

(23)

Moment of inertia of the ellipsoidal nucleus

moment of inertia $I = \int d\vec{r} \rho(\vec{r}) r^2 \sin^2 \theta$

$$\rho(\vec{r}) = \text{constant} = M/V ; 0 \leq R \leq R_{\text{av}} (1 + \beta \gamma_{20})$$

$$V = \int d\vec{r} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta \int_0^{R_{\text{av}}(1+\beta\gamma_{20})} r^2 dr$$

$$= 2\pi \int_0^{\pi} \sin \theta d\theta \frac{R_{\text{av}}^3}{3} (1 + \beta \gamma_{20})^3$$

$$= \frac{2\pi}{3} R_{\text{av}}^3 \int_0^{\pi} \sin \theta d\theta \left(1 + \beta \left(\frac{5}{16\pi} \right)^{1/2} (3\cos^2 \theta - 1) \right)^3$$

$$= \frac{2\pi}{3} R_{\text{av}}^3 \int_{-1}^1 d\mu \left(1 + \beta \left(\frac{5}{16\pi} \right)^{1/2} (3\mu^2 - 1) \right)^3$$

$$= \frac{4\pi}{3} R_{\text{av}}^3 \int_0^1 d\mu \left\{ 1 + \beta \left(\frac{5}{16\pi} \right)^{1/2} (3\mu^2 - 1) \right\}^3$$

$$= \left(1 + \frac{3}{4\pi} \beta^2 + \frac{\sqrt{5}}{20\pi^{3/2}} \beta^3 \right) \frac{4\pi}{3} R_{\text{av}}^3$$

$$J = \int d\vec{r} g(\vec{r}) r^2 \sin^2 \theta$$

$$= \frac{M}{V} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \sin^2 \theta \int_0^{R_{av}(1+\beta Y_{20}(\theta))} r^2 dr r^2$$

$$= \frac{M}{V} 2\pi \int_{-1}^1 d\mu (1-\mu^2) \frac{R_{av}^5}{5} (1+\beta Y_{20})^5$$

$$= \frac{M}{V} \frac{2\pi}{5} R_{av}^5 \int_{-1}^1 d\mu (1-\mu^2) \left(1 + \left(\frac{5}{16\pi}\right)^{1/2} \beta (3\mu^2 - 1)\right)^5$$

$$= \frac{M}{V} \frac{4\pi}{5} R_{av}^5 \int_0^1 d\mu (1-\mu^2) \left(1 + \left(\frac{5}{16\pi}\right)^{1/2} \beta (3\mu^2 - 1)\right)^5$$

$$J = \frac{2}{5} M R_{av}^2 \left\{ 1 - \frac{1}{2} \sqrt{\frac{5}{11}} \beta + \frac{71}{288\pi} \beta^2 + O(\beta^3) \dots \right\}$$

$$= \frac{2}{5} M R_{av}^2 \left\{ 1 - 0.63\beta + 0.91\beta^2 \dots \right\}$$

<1%

$$= \frac{2}{5} M R_{av}^2 \left\{ 1 - 0.18 + 0.07 \dots <1\% \right\}$$

$$J = \frac{M 4\pi R_{av}^5 \int_0^1 d\mu (1-\mu^2) \left(1 + \left(\frac{5}{16\pi}\right)^{1/2} \beta (3\mu^2 - 1)\right)^5}{\frac{4\pi R_{av}^3 \int_0^1 d\mu (1 + \left(\frac{5}{16\pi}\right)^{1/2} \beta (3\mu^2 - 1))^3}$$

Rotational Bands

$$E = \frac{1}{2I} I(I+1) \hbar^2$$

$$E(0^+) = 0 \quad \text{ground state}$$

$$E(2^+) = 3\hbar^2/I$$

$$E(4^+) = 10\hbar^2/I$$

$$E(6^+) = 42\hbar^2/I$$

$$E(8^+) = 72\hbar^2/I$$

Using above I , relative spacing is fine
but absolute is too low by $2-3 \times$
 $\Rightarrow I$ is too high

Using a fluid model

$$I_{\text{fluid}} = \frac{2}{5} M R^2$$

} no idea how
this calculation
is done

ΔE too high by about $2-3$

(shell of deformation circling)
a static core

$$\text{So } I_{\text{fluid}} < I_N < I_{\text{rigid}}$$

Can model this as a surface that drags
part of the inner core with it

Ch. 6 Radioactive Decay

6.1 The Radioactive Decay Law

Problem: Given N_0 particles at $t=0$, each with an independent probability of decaying per unit time λ , how many particles exist at time t ?

