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Correspondence principle and scattering from potential steps

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The correspondence principle provided a powerful constraint in the development of quantum theory. For example, in Ref. 1 we read, "In the limit where $\hbar \rightarrow 0$, the laws of quantum mechanics must reduce to those of classical mechanics. The correspondence principle, which played such an important role in the establishment of the theory, had as its exact goal the fulfilment of this fundamental requirement." Students are often puzzled by an apparent violation of this principle in scattering from a potential step. Consider a plane wave, corresponding to a particle of mass m and energy E , incident from the left on the potential step of height V_o ($< E$) depicted in Fig. 1. The standard treatment² shows that the probability of reflection is equal to

$$\frac{|k_1 - k_2|^2}{|k_1 + k_2|^2}, \quad (1)$$

where

$$k_1^2 = 2mE/\hbar^2,$$

$$k_2^2 = 2m(E - V_o)/\hbar^2.$$

If we put $(V_o/E) = \alpha$, we may write (1) as

$$\frac{|1 - (1 - \alpha)^{1/2}|^2}{|1 + (1 - \alpha)^{1/2}|^2}. \quad (2)$$

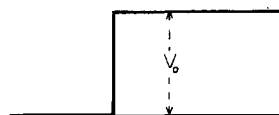
Therefore, if $\hbar \rightarrow 0$, or, more generally, if the de Broglie wavelength $\lambda = [2\pi\hbar/(2mE)^{1/2}]$ tends to zero with α held constant, then the probability of reflection stays constant and certainly does not tend to the classical value of zero.

This disagreement arises from the fact that the discontinuous step potential is never realized in nature; it is an approximation to a continuous potential, and the approximation is good so long as the distance over which the potential varies from zero to V_o is small compared with λ . As $\hbar \rightarrow 0$, λ also tends to zero, so that the approximation must ultimately become invalid in this limit. It is made clear elsewhere in Ref. 1 (see Ref. 3) that in order that the classical approximation be justified, the potential must vary sufficiently slowly. To be more precise, the potential must not vary appreciably over a distance comparable with λ . As $\hbar$ (and λ) $\rightarrow 0$ this condition will clearly always be met for a continuous potential.

These points can be illustrated by considering different limits of scattering from the following potential, illustrated in Fig. 2:

$$V(x) = \begin{cases} 0 & x < 0 \\ V_o x/a & 0 \leq x \leq a \\ V_o & x > a. \end{cases}$$

Fig. 1. Potential step.



In the regions $x < 0$, $x > a$ the wave function is given by ϕ_1 , ϕ_2 , respectively, where

$$\phi_1 = \exp(ik_1x) + R \exp(-ik_1x),$$

$$\phi_2 = T \exp(ik_2x).$$

The probability of reflection is $|R|^2$.

In the region $0 \leq x \leq a$, the Schrödinger equation is

$$-(\hbar^2/2m)(d^2\phi/dx^2) + (V_o x/a)\phi = E\phi.$$

As shown in Ref. 4 we may write the solution of this equation as

$$w(y) = Ff(y) + Gg(y),$$

where

$$w(y) = \phi[x(y)]$$

and

$$y = (2ma^2/V_o^2\hbar^2)^{1/3} [(V_o x/a) - E].$$

The functions f and g are solutions of Airy's equation⁵ given by Eqs. (17) and (18) of Ref. 4.

When $x = 0$ and $x = a$ we have, respectively,

$$y = -(2ma^2/V_o^2\hbar^2)^{1/3} E = y_o \text{ (say)}, \quad (3)$$

$$y = -(2ma^2/V_o^2\hbar^2)^{1/3} (E - V_o) = y_a \text{ (say)}. \quad (4)$$

We impose continuity of ϕ and $d\phi/dx$ at $x = 0$ and $x = a$ and from the resulting four equations we eliminate F , G , and T , obtaining the following expression for the probability of reflection:

$$|R|^2 = \frac{|(p - q) + i(r + s)|^2}{|(p + q) + i(r - s)|^2}, \quad (5)$$

where

$$p = k_2[f(y_o)g(y_a) - f(y_a)g(y_o)] (a\hbar^2/2mV_o)^{1/3}, \quad (6)$$

$$q = (1/k_1) [f'(y_o)g'(y_a) - f'(y_a)g'(y_o)] \times (2mV_o/a\hbar^2)^{1/3}, \quad (7)$$

$$r = [f(y_o)g'(y_a) - f'(y_a)g(y_o)], \quad (8)$$

$$s = (k_2/k_1) [f'(y_o)g(y_a) - f(y_a)g'(y_o)]. \quad (9)$$

We now consider two different limits of R , in each case keeping α constant.

(A) We consider the limit in which the ratio $\rho = (2\pi a/\lambda)$ tends to zero, i.e., the distance over which the potential changes from zero to V_o becomes negligible compared with a de Broglie wavelength.

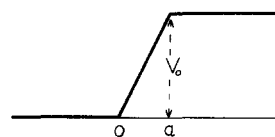


Fig. 2. Potential $V(x)$.

