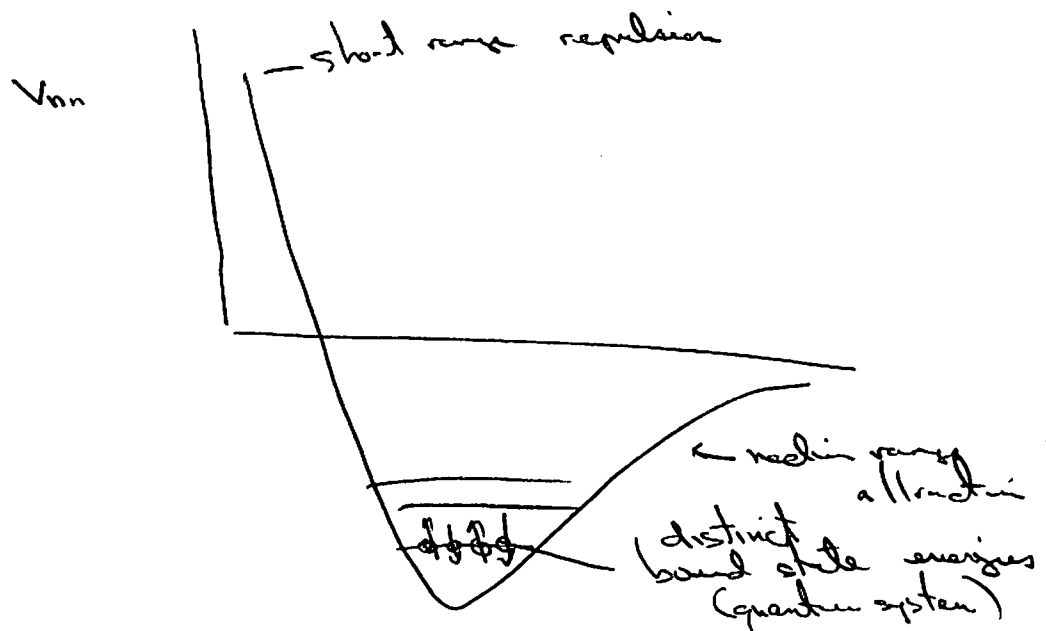


Ch 3 KII
Ch 10 online

The nucleus,
made up of protons & neutrons (collective name nucleons)
The force that binds nucleons is relatively short
ranged attractive (extends about a nuclear diameter)
The force that keeps the nucleons from collapsing in
on each other is short ranged ($<$ nuclear diameter)
repulsive



The n-n, n-p, p-p nuclear force is, more or
less, the same. p-p have an additional
repulsion (Coulombic)

Because of this nuclei are tightly bound, very dense
objects, with the nucleons in very close
proximity. (basically a constant potential)

nucleons are $spin\ 1/2$ particles (fermions)
Hence, the Pauli Exclusion Principle applies
(no two fermions can have the same q.n.'s)

Hence, as we build nuclei from constituent nucleons, nuclei with the same ~~sum~~ # of neutrons & protons are favored.

At some point, the Coulomb repulsion, which extends over the whole nucleus, destabilizes it. At some value of $A \gtrsim 200$, nuclei become unstable.

Moderate to large nuclei are also, neutron rich, offsetting Coulomb repulsion, by adding more neutrons, pushing the protons on average, further apart.

As a consequence of the contact nature of the potential, it is reasonable to talk about the "radius" of the nucleus.

3.1 The nuclear "radius"

Like the Δm , the radius is not a precisely defined quantity, and it depends on how one measures it. For example, we could measure the radius of charged protons by scattering electrons from it.

How does this work?

(3)

Consider an incident electron $\psi_i = e^{i\vec{k}_i \cdot \vec{x}}$
 and the scattered electron $\psi_f = e^{i\vec{k}_f \cdot \vec{x}}$
 scattering from a charge potential $V(\vec{x})$, made up
 of a contribution from all the protons in the
 nucleus.

