corresponds directly to the chapter number in the online book.
The section numbers, and equation numbers correspond directly to those in the online book.*

11.0: Introduction...

Try to imagine the 19th and early 20th centuries before the advent of Quantum Mechanics...

Measurements $\implies$ Phenomenology $\implies$ Empirical relationships (Balmer series) $\implies$ Crude theories (Thomson model, Bohr-Rutherford model) $\implies$ Radical ideas (Planck, de Broglie) $\implies$ until finally, a great breakthrough ...

The Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2 \dots$$

That's the way science works...sometimes.

11.0: ...Introduction...

Now we had a well-defined theory on which to base the basic understanding of atomic structure. With a solid fundamental theory, we now have a well-defined systematic approach:

- Solve the Schrödinger equation for the H atom, treating p, e^- as fundamental point-charge particles, a good approximation since the electron wavefunctions overlap minimally with the very small, but finite-sized proton.
- Build up more complex atoms using the rules of Quantum Mechanics. Everything is treated theoretically, as a perturbation, since the Coulomb force is relatively weak. This is expressed by the smallness of the fine-structure constant:

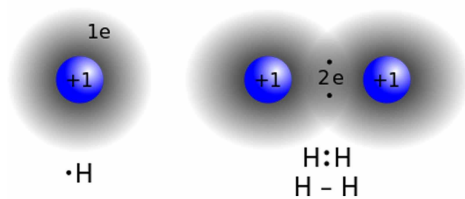
$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = 7.297\,352\,5698(24) \times 10^{-3} = \frac{1}{137.035\,999\,074(44)} \quad (11.1)$$

11.0: ...Introduction...

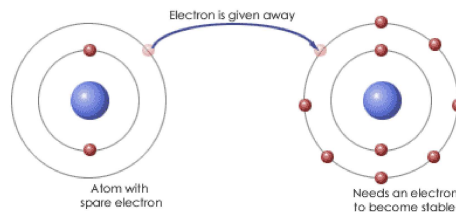
- The interaction between atoms is studied theoretically by *Molecular Theory*. Molecular Theory treats the force between atoms as a *derivative force*, a remnant of the more basic Coulomb force that binds the atom, and the energy benefit of closing atomic shells, or simple electrostatic attraction:

Chemical bonding:

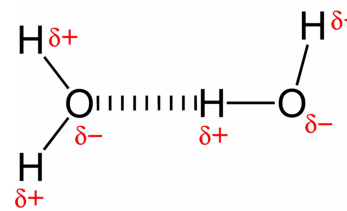
covalent
sharing e^- 's
to close shells



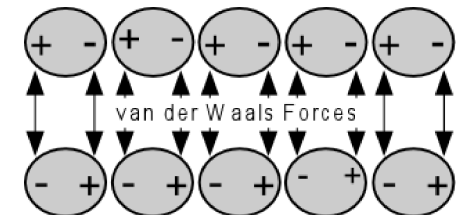
ionic
taking e^- 's
to close shells



H-bonds
electrostatic
bonds from
polar H-atoms



Van der Waal's
dipole-dipole
and higher
multipoles



11.0: ...Introduction...

Theoretical *ab initio* nuclear physics, on the other hand, is just at its infancy, because even the simplest problem, like finding the mass of a nucleon, requires supercomputers, months of time, and tricky algorithms.

For example, *ab initio* calculation of the nucleon mass (the calculation is not refined enough to tell the difference between a neutron and a proton!) has a 2% uncertainty, and even that is considered a HUGE success.

Only about 1% of the mass of a nucleon is made up of the rest-mass energies of its constituent particles with mass, the quarks. The rest is the kinetic energy of those quarks and the gluons. What a difficult calculation. Perturbation methods do not work at all!

However, that will not deter us from building up nuclei from its constituent components ...

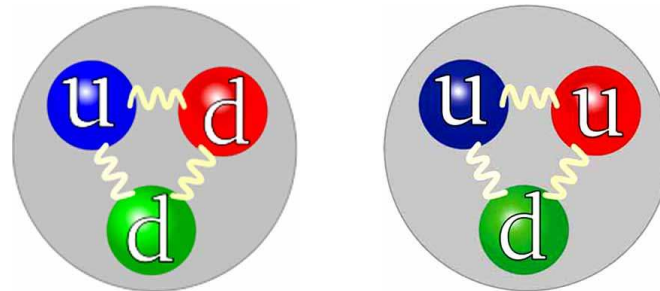
11.0: ...Introduction...