Assume N is large, so that we can use calculus.

$$\frac{dN(t)}{dt} = -\lambda N \quad (1)$$

$\uparrow$ change in N of particle $\uparrow$ losing particles $\uparrow$ per unit time

depends proportionally on N of particles

$$\frac{dN}{N} = -\lambda dt \quad (2)$$

Integrate $\log N \Big|_{N_0}^N = -\lambda t$

$$\log N - \log N_0 = -\lambda t$$

$$\log N/N_0 = -\lambda t \quad (3)$$

$$\textcircled{3} \Rightarrow \frac{N}{N_0} = e^{-\lambda t}$$

$$\boxed{N = N_0 e^{-\lambda t}} \quad \textcircled{4}$$

$\textcircled{4} \Rightarrow$ The radioactive decay law.

Interpretations

$\frac{N}{N_0}$ is probability that a particle exists after time t .

Mean life?

τ = mean life of a particle

$$= \frac{\int_0^{\infty} dt \, p(t) \, t}{\int_0^{\infty} dt \, p(t)}$$

$$= \frac{\int_0^{\infty} dt \, t \, e^{-\lambda t}}{\int_0^{\infty} dt \, e^{-\lambda t}}$$

Set $x = \lambda t$

$$= \frac{\frac{1}{\lambda^2} \int_0^{\infty} dx \, x \, e^{-x}}{\frac{1}{\lambda} \int_0^{\infty} dx \, e^{-x}}$$

$$\tau = \frac{1}{\lambda} \quad (5)$$

τ is called "the mean life"

Half life

$$N = N_0 e^{-\lambda t}$$

After how much time with $N = N_0/2$,
to half its value

$$\frac{N_0}{2} = N_0 e^{-\lambda t_{1/2}}$$

$$\frac{1}{2} = e^{-\lambda t_{1/2}}$$

$$\log 2 = -\lambda t_{1/2}$$

$$t_{1/2} = \frac{1}{\lambda} \log 2 \quad (6)$$

$$t_{1/2} = \frac{0.693}{\lambda}$$

Note: don't confuse
 $t_{1/2}$ & τ
RTD formulas
give $t_{1/2}$

Activity

Generally, if we have a sample of radioactive material, we don't know what N_0 is. We can, however, count the number of decays between $t \pm t$ and $t + \Delta t$

$$\Delta N = N(t) - N(t + \Delta t) = N_0 e^{-\lambda t} - N_0 e^{-\lambda(t + \Delta t)}$$