The scattering amplitude is prop to the square of

$$F(\vec{k}_i, \vec{k}_f) = \langle e^{i\vec{k}_f \cdot \vec{x}} | V(\vec{x}) | e^{i\vec{k}_i \cdot \vec{x}} \rangle$$

Evaluation

complex conjugate

$$\begin{aligned} F(\vec{k}_i, \vec{k}_f) &\propto \int d\vec{x} e^{-i\vec{k}_f \cdot \vec{x}} V(\vec{x}) e^{i\vec{k}_i \cdot \vec{x}} \\ &\propto \int d\vec{x} V(\vec{x}) e^{i(\vec{k}_i - \vec{k}_f) \cdot \vec{x}} \end{aligned}$$

$\vec{k}_i - \vec{k}_f = \vec{q}$ called the
momentum
transfer

$$\text{Thus } F(\vec{k}_i, \vec{k}_f) = F(\vec{q}) \propto \int d\vec{x} V(\vec{x}) e^{i\vec{q} \cdot \vec{x}}$$

anyone recognise
this?

(This is a Fourier Transform)

(4)

Potential of a charge distribution

$$dV(\vec{x}) = \frac{-Ze^2 g_p(\vec{x}') d\vec{x}'}{4\pi\epsilon_0 |\vec{x} - \vec{x}'|}$$

↑
contribution
to V at
 $\vec{x}$

↑
from an element of charge at
 $\vec{x}'$

$g_p(\vec{x}')$ is the density of
protons, normalized with

$$1 = \int d\vec{x}' g_p(\vec{x}')$$

This procedure is based on the assumption that individual
proton charges can be considered to be smeared out.

In other words, the wavelength of the electron
is large enough so that each proton "washes" the nucleus many
times before the electron passes by.

$$\lambda = \frac{h}{p} = \frac{hc}{cp} \approx \frac{hc}{E_e} = \frac{197 \text{ MeV} \cdot \text{\AA}}{E_e} \gg R_N$$

assum
relativistic

possible with MeV
electrons

(5)

$$\text{So, } F(\vec{q}) \propto \int d\vec{x} e^{i\vec{q}\cdot\vec{x}} \int d\vec{x}' \left(\frac{-Ze^2}{4\pi\epsilon_0} \right) \frac{\rho_p(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

Normalize so that

$$F(\vec{q}) = 1 \quad \text{when } \vec{q} = 0$$

$\vec{q} = \vec{k}_i - \vec{k}_f$ no effect
 $\Rightarrow$ potential has no effect on
 electron wave function

$$F(\vec{q}) =$$

$$|\langle \psi_i | \psi_f \rangle|^2 = 1 \quad \text{by normalization}$$

It's a long calculation, but we will do it some point
 } (extra credit: Find normalization factor of above, so that $F(0) = 1$)

$$F(\vec{q}) = \int d\vec{x}' e^{i\vec{q}\cdot\vec{x}'} \rho_p(\vec{x}')$$

Fourier transform of the
 charge density.

So, electron scattering experiments
 yield the shape of the charge distribution
 through the Fourier transform

(6)

If the $f_p(\vec{r})$ is spherically symmetric

$$F(\vec{q}) = \int d\vec{r}' e^{i\vec{q} \cdot \vec{r}'} f_p(\vec{r}')$$

$$\text{Let } \vec{q} = q\hat{z} \quad \int_0^\infty r'^2 dr' \int_0^\pi \int_0^{2\pi} d\phi \sin\theta e^{iqr'\cos\theta} f_p(r')$$

$$= 2\pi \int_0^\infty r'^2 dr' f_p(r') \int_{-1}^1 e^{iqr'\mu} d\mu$$

$$= 2\pi \int_0^\infty r'^2 dr' f_p(r') \int_{-1}^1 e^{iqr'\mu} d\mu$$

$$= 2\pi \int_0^\infty r'^2 dr' f_p(r') \frac{1}{iqr'} \left(e^{iqr'\mu} \Big|_{-1}^1 \right)$$

$$= \frac{2\pi}{iq} \int_0^\infty r' dr' f_p(r') \underbrace{(e^{iqr'} - e^{-iqr'})}_{2i \sin qr'}$$

$$= \frac{4\pi}{q} \int_0^\infty r' dr' f_p(r') \sin qr'$$

Look at charts on p. 46/47

Show chart p 49

Conclusion

- 1) Central density roughly constant
- 2) "Skin depth" roughly constant as well.

$$R = R_0 A^{1/3}$$

typical characterization

$$R_0 \doteq 1.22 \text{ fm}$$

If the entire earth collapsed to its nuclear density,
what volume would it occupy?

(8)

Must turn in by Monday for extra credit

$$F(\vec{q}) = N \int d\vec{x} e^{i\vec{q}\cdot\vec{x}} \int d\vec{x}' \frac{f_0(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

A non-linear

collapses to

$$F(\vec{q}) = \int d\vec{x} e^{i\vec{q}\cdot\vec{x}} f_0(\vec{x})$$

the explicit value of
what is $N(\vec{q})$?