The nucleon building blocks...

Name	up quark (u)	down quark (d)
mass (MeV)	1.7 – 3.1	4.1 – 5.7
charge (e)	$+2/3$	$-1/3$
spin	$1/2$	$1/2$

The nuclear building blocks...

Name	neutron	proton
mass (MeV)	939.565378(21)	938.272046(21)
charge (e)	0	1
constituents	$2d + 1u$	$2u + 1d$
I^π	$1/2^+$	$1/2^+$



11.0: ...Introduction...

Please review the following lecture notes:

10.0 Introduction	Topic	Slide # pages
	Characteristics of nucleons	4–6
	The nucleon-nucleon force	7–9
	Some nomenclature	10
	Nucleus formation	11-12

11.0: The nuclear force (major observations)...

- A major part of the nucleon-nucleon potential is central, a trade off between the strong very short-ranged repulsive force, that arises from the exchange of a virtual vector meson (meaning spin-1), and a slightly longer range attractive force, that arises from the exchange of a virtual scalar meson (meaning spin-0). Hence the nuclear force has a strong central component. (Think Y_{lm_l} !)

Recall:

$$V_{nn}(r) = \frac{\hbar c}{r} \left\{ \alpha_V \exp[-(m_V c/\hbar)r] - \alpha_S \exp[-(m_S c/\hbar)r] \right\} \quad (11.2)$$

11.0: ...The nuclear force (major observations)...

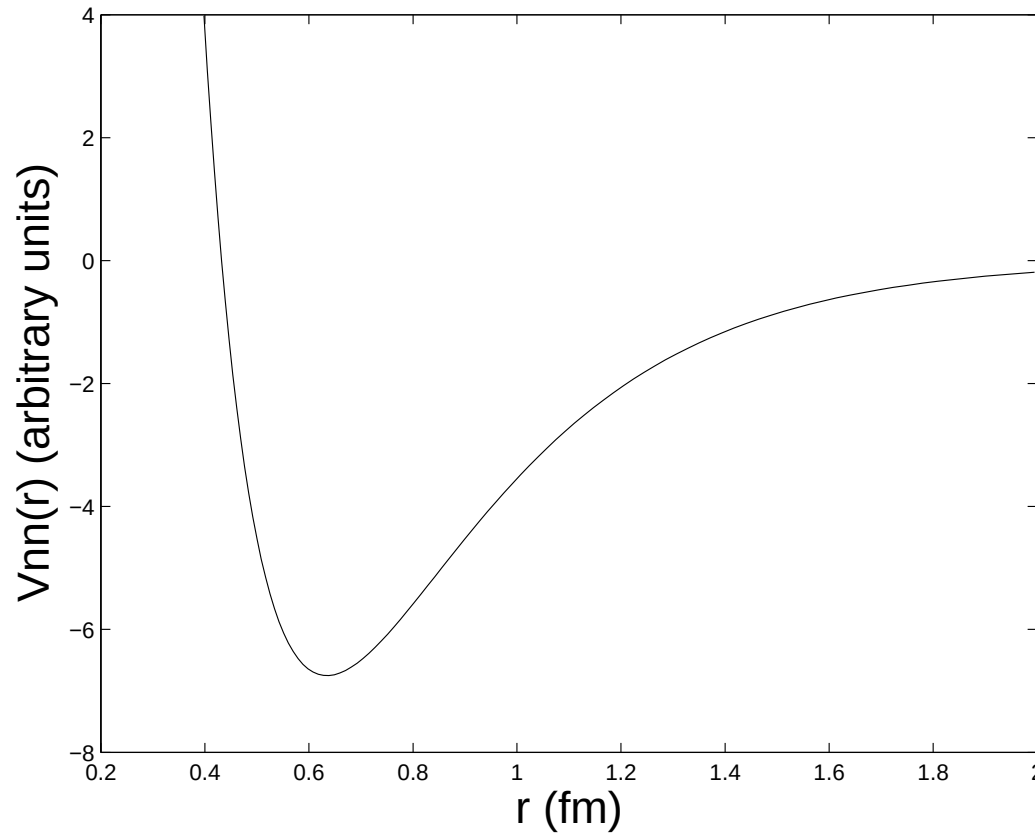


Figure 11.1: A sketch of the central part $V_{nn}(r)$ of the nucleon-nucleon potential.