↑
↑

what we measure
time of measurement

$$= N_0 e^{-\lambda t} (1 - e^{-\lambda \Delta t})$$

If $\lambda \Delta t \ll 1$, or $t \ll t_{1/2}$

$$\Delta N = N_0 e^{-\lambda t} (1 - [1 - \lambda \Delta t])$$

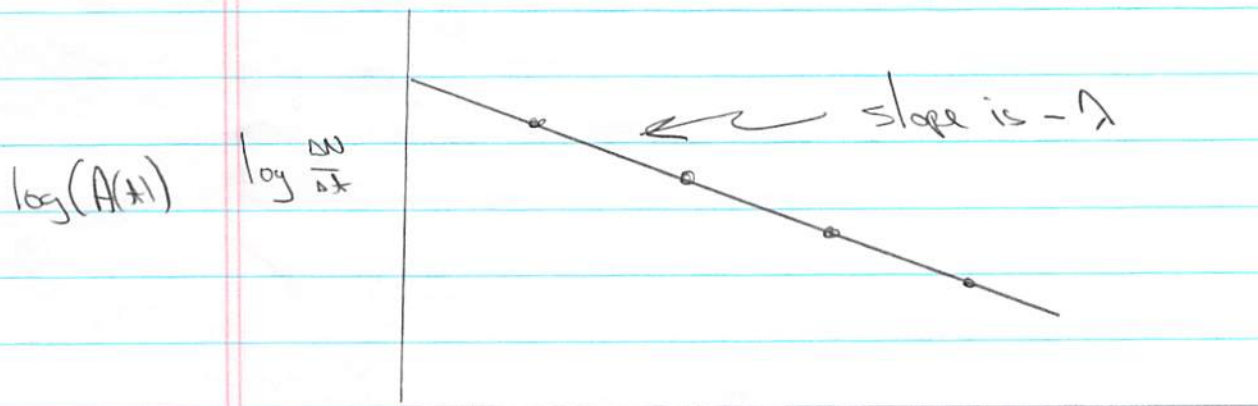
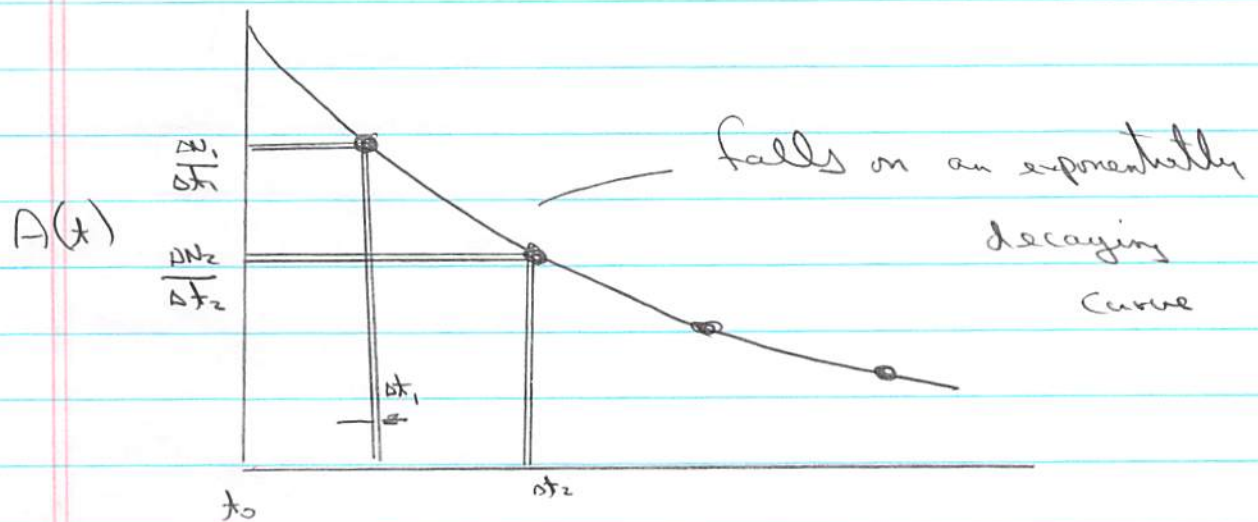
$$= N_0 e^{-\lambda t} \lambda \Delta t$$

$$\text{Or } \frac{\Delta N}{\Delta t} = \underbrace{N_0 \lambda e^{-\lambda t}}_{\text{This is called the "Activity" } A.}$$

It is a fn of t .

Note that $A(t) = A_0 e^{-\lambda t}$

A typical activity measurement



How can I measure λ if $t_{1/2}$ is very very long?

If λ is very small A is constant with time

Go back to (7)

$$\Delta N / \Delta t = N_0 e^{-\lambda t} \lambda \quad (7)$$

$$\frac{(\Delta N / \Delta t)}{N_0 e^{-\lambda t}} = \lambda$$

} measure independently
(chemical analysis)