Turn in before lecture Monday Jan 12, for a c.

example

$$f_0(\vec{x}) = \frac{1}{4\pi R_0^2} \delta(r-R_0) \quad \left. \vphantom{\frac{1}{4\pi R_0^2}} \right\} \begin{array}{l} \text{shell} \\ \text{radius } R_0 \end{array}$$

$$\begin{aligned} & \int d\vec{x} \frac{1}{4\pi R_0^2} \delta(r-R_0) \\ &= \underbrace{\int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta}_{4\pi} \int_0^\infty r^2 dr \frac{1}{4\pi R_0^2} \delta(r-R_0) \end{aligned}$$

$$= \frac{1}{R_0^2} \int_0^\infty r^2 dr \delta(r-R_0) = \frac{1}{R_0^2} R_0^2 \cdot 2 \quad \checkmark$$

$$F(\vec{q}) = \int_{\vec{r}} e^{i\vec{q}\cdot\vec{r}} \frac{1}{4\pi R_0^2} \delta(r-R_0) d\vec{r}$$

$$= \frac{1}{4\pi R_0^2} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \int_0^\infty r dr e^{iqr \cos\theta} \delta(r-R_0)$$

Let $\vec{q} = q\hat{z}$ $\vec{r} = r\cos\theta\hat{z} + r\sin\theta\hat{x} + r\sin\theta\hat{y}$

$\vec{q}\cdot\vec{r} = qr\cos\theta$

$$= \frac{1}{2R_0^2} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta R_0^2 e^{iqR_0\cos\theta}$$

$$= \frac{1}{2} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta e^{iqR_0\cos\theta}$$

$$= \frac{1}{2iqR_0} \left[-e^{iqR_0\cos\theta} \right]_0^\pi$$

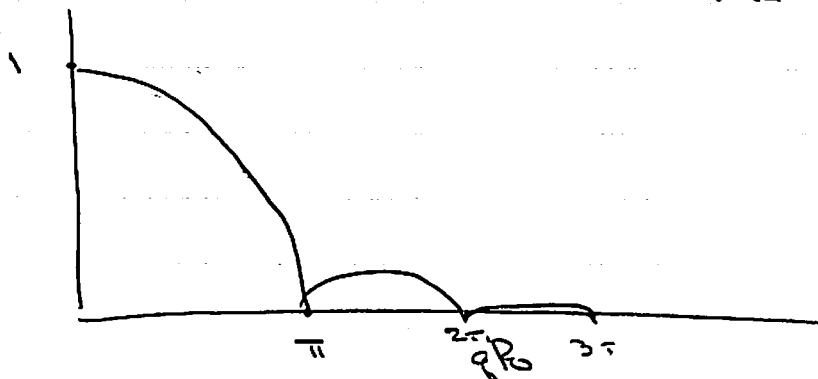
$$= \frac{1}{2iqR_0} 2i \sin qR_0$$

$$= \frac{1}{qR_0} \sin qR_0$$

$|F(q)|^2 = \frac{\sin^2 qR_0}{(qR_0)^2}$

note $F(\vec{q}) = F(q)$

$F(0) = 1$ since $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$



Spherically symmetric distribution

$$\rho(\vec{r}) = \rho(r)$$

$$F(\vec{q}) = \int d\vec{r} e^{i\vec{q}\cdot\vec{r}} \rho(\vec{r})$$

$$= \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \int_0^\infty r^2 dr e^{i\vec{q}\cdot\vec{r}} \rho(r)$$

Let $\vec{q} = q\hat{z}$ align along z-axis

$$\int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^\infty r^2 dr e^{iqr\cos\theta} \rho(r)$$

$$= \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^\infty r^2 dr e^{iqr\cos\theta} \rho(r)$$

$$= \int_0^\infty r^2 dr \rho(r) \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta e^{iqr\cos\theta}$$

$$= \int_0^\infty r^2 dr \rho(r) \left[\frac{1}{iqr} e^{iqr\cos\theta} \right]_{\theta=0}^{\theta=\pi}$$

$$= 2\pi \int_0^\infty r^2 dr \rho(r) \left[\frac{1}{iqr} 2\cos\theta \right]_{\theta=0}^{\theta=\pi}$$

$$F(q) = \frac{4\pi}{q} \int_0^\infty r^2 dr \rho(r) \sin qr$$

example

$$\rho(r) = \frac{1}{4\pi R_0^3} \rho(r-R_0)$$

$$F(q) = \frac{1}{q R_0^3} R_0 \sin q R_0$$

$$= \frac{\sin q R_0}{q R_0^2} \quad \left. \vphantom{\frac{\sin q R_0}{q R_0^2}} \right\} \text{same result as before}$$