11.0: ...The nuclear force (major observations)...

- Only the *hadrons* (meaning bound states of quarks), feel the strong force. The nucleons are *baryons* meaning bound states of 3 quarks, and the *mesons* are bound states of a quark and an anti-quark. Note that, historically the μ was referred to as a mu-meson, though it is not a meson, it is a lepton. In modern language, it is called *muon*.
- The nuclear force is “almost” independent of type of nucleon. That is, the strong component of the n - n , n - p , & p - p , can be (usually) treated as the same.

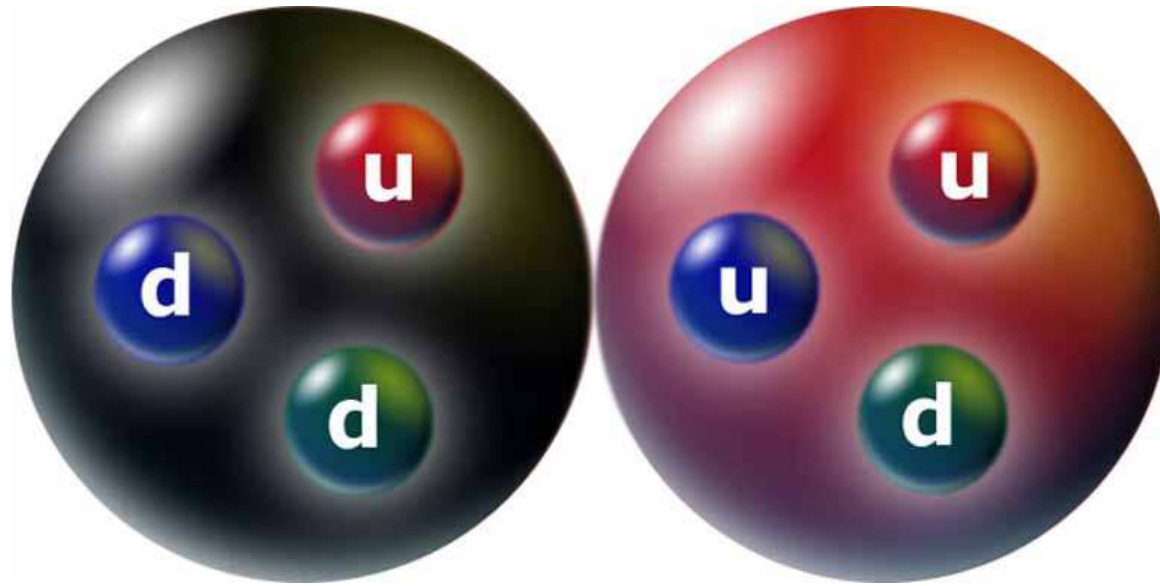
11.0: ...The nuclear force (major observations)

- The nucleon-nucleon force has a very significant *spin-orbit* component, that can be modeled as $V_{\text{so}}(r)(\vec{l} \cdot \vec{s})$. Hence the nuclear force has a significant spin-orbit component.
- The nucleon-nucleon force has a very significant *spin-spin* component, that can be modeled as $V_{\text{ss}}(r)(\vec{s}_1 \cdot \vec{s}_2)$. Hence the nuclear force also has an important spin-spin component.
- Since the nucleon-nucleon force is non-central, $V_{\text{nn}}(\vec{x}) \neq V_{\text{nn}}(|\vec{x}|)$ the nuclear force is non-central.
- Including the non-central and Coulomb components, the complete nucleon-nucleon potential may be written:

$$V_N(\vec{x}) = V_{nn}(r) + V_C(r) + V_{\text{so}}(\vec{x}) + V_{\text{ss}}(\vec{x}) , \quad (11.3)$$

- $\vec{I} = \sum_{i=1}^A \vec{j}_i$ is conserved, and may be measured.
- $\vec{L} = \sum_{i=1}^A \vec{l}_i$ and $\vec{S} = \sum_{i=1}^A \vec{s}_i$ are not conserved, and can not be measured unambiguously.
However, they can be inferred (somewhat) from parity assignments.
- This has some fascinating consequences!