Single parent, single stable daughter

N_1 parent



$$(8) \quad dN_1 = -\lambda N_1 dt$$

$$(9) \quad dN_2 = \lambda N_1 dt$$

Note (8) + (9) $\Rightarrow$

$dN_1 + dN_2 = 0$ i.e. total number of family members stay the same

$$(10) \quad N_1 = N_1(0) e^{-\lambda t} \quad \text{from before}$$

$$dN_2 = \lambda N_1 dt$$

$$\dot{N}_2 = \lambda N_1(0) e^{-\lambda t}$$

Integrate

$$N_2 = N_2(0) + \lambda N_1(0) \int_0^t e^{-\lambda t} dt$$

↑
could be
some
around
of the
start

$$= N_2(0) - N_1(0) e^{-\lambda t} \Big|_0^{\infty}$$

$$\boxed{N_2 = N_2(0) + N_1(0)(1 - e^{-\lambda t})} \quad (11)$$

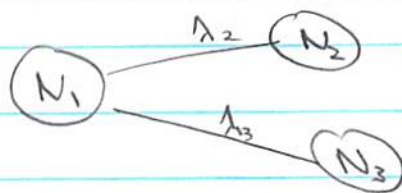
$$N_1 = N_1(0) e^{-\lambda t}$$

$$N_2 = N_2(0) + N_1(0)(1 - e^{-\lambda t})$$

$$N_1 + N_2 = N_1(0) + N_2(0) \quad \text{as expected}$$



Case: Single parent, two stable daughters



$$(12) \quad dN_1 = -\lambda_2 N_1 dt - \lambda_3 N_1 dt$$

$$(13) \quad dN_2 = \lambda_2 N_1 dt$$

$$(14) \quad dN_3 = \lambda_3 N_1 dt$$

$$\sum dN_i = 0$$

(stay within the family)

$$(13) \quad dN_1 = -(\lambda_2 + \lambda_3) N_1 dt$$

$$(14) \quad N_1 = N_1(0) e^{-(\lambda_2 + \lambda_3)t}$$

$$(13) \quad dN_2 = \lambda_2 N_1(0) e^{-(\lambda_2 + \lambda_3)t} dt$$

$$N_2 = N_2(0) + N_1(0) \lambda_2 \int_0^t e^{-(\lambda_2 + \lambda_3)t} dt$$

$$(15) \quad N_2 = N_2(0) + N_1(0) \frac{\lambda_2}{(\lambda_2 + \lambda_3)} (1 - e^{-(\lambda_2 + \lambda_3)t})$$

Similarly

$$(16) \quad N_3 = N_3(0) + N_1(0) \frac{\lambda_3}{\lambda_2 + \lambda_3} (1 - e^{-(\lambda_2 + \lambda_3)t})$$

$$(14) + (15) + (16) \Rightarrow$$

$$N_1 + N_2 + N_3 = N_1(0) + N_2(0) + N_3(0)$$

6.2 Quantum Theory of Radiative Decay

Fermi's Golden Rule #2

Transition Rate

$$\textcircled{17} \quad \lambda = \frac{2\pi}{\hbar} \left| \underbrace{\int d\mathbf{x} \psi_f^* V_p \psi_i}_{\text{transition matrix element}} \right|^2 \underbrace{g(\epsilon)}_{\text{density of final states}}$$

Case of decay (example, excited state $\rightarrow$ gs with γ)



$$E_f = E_i + E_R$$

$\nwarrow$ recoil energy

If the energy of the excited state is uncertain, we know from

$$\Delta E \Delta t \geq \hbar/2$$

So, N^* has a finite lifetime

In fact, there is an inverse relationship $\Delta E \uparrow \Delta t \downarrow$

The final nuclear state is

$$\psi_f(\vec{r}, t) = \psi_f(\vec{r}) e^{-iE_f t/\hbar} e^{-t/\tau} \quad \leftarrow \text{mean life}$$

$$\text{So, } |\psi_f(\vec{r}, t)|^2 = |\psi_f(\vec{r})|^2 e^{-t/\tau}$$

↗ finite lifetime

Since E_f is uncertain Δ is finite
The larger ΔE_f is, the larger Δ is,
that is, the transition is faster.

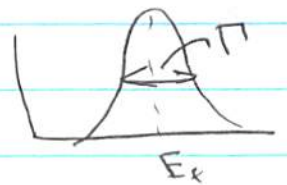
The probability dist'n implied is

$$(18) \quad p(E) dE = \frac{\Gamma}{2\pi} \left(\frac{dE}{(E-E_f)^2 + \left(\frac{\Gamma}{2}\right)^2} \right)$$

where $\Gamma = \frac{\hbar}{\tau}$

From this it can be shown that

$$\langle E \rangle = E_f$$



Q. If $10^{-12} \text{ s} \leq \tau_N \leq \infty$, what range of Γ do we expect

$$\Gamma = \frac{\hbar}{\tau}$$

$$\text{If } \tau > 10^{-12} \text{ s}$$

$$\Gamma < \frac{\hbar c}{c R_{\min}}$$

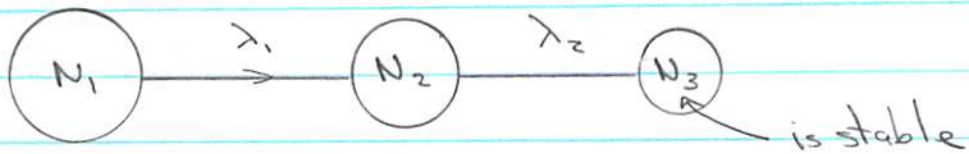
$$< \frac{197 \text{ MeV} \times 10^{-15} \text{ m}}{3 \times 10^8 \frac{\text{m}}{\text{s}} \times 10^{-12} \text{ s}}$$