Generally

$$\int_{\text{all } \mu} d\vec{x} f(\cos\theta, r) \rightarrow 2\pi \int_{-1}^1 d\mu \int_0^\infty r dr f(\mu, r)$$

you will use this, like for example

$$\int_0^\pi d\theta \sin\theta \rightarrow \int_{-1}^1 d\mu$$

$$\mu = \cos\theta \quad d\mu = -\sin\theta d\theta$$

estimate

How do we calculate the effect on the 1s state?

For a point nucleus

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

For a sphere of uniform charge density

$$V_0(r) = -\frac{Ze^2}{4\pi\epsilon_0 R_N} \left\{ \frac{3}{2} - \frac{1}{2} \left(\frac{r}{R_N} \right)^2 \right\} \quad r \leq R_N$$

↑
radius of the nucleus

$$V_0(r) = -\frac{Ze^2}{4\pi\epsilon_0 r} \quad r \geq R_N$$

note $V_0(r)$ is continuous at R_N (so is $V_0'(r)$)

So, if we treat the V_0 solution as approximately true we can estimate the shift in energy as

$$\Delta E = \langle \psi_{1s} | \underbrace{V_0(r) - V(r)}_{V_0 = V \text{ for } r \geq R_N} | \psi_{1s} \rangle$$

$$V_0 = V \text{ for } r \geq R_N$$

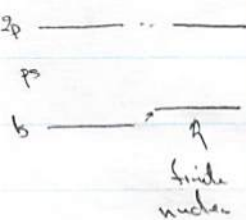
so, this integral only spans

$$\int_0^{R_N} dr$$

} further
justification
for say

$$\Delta E_{2p} \approx 0$$

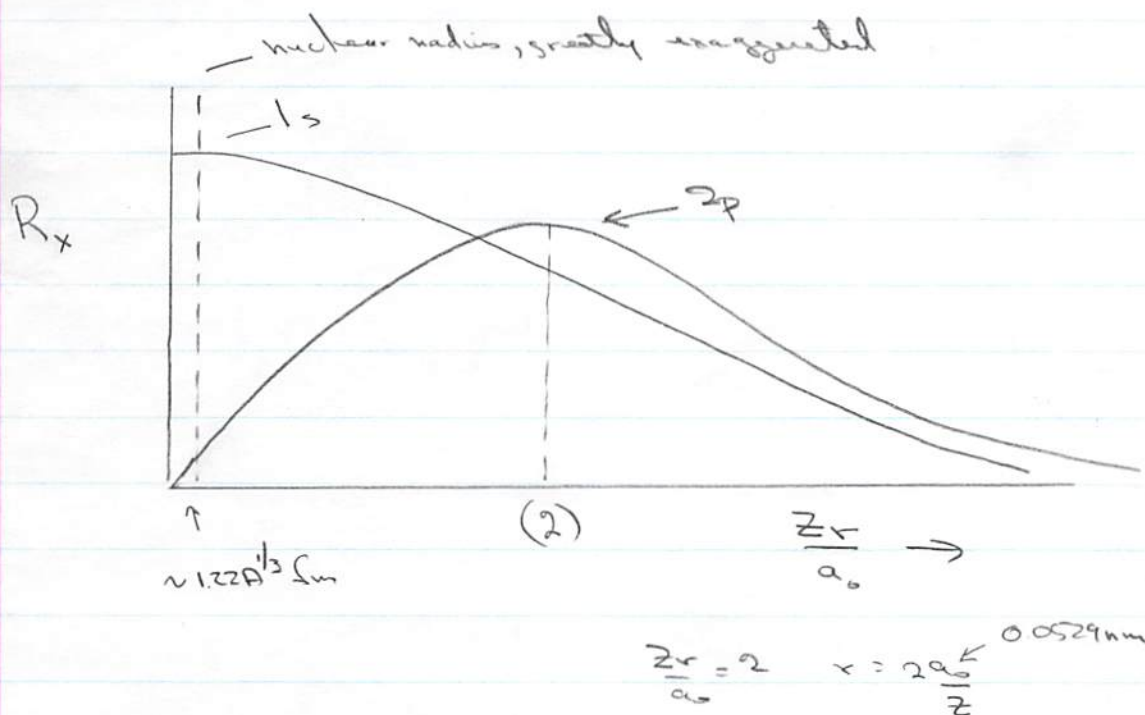
Determination of R_N from shifts in atomic energy level