11.1: The deuteron...



Name	deuteron aka ${}^2\text{H}$ or D
mass (MeV)	1875.612859(41)
charge (e)	1
constituents	$1n + 1p$
I^π	1^+
Binding energy (MeV)	2.224526624(39)

11.1: ...The deuteron...

Isotopes/isobars/exotics related to D, and some other data.:

Isotope	I^π	Atomic mass (u)	Natural abundance/mean life
^1H	$\frac{1}{2}^+$	1.00782503207(10)	0.999885(70)
^2H , D	1^+	2.0141017778(4)	0.000115(70)
^3H , T	$\frac{1}{2}^+$	3.0160492777(25)	4500(8)d [12.32(2) years]
dineutron	?		very short!
n_m		where $m \geq 3$	are not expected to be found
diproton			does not exist (Coulomb repulsion)

Note: The dineutron was (unambiguously) found at Michigan State in 2012.

11.1: ...The deuteron...

Some other facts...

- Heavy water (D_2O) is produced through electrolysis of water. One in every 3200 molecules of H_2O , is HDO .
- D_2O is chemically very similar to H_2O , but not identical. Drinking D_2O , instead of H_2O , for 10–14 days is lethal.
- The D is the most ideal laboratory we have for studying nuclear physics, like the H-atom in atomic physics, and the H_2 molecule, for molecular physics.
- The D has no excited states.

Hence, let us investigate the properties of the D as an individual nucleus.

11.1: ...The deuteron...

Using the mass-binding energy relation, that is

$$B_N(Z, A) = [Zm(^1H) + Nm_n - m(^AX)]c^2 ,$$

we can evaluate the binding energy of the deuteron,

$$B_N(1, 2) = [m(^1H) + m_n - m(D)]c^2 .$$

Modern data for $B_N(1, 2)$:

Source	$B_N(1, 2)$ (MeV)
Accepted value	2.224566(1) (MeV)
$H(n, \gamma)D$	2.224569(2) (MeV)
$D(\gamma, n)H$	2.224(2) (MeV)

Note that the D is very weakly bound. $B(D) \approx 1.1$ MeV, whereas $B(Z, A) \approx 8$ MeV, for most of the periodic table ($A > 20$).

The D has only one bound state...

11.1: ...The deuteron...

Why does the deuteron have only one bound state?

In NERS311 we discussed the s -state solutions to the 3D-Schrödinger equation. We discovered that, unlike the 1D-square well, which always has a bound state no matter the depth of the well, the 3D requires the potential depth to satisfy the following:

$$|V_0| > \frac{\pi^2 \hbar^2}{8MR^2} .$$

In this case, M is the reduced mass, and R is the radius of the deuteron. Putting in values, $V_0 \approx 18$ binds the deuteron with 0 binding energy.

11.1: ...The deuteron...