$$\approx 66 \times 10^{-11} \text{ MeV}$$

$$\approx 10^{-3} \text{ eV}$$

Typical spacing 10^{-3} MeV and larger.
Hence no danger of overlap.

Decay chains (cont from Feb. 13th)



Rate equations

$$dN_1 = -\lambda_1 N_1 dt \quad (1)$$

$$dN_2 = \lambda_1 N_1 dt - \lambda_2 N_2 dt \quad (2)$$

$$dN_3 = \lambda_2 N_2 dt \quad (3)$$

$$+ \quad d(N_1 + N_2 + N_3) = 0$$

$$\Rightarrow N_1 + N_2 + N_3 = N_1(0) + N_2(0) + N_3(0)$$

Krone's discussion, special case $N_1(0) = N_0$

$$N_2(0) = N_3(0) = 0$$

Integrating directly

$$N_1 = N_0 e^{-\lambda_1 t} \quad (4)$$

From (2) $\dot{N}_2 = \lambda_1 N_1 - \lambda_2 N_2$

$$\dot{N}_2 + \lambda_2 N_2 = \lambda_1 N_0 e^{-\lambda_1 t} \quad (5)$$

$$(5) * e^{\lambda_2 t} \Rightarrow \frac{d(N_2 e^{\lambda_2 t})}{dt} = \lambda_1 N_0 e^{-\lambda_1 t + \lambda_2 t}$$

$$\int_0^t \frac{d(N_2 e^{\lambda_2 t})}{dt} dt = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} \Big|_0^t$$

$$N_2 e^{\lambda_2 t} = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} \left(e^{(\lambda_2 - \lambda_1)t} - 1 \right)$$

$$N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) \quad (6)$$

Similarly, from (3)

$$\dot{N}_3 = \frac{\lambda_2 \lambda_1 N_0}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right)$$

$$N_3 = \frac{\lambda_2 \lambda_1 N_0}{\lambda_2 - \lambda_1} \left(\frac{1}{\lambda_1} (1 - e^{-\lambda_1 t}) - \frac{1}{\lambda_2} (1 - e^{-\lambda_2 t}) \right)$$

$$N_3 = \frac{N_0}{\lambda_2 - \lambda_1} \left(\lambda_2 - \lambda_1 + \lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t} \right)$$

$$N_3 = N_0 \left(1 + \frac{\lambda_1 e^{-\lambda_2 t}}{\lambda_2 - \lambda_1} - \frac{\lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} \right) \quad (7)$$

$$(4) + (6) + (7) \Rightarrow$$

$$N_1 + N_2 + N_3 = N_0 \left(\overbrace{e^{-\lambda_1 t}}^{N_1} + \overbrace{\frac{\lambda_1 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} - \frac{\lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1}}^{N_2} + 1 + \cancel{\frac{\lambda_1 e^{-\lambda_2 t}}{\lambda_2 - \lambda_1}} - \frac{\lambda_2 e^{-\lambda_2 t}}{\lambda_2 - \lambda_1} \right)$$

$$= N_0 \left(\cancel{e^{-\lambda_1 t}} + \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \cancel{e^{-\lambda_1 t}} + 1 \right)$$

$$= N_0$$

$$N_1 + N_2 + N_3 = N_0 \quad \text{cross check works}$$

Investigate N_2 under some limiting cases

$$\text{From (6)} \quad N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \quad (6)$$

Case $\lambda_1 \ll \lambda_2$ N_2 's creation rate $\ll$ its decay rate

$\frac{1}{\lambda_1} t \ll \frac{1}{\lambda_2} \Rightarrow$ not much of N_1 has been depleted

$$N_2 \rightarrow \frac{\lambda_1 N_0}{\lambda_2} (1 - e^{-\lambda_2 t})$$

$$A_2 = \lambda_2 N_2 = A_1 (1 - e^{-\lambda_2 t})$$

for large t , but still $\lambda_1 t \ll 1$

$A_2 = A_1$, another example of secular equilibrium

Activity of 2 = Activity of 1

(11)