From (3) and (4) we see that

$$y_o = -(\rho/\alpha)^{2/3}, \quad y_a = -(1 - \alpha)(\rho/\alpha)^{2/3}. \quad (10)$$

The behavior of $f(y)$ and $g(y)$ and their derivatives as $y \rightarrow 0$ is given by^{4,5}

$$f(y) \rightarrow 1, \quad g(y) \sim y, \quad f'(y) \sim y^2/2, \quad g'(y) \rightarrow 1.$$

Hence Eqs. (6)–(9) give the following behavior as $\rho \rightarrow 0$:

$$\begin{aligned} p &\sim (1 - \alpha)^{1/2} \rho, \\ q &\sim (2 - \alpha)\rho/2, \\ r &\sim 1, \\ s &\sim -(1 - \alpha)^{1/2}. \end{aligned}$$

Substituting these in (5) and taking the limit $\rho \rightarrow 0$, we see that we reproduce the result (2), i.e., the approximation involved in the step potential is valid in this limit.

(B) Second we consider the classical limit in which ρ tends to infinity. In this case the de Broglie wavelength becomes much less than the distance over which the potential changes.

Equation (10) shows that we need to know the asymptotic behavior of f and g and their derivatives for large negative values of their argument. This is given by⁵

$$\begin{aligned} f(-y) &\sim (1/c)y^{-1/4} \cos[(2/3)y^{3/2} - (\pi/12)], \\ f'(-y) &\sim (1/c)y^{1/4} \sin[(2/3)y^{3/2} - (\pi/12)], \end{aligned}$$

$$\begin{aligned} g(-y) &\sim -(4/3)^{1/2} c y^{1/4} \sin[(2/3)y^{3/2} + (\pi/12)], \\ g'(-y) &\sim (4/3)^{1/2} c y^{1/4} \cos[(2/3)y^{3/2} + (\pi/12)], \end{aligned}$$

where

$$c = 3^{-1/6} \pi^{1/2} / \Gamma(2/3).$$

From Eqs. (6)–(10) we deduce that as $\rho \rightarrow \infty$,

$$\begin{aligned} p &\sim q \sim (1 - \alpha)^{1/4} \sin \theta, \\ r &\sim (-s) \sim (1 - \alpha)^{1/4} \cos \theta, \end{aligned}$$

where

$$\theta = 2\rho [1 - (1 - \alpha)^{3/2}] / (3\alpha).$$

Substituting these values in (5) we see that in this case the limit agrees with the classical result: there is no reflection.

¹A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1964), Vol. 1, p. 214.

²Reference 1, p. 80.

³Reference 1, p. 220.

⁴D. Branson, *Am. J. Phys.* **47**, 1000 (1979).

⁵M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Sec. 10.4.

Directional alignment properties of the bifocal lens

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The imaging properties of the bifocal lens permits its use in optical alignment applications. These include supplementation of, or substitution for, a pointing telescope in astronomical, navigation, and surveying instruments. We consider here the case where the two lens components, of focal length f_1 and f_2 , are equal throughout and where both lens components simultaneously participate in the imaging process. Under these conditions, two images of the object are formed (Fig. 1). The direction of the object is determined by orientation of the lens system, which is adjusted so that the separate images superimpose.

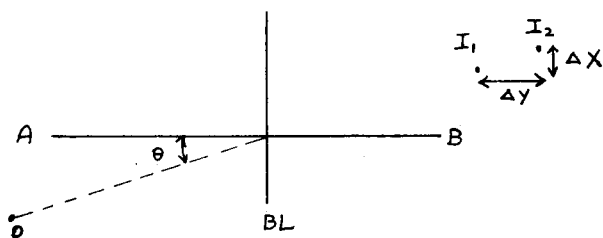


Fig. 1. Diagrammatic representation of the alignment procedure described in the text. O is a specific point object. BL is the bifocal lens system, where the two lens components have the same optic axis AB. I_1 and I_2 are the corresponding images of the point object. θ is the off-axis angle of the point object. ΔX is the relative transverse displacement of the two images. ΔY is the relative longitudinal displacement of the two images.

In a complete development of this procedure for the two lens components of different focal length which are located at the same plane with coincident optic axes, it can be shown that the condition for the superposition of the two images of a specific point on an object is that this point must be on the mutual optic axis of the two lens components. The condition that ΔX in Fig. 1 vanish is that the off-axis angle θ be zero. The sensitivity of the alignment procedure increases with the difference in focal length of the two lens components, and is maximum when the object is near the focal point of one of the lens components.

In this complete analysis, the relative longitudinal displacement, ΔY (Fig. 1), must be considered. The blurring effects of this condition on the image superposition procedure can be minimized by an appropriate overlap of the depth of focus for the two images produced by the two lens components. This requirement can be met with a suitable choice of focal lengths and system aperture. An important part of the overall optical system is the human eye. Hence the difference in the position of the focused images within the eye is crucial in this development. Preliminary calculations of system requirements, based on diffraction limited boundary conditions, yield results that are both encouraging and ambiguous. The degree of tolerance by the dynamically focusing eye to out-of-focus images in the comparison and superposition of those images is not easily taken into account in these calculations. However, the experimental