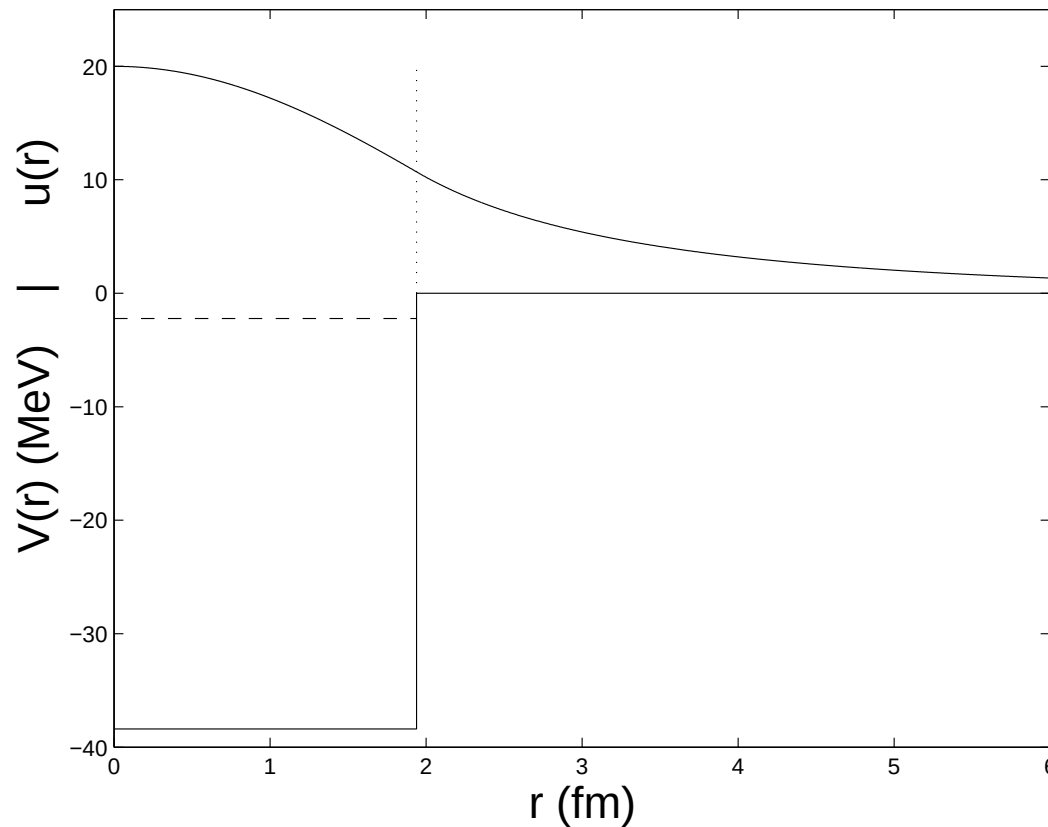


Figure 11.2: Spherical-well potential model for the deuteron. The single bound level is shown at $E = -2.224$ MeV. The loose binding of the deuteron is indicated by the long exponential tail outside of the radius of the deuteron.

11.1: ...The deuteron...

Using the potential-well central potential as a 3D model of the nucleus, we know that the solutions are of the form:

$$\psi(r, \theta, \phi)_{n,l,m_l} = R_{nl}(r)Y_{l,m_l}(\theta, \phi)$$

$Y_{l,m_l}(\theta, \phi)$ is the spherical harmonic, with orbital and magnetic quantum numbers, l and m_l , and $R_{nl}(r)$ that depends on the principle quantum number n , as well as l . This is the radial part of the wave function, and satisfies the differential equation:

$$\left[-\frac{\hbar^2}{2m} \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) + V(r) + \frac{\hbar^2 l(l+1)}{2mr^2} \right] R_{nl}(r) = E_{nl} R_{nl}(r)$$

where m is the reduced mass, *viz.*

$$m \equiv 1 / \left(\frac{1}{m_n} + \frac{1}{m_p} \right)$$

11.1: ...The deuteron...

Substituting

$$R_{nl}(r) = \frac{u_{nl}(r)}{r}$$

we have a simpler differential equation,

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V(r) + \frac{\hbar^2 l(l+1)}{2mr^2} \right] u_{nl}(r) = E_{nl} u_{nl}(r)$$

For $V(r) = -V_0 \theta(R_N - r)$, the solutions are known in terms of “spherical Bessel functions”.

However, in the case of the s states, namely, $l = 0$, we have the further simplification,

$$\text{for } (r < R_N) : u_n'' + k_n^2 u = 0 \quad k_n^2 = \frac{2m(E_n + V_0)}{\hbar^2}$$

$$\text{for } (r > R_N) : u_n'' - K_n^2 u = 0 \quad K_n^2 = \frac{-2mE_n}{\hbar^2}$$

11.1: ...The deuteron...