Case $\lambda_1 > \lambda_2$ (parent dies more quickly
than the daughter)From (6) for $t \gg \frac{1}{\lambda_1}$ parent has
vanished

$$N_2 \rightarrow \frac{\lambda_1 N_0 e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}$$

ratio of activities

From (6) & (4)

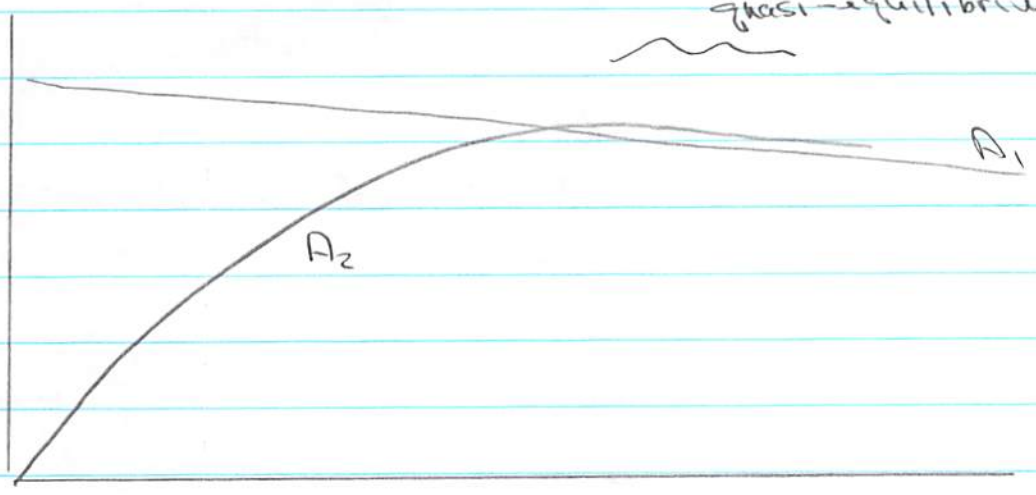
$$\begin{aligned} \frac{A_2}{A_1} &= \frac{\lambda_2 N_2}{\lambda_1 N_1} = \frac{\lambda_2 \lambda_1 N_0 (e^{-\lambda_1 t} - e^{-\lambda_2 t})}{\lambda_2 - \lambda_1} \\ &\quad \frac{\lambda_1 N_0 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} \\ &= \frac{\lambda_2}{\lambda_2 - \lambda_1} (1 - e^{-(\lambda_1 - \lambda_2)t}) \end{aligned}$$

$$t \gg \frac{1}{\lambda_1}, \frac{1}{\lambda_2}$$

$$\frac{A_2}{A_1} \rightarrow \frac{\lambda_2}{\lambda_2 - \lambda_1} \quad \text{a constant}$$

$\lambda_1 \ll \lambda_2$ much longer lived parent

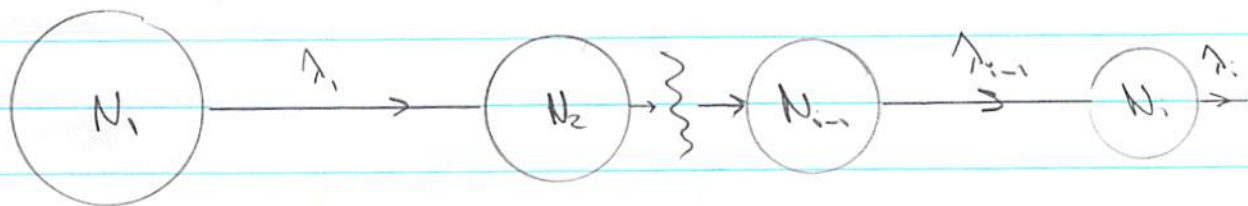
quasi-equilibrium



Series of decays

$$dN_1 = -\lambda_1 N_1 dt$$

$$dN_i = \lambda_{i-1} N_{i-1} dt - \lambda_i N_i dt$$



$$A_n = N_0 \sum_{i=1}^n c_i e^{-\lambda_i t} = N_0 (c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t} + \dots)$$

$$n \geq 2$$

$$c_m = \frac{\prod_{i=1}^m \lambda_i}{\prod_{i=1}^m (\lambda_i - \lambda_m)} = \frac{\lambda_1 \lambda_2 \dots \lambda_m}{(\lambda_1 - \lambda_m)(\lambda_2 - \lambda_m) \dots (\lambda_{m-1} - \lambda_m)}$$