Recall from 311, let's look at the $1s$ & $2p$
Radial wave function of a hydrogenic atom

$$\textcircled{1} \quad \begin{aligned} R_{1s}(r) &= 2 \left(\frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0} \\ R_{2p}(r) &= \frac{1}{\sqrt{3}} \left(\frac{Z}{2a_0} \right)^{3/2} \left(\frac{Zr}{a_0} \right) e^{-Zr/2a_0} \end{aligned} \quad \left. \begin{array}{l} \text{K2} \\ \text{p.33} \end{array} \right\}$$

What effect does the finite nuclear size



So, we expect that effect, if there is any, will affect the $1s$ the most and the $2p$ negligibly (to be investigated in A1)

$$\Delta E = \langle \Psi_{1s} | \underset{\substack{\uparrow \\ \text{point}}}{V_1(r)} - \underset{\substack{\uparrow \\ \text{finite} \\ \text{nucleus}}}{V_0(r)} | \Psi_{1s} \rangle \quad (2)$$

Since V_1, V_0 are spherically symmetric

$$\Delta E = \langle R_{10}(r) | V_1(r) - V_0(r) | R_{10} \rangle \quad (3)$$

point source $V_1(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$

finite sphere of uniform charge throughout the volume $(4) \quad V_0(r) = -\frac{Ze^2}{4\pi\epsilon_0 R_N} \left\{ \frac{3}{2} - \frac{1}{2} \left(\frac{r}{R_N} \right)^2 \right\} \quad r \leq R_N$
 $= V_1(r) \quad r \geq R_N$

Sketch of calculation

Inside:

— radial component of the $\vec{E}$ field vector

$$E_r = -\frac{Ze^2}{4\pi\epsilon_0} \frac{1}{r^2} \left(\frac{r}{R_N} \right)^3$$

$$= -\frac{Ze^2}{4\pi\epsilon_0} \frac{r}{R_N^3} \left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R_N^3} \right)$$

$$E_r = -\frac{dV_0}{dr}$$

Integrate E_r to get

$$V_0(r) = \frac{1}{2} \frac{Ze^2}{4\pi\epsilon_0} \frac{r^2}{R_0^3} + C$$

$$\text{Set } V_0(R_0) = V_0(R_0)$$

and set (4)

① (4) → (3)

$$\textcircled{5} \quad \Delta E = \frac{Ze^2}{4\pi\epsilon_0} \int_0^{R_0} r^2 dr \left[\underbrace{2\left(\frac{Z}{a_0}\right)^{3/2}}_{R_0^2(a_0)} - \frac{Zr}{a_0} \right]^2 \left\{ \underbrace{\frac{1}{r}}_{V_1} - \underbrace{\frac{1}{R_0} \left(\frac{3}{2} - \frac{1}{2} \left(\frac{r}{R_0} \right)^2 \right)}_{V_0} \right\}$$

$$\frac{ZR_0}{a_0} \lesssim 0.01 \quad (0.0112 \text{ for } {}^{203}\text{Pb})$$

$$\text{So } e^{-Zr/a_0} \rightarrow 1$$

$$\textcircled{6} \quad \Delta E = \frac{Ze^2}{4\pi\epsilon_0} \frac{4}{a_0^3} \int_0^{R_0} r^2 dr \left\{ \frac{1}{r} - \frac{3}{2R_0} + \frac{1}{2} \frac{r^2}{R_0^3} \right\}$$