We are now in familiar territory! Since the deuteron is known to have only one bound state, we drop the subscript n , and write the solution as:

$$u(r) = u_{<}(r)\theta(R_N - r) + u_{>}(r)\theta(r - R_N)$$

$$u_{<}(r) = A \sin kr$$

$$u_{>}(r) = Ce^{-Kr}$$

$$K = -k \cot kR_N$$

The last condition, that also quantizes the energy, relating it to V_0 , is obtained by matching the logarithmic derivative at $r = R_N$, namely,

$$\left. \frac{u_{<}(r)'}{u_{<}(r)} \right|_{r=R_N} = \left. \frac{u_{>}(r)'}{u_{>}(r)} \right|_{r=R_N}$$

11.1: ...The deuteron...

After some work, it can be shown that the full solution is:

$$u(r) = \sqrt{\frac{2}{R_N} \frac{\beta}{1 + \beta}} [\theta(R_N - r) \sin kr + \theta(r - R_N) \sin \alpha \exp(\beta - Kr)]$$

where $\alpha = kR_N$, and $\beta = KR_N$.

The clever use of the $K = -k \cot kR_N$ is essential to collapse it to this nice, compact form.

To use Mathematica effectively, it is best to use the following dimensionless form:

$$\tilde{u}(r) = \sqrt{\frac{2\beta}{1 + \beta}} [\theta(1 - x) \sin \alpha x + \theta(x - 1) \sin \alpha \exp(\beta(1 - x))]$$

and all expectation values can be expressed as follows: $\langle r^n \rangle = R_N^n \langle x^n \rangle$.

11.1: ...The deuteron...

For example, it can be shown that:

$$\langle r^2 \rangle = R_N^2 \langle x^2 \rangle = R_N^2 \left(\frac{\beta}{1 + \beta} \right) \left[\frac{1}{3} + \frac{1}{\beta} + \frac{1}{\beta^2} \left(1 - \frac{\beta^2}{2\alpha^2} \right) \frac{1}{2\beta} \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) \right]$$

which obtains the two physical limits:

- $\lim_{\beta \rightarrow \infty} \langle r^2 \rangle \longrightarrow \frac{R_N^2}{3} ,$

which is the expected value for the ground state of an infinitely deep well, and

- $\lim_{\beta \rightarrow 0} \langle r^2 \rangle \longrightarrow +\infty ,$

since, in the limit, the quantum state becomes unbound.

11.1: ...The deuteron...

The spin and parity of the deuteron

$$I^\pi(D) = 1^+.$$

$$\vec{I} = \vec{s}_n + \vec{s}_p + \vec{l}.$$

$\therefore I = 1 \implies$, there are only 4 possible ways to make a D:

S	l	I	π	?
$\uparrow\uparrow$	0	1	+1	yes
$\uparrow\uparrow$	1	1	-1	no
$\downarrow\uparrow$	1	1	-1	no
$\uparrow\uparrow$	2	1	+1	yes

We can rule out the $l = 1$ states due to the parity relation, $\pi = (-1)^l$. Without further knowledge, we can not rule out the $l = 2$ state.

This calls for an experiment! Two ways of doing this are...

11.1: ...The deuteron...

Measure the magnetic dipole of the deuteron

If $l = 0$, only S can contribute to the magnetic moment

Following the discussion in Chapter 10, the, if the deuteron were in a pure $L = 0$ s state, its magnetic moment would be given by:

$$\mu_D \stackrel{?}{=} \frac{\mu_N}{2}(g_{sn} + g_{sp}) . \quad (11.4)$$

$$g_{sn} = -3.82608545(46) \ \& \ g_{sp} = 5.585694713(90) \implies (g_{sn} + g_{sp}) = 1.75960926(10)$$

$$\text{Experiment} \implies 1.714876(2), \Delta = 0.044733(2).$$

Conclusion ... the deuteron can not be a pure s state!

$$\mu_D \neq \frac{\mu_N}{2}(g_{sn} + g_{sp}) .$$

11.1: ...The deuteron...

Determining the s/d mixture of the deuteron, from its μ

Since $V_{nn}(\vec{x}) \neq V_{nn}(|\vec{x}|)$, a unique $\langle \vec{l} \rangle$ is not guaranteed!