The Bateman equations

illustration



$$A_2 = N_0 (c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t})$$

$$c_1 = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \quad ; \quad c_2 = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}$$

$$A_2 = \frac{\lambda_1 \lambda_2 N_0}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right)$$

$$= \lambda_2 N_2$$

↖ from (6)

Chapter 3 α -decay

The α -particle is the nucleus of the He-atom.

$2p + 2n$. First double magic nucleus.

It is very stable.

It's mass defect $\Delta = (m - A)c^2$ ^{in (u)}

is 0.002603

while its closest neighbors

${}^3\text{He}$	has	0.016029
${}^3\text{H}$		0.016049
${}^5\text{He}$		0.0121
${}^5\text{Li}$		0.0125

	${}^5\text{Li}$	
	0.0125	
${}^3\text{He}$	${}^4\text{He}$	${}^5\text{He}$
0.01603	0.002603	0.0121
	${}^3\text{H}$	
	0.01605	

α decay is
one means by
which a
proton rich heavy
nucleus can
get closer
to the line
of stability

3.2 Basic α -decay processes

e.g. ${}^{232}_{92}\text{U} \rightarrow {}^{228}_{90}\text{Th} + \alpha$
 $Z/A = 92/232 = 0.3966$



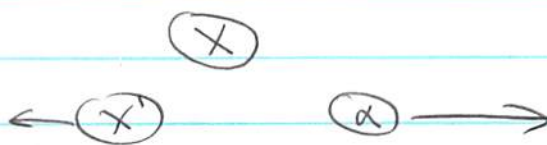
${}^{228}_{90}\text{Th}$ $Z/A = 90/228 = 0.3947$

energetics

$$m_x c^2 = m_{x'} c^2 + \overline{T}_{x'} + m_\alpha c^2 + \overline{T}_\alpha$$

$\uparrow$ recoil KE $\uparrow$ α 's KE

$Q = (m_x - m_{x'} - m_\alpha) c^2$ is the energy available to go into the KE of the resultant particles



$$p_{x'} = p_\alpha$$

$$Q = T_{x'} + T_\alpha$$

$$p_{x'} = p_\alpha \rightarrow 2m_{x'} T_{x'} = 2m_\alpha T_\alpha$$

 $\Rightarrow$


$$T_\alpha = \frac{Q}{(1 + m_\alpha/m_{x'})} \quad (1)$$

$$T_{x'} = \frac{m_\alpha}{m_{x'}} \frac{Q}{(1 + m_\alpha/m_{x'})} \quad (2)$$

Fully relativistic calculation

$$\overline{T}_\alpha = \frac{Q \left(1 + \frac{Q}{2m_\alpha c^2} \right)}{\left(1 + \frac{m_\alpha}{m_{x'}} + \frac{Q}{m_{x'} c^2} \right)} \quad (3)$$

$$\overline{T}_{x'} = \frac{\frac{m_\alpha}{m_{x'}} Q \left(1 + \frac{Q}{2m_\alpha c^2} \right)}{\left(1 + \frac{m_\alpha}{m_{x'}} + \frac{Q}{m_{x'} c^2} \right)} \quad (4)$$

(correction)
m_{x'} was posted

One can show, in the least favorable case, that the relativistic correction is about 2.5×10^{-4} (low A, getting much better as A increases.)

Back to ①, ②

$$\frac{m_\alpha}{m_{x'}} \approx \frac{4}{A}$$

$$\textcircled{5} \quad T_\alpha \approx \frac{Q}{1 + \frac{4}{A}} \approx Q \left(1 - \frac{4}{A}\right) \quad \text{KII (8.7)}$$

$$\textcircled{6} \quad T_{x'} \approx \frac{4}{A} Q$$

The recoil energy is not insignificant

For a typical α -emitter ($A=200$) $Q=5\text{MeV}$

$$T_{x'} = \frac{1}{50} 5000\text{keV} \approx 100\text{keV}$$

↗
This will break
crystal bonds