Assume the D is made up of a mixture of s and d states $\implies$

$$\psi_D = a_s \psi_s + a_d \psi_d \quad \text{where } |a_s|^2 + |a_d|^2 = 1. \quad (11.5)$$

Taking expectation values:

$$\mu_D = |a_s|^2 \mu_s + |a_d|^2 \mu_d, \quad (11.6)$$

Where μ_s is the s -state contribution to μ_D , and μ_d is the d -state contribution (yet to be determined) to μ_D .

We already know that

$$\mu/\mu_N = (g_{sn} + g_{sp})/2 = 0.87980463(5) \text{ from (11.4)/2.}$$

Before we tackle μ_d , let us isolate its contribution to μ_D .

11.1: ...The deuteron...

We know the measured value of μ_d to be 0.857438(1) in units of μ_N . Let us call this value μ_D^M .

From (11.4), we can write

$$\mu_D^M = (1 - p_d)\mu_s + p_d\mu_d ,$$

or, solving for p_d , the fractional probability of finding the D in the d state (which is really the point of all of this!):

$$p_d = \frac{\mu_s - \mu_D^M}{\mu_s - \mu_d} ,$$

that isolates p_d , what we want to know, from μ_d , which is the only thing we do not know yet.

So ... let's get to work!

11.1: ...The deuteron...

Let's consider the general expression for extraction of the magnetic moment from a wave-function with a total spin of $\vec{I}$. It takes the form,

$$\mu = \frac{1}{I + 1} \langle I, M_I = I | \vec{\mu} \cdot \vec{I} | I, M_I = I \rangle$$

In other words, the magnetic is the projection of $\vec{\mu}$ on $\vec{I}$, in the case where $\vec{I}$ has its maximum projection along the positive $\hat{z}$ axis.

That was just definition.

From now on, we shall use the shorthand:

$$\mu = \frac{1}{I + 1} \langle \vec{\mu} \cdot \vec{I} \rangle$$

Putting in the gyromagnetic factors for both orbital and spin, as well as both nucleons,

$$\mu = \frac{1}{I + 1} \left(\frac{1}{2} (g_{lp} + g_{ln}) \langle \vec{L} \cdot \vec{I} \rangle + \frac{1}{2} (g_{sp} + g_{sn}) \langle \vec{S} \cdot \vec{I} \rangle \right)$$

The $(1/2)$ on the orbital terms comes from the equal partitioning of the $\vec{L}$ between the n and p , and μ is in units of μ_N .

11.1: ...The deuteron...

Now, we substitute for the orbital gyromagnetic factors $g_{lp} = 1$ (point-like charged proton) and $g_{ln} = 0$ (point-like neutral neutron), and simplify to get:

$$\mu = \frac{\langle \vec{L} \cdot \vec{I} \rangle + (g_{sp} + g_{sn}) \langle \vec{S} \cdot \vec{I} \rangle}{2(I + 1)}$$

Using the vector identities

$$\vec{I} - \vec{L} = \vec{S} \implies \langle |\vec{I} - \vec{L}|^2 \rangle = \langle |\vec{S}|^2 \rangle \implies \langle \vec{L} \cdot \vec{I} \rangle = [I(I + 1) + L(L + 1) - S(S + 1)]/2$$

$$\vec{I} - \vec{S} = \vec{L} \implies \langle |\vec{I} - \vec{S}|^2 \rangle = \langle |\vec{L}|^2 \rangle \implies \langle \vec{S} \cdot \vec{I} \rangle = [I(I + 1) - L(L + 1) + S(S + 1)]/2$$

and substituting above, gives the general expression:

$$\mu = \frac{I(I + 1) + L(L + 1) - S(S + 1) + (g_{sp} + g_{sn})[I(I + 1) - L(L + 1) + S(S + 1)]}{4(I + 1)}$$

Substituting ($I = 1, L = 0, S = 1$) obtains μ_s from before.

Substituting ($I = 1, L = 2, S = 1$) obtains:

$$\mu_d = \frac{3}{4} - \frac{1}{2}\mu_s \implies p_d = \frac{2}{3} \left(1 - \frac{\mu_D - \frac{1}{2}}{\mu_s - \frac{1}{2}} \right) \implies p_d = 0.0392599(17); p_s = 0.9607401(17);$$

11.1: ...The deuteron...

We conclude that the deuteron is $\approx 96\%$ s and 4% d .

This is a clear example of the non-conservation of orbital angular momentum when non-central forces and spins are involved.

11.1: ...The deuteron...

The quadrupole moment of the deuteron

Another experiment that isolates the d component of the deuteron wave function, is the measurement of the quadrupole moment.

An s state is spherical, and hence its quadrupole moment (Q) vanishes.

The Q of the deuteron is measured to be $Q = 0.002860(15)$ b.

A “b” is a “barn”. A barn is defined as 10^{-28} m², about the cross sectional area of a typical heavy nucleus. This unit of measure is a favorite among nuclear physicists.

11.1: ...The deuteron...

Using (11.5) in the definition of the quadrupole moment given by (??), that is,

$$Q = \int d\vec{x} \psi_N^*(\vec{x})(3z^2 - r^2)\psi_N(\vec{x}) ,$$

results in

$$Q = \frac{\sqrt{2}}{10} a_s^* a_d \langle R_s | r^2 | R_d \rangle - \frac{1}{20} |a_d|^2 \langle R_d | r^2 | R_d \rangle , \quad (11.7)$$

where R_s and R_d are the radial components of the deuteron's s and d wavefunctions.

For consistency, a_s appears in (11.7) as its complex conjugate. However, the a_s and a_d constants may be chosen to be real for the application, and thus we can replace $a_s^* = a_s$ and $|a_d|^2 = a_d^2$.

The deuteron's radial wavefunctions are unknown and unmeasured. However, reasonable theoretical approximations to these can be formulated¹, and yield results consistent with the 96%/4% s/d mixture concluded from the deuteron's magnetic moment measurements.

¹Extra credit to any student who comes up with a decent approximation to the deuteron's wavefunctions, and calculates consistent results!

Chapter 11: The Force Between Nucleons...

Things to think about ...

Chapter 11.0: Introduction ...

1. How are the Quantum Mechanics in Nuclear (312) and Atomic (311) Physics the same?
2. How are they different?
3. What is a “derivative” force?
4. What makes up nucleons?
5. Sketch the Strong central part of the force between the nucleons in a D and recall its functional form.
6. How does it bind? That is, how can it be attractive, and repulsive at the same time?
7. What is a fermion, boson, hadron, baryon, lepton, muon, meson, quark, gluon?
8. What do $\vec{s}$, $\vec{l}$, $\vec{j}$, $\vec{S}$, $\vec{L}$, $\vec{I}$, I^π mean? How do you calculate/determine/measure them? How do you add them?
9. What are the central and non-central components of the nuclear force? What are the implications of the non-central parts?

Chapter 11: ...The Force Between Nucleons

Chapter 11.1: The Deuteron (D) ...

1. What is it? How is it made up?
2. What about a dineutron, and the diproton? Do they exist? Why or why not?
3. What is “Fermi pressure”, *a.k.a.* degeneracy or Pauli Exclusion pressure?
4. What is the binding energy of the deuteron? How would you determine it from nuclear data tables?
5. Why does it only have one bound state (the ground state)?
6. What is the I^π , L , S of the D?
7. What is the magnetic moment of the D? What things make it up?
8. How is the magnetic moment used to determine the s/d orbital spin mixture of the D?
9. Show that $\langle \vec{L} \cdot \vec{I} \rangle = [I(I + 1) + L(L + 1) - S(S + 1)]/2$ and $\langle \vec{S} \cdot \vec{I} \rangle = [I(I + 1) - L(L + 1) + S(S + 1)]/2$
10. Evaluate the above for the D.

Chapter 11: ...The Force Between Nucleons

Chapter 11.1: The Deuteron (D) ...

11. What is the quadrupole moment of the D? How does this show conclusively, that the D has an s/d orbital angular momentum admixture.
12. In the nuclear context, what is a barn? an outhouse? a shed?
13. Estimate the volume of a ^{208}Pb nucleus in the unit “barn-yards”.