$$\Delta E = \frac{2}{5} \frac{Z^4 e^2}{4\pi\epsilon_0} \frac{R_0^2}{a_0^3}$$

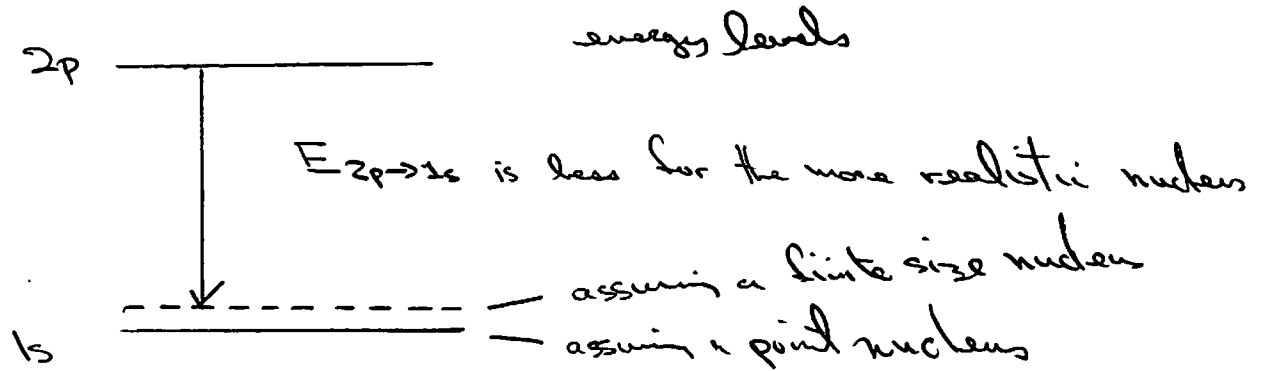
$$\Delta E / E \sim 10^{-4} \text{ or so}$$

This is a very small difference!

Since the hydrogenic wave fns are approximate
(relativistic effect, shielding by other e^-),
comparison of ΔE_{mass} to ΔE_{theory} is not possible.
Instead...

How do we measure the effect? directly

It has the following effect



But! We don't have a point nucleus to play with!

So, look at 2 different isotopes, same Z , different A (or N) How does this work?

Let $E_K(A)$ stand for the E of the $2p \rightarrow 1s$ transition for isotope A

$$\begin{aligned}
 E_K(A) - E_K(A') &= (E_{2p}(A) - E_{1s}(A)) \\
 &\quad - (E_{2p}(A') - E_{1s}(A')) \\
 &= [E_{1s}(A') - E_{1s}(A)] \\
 &\quad - \underbrace{[E_{2p}(A') - E_{2p}(A)]}_{\text{assuming 0}}
 \end{aligned}$$

$$E_K(A) - E_K(A') = E_{1s}(A') - E_{1s}(A)$$

Using a spherical ball of charge.

$$E_K(A) - E_K(A') = - \frac{2}{5} \frac{Z_0^2}{4\pi\epsilon_0} \frac{R_0^2}{Q_0^3} (A^{2/3} - A'^{2/3})$$

this is called the isotope shift

Graph on page 54

10% off Krane p. 51

difference due to
relativistic effects
and other effects

μ measurements $R_0 = 1.25$

MUONS

Back to Rutherford scattering

Recall ex. problem

$$F(\vec{q}) = N \int_{\text{all space}} d\vec{x} e^{i\vec{q} \cdot \vec{x}} \int_{\text{all space}} d\vec{x}' \frac{\rho_p(\vec{x}')}{|\vec{x} - \vec{x}'|} \quad (1)$$

What must N be so that

$$F(\vec{q}) = \int_{\text{all space}} d\vec{x} e^{i\vec{q} \cdot \vec{x}} \rho_p(\vec{x}) \quad (2)$$

Reorganize (1)

$$\begin{aligned} F(\vec{q}) &= N \int_{\text{all space}} d\vec{x}' \rho_p(\vec{x}') \int_{\text{all space}} d\vec{x} \frac{e^{i\vec{q} \cdot \vec{x}}}{|\vec{x} - \vec{x}'|} \\ &= N \int_{\text{all space}} d\vec{x}' e^{i\vec{q} \cdot \vec{x}'} \rho_p(\vec{x}') \int_{\text{all space}} d\vec{x} \frac{e^{i\vec{q} \cdot (\vec{x} - \vec{x}')}}{|\vec{x} - \vec{x}'|} \\ &= N \int_{\text{all space}} d\vec{x}' e^{i\vec{q} \cdot \vec{x}'} \rho_p(\vec{x}') \int_{\text{all space}} d\vec{x} \frac{e^{i\vec{q} \cdot \vec{x}}}{|\vec{x}|} \end{aligned}$$

$\vec{x} - \vec{x}' \rightarrow \vec{x}$

$$|\vec{x}| = r$$

$$\text{Let } \vec{q} = q \hat{z}$$

$$\begin{aligned}
 &= N \int d\vec{x}' e^{i\vec{q}\cdot\vec{x}'} f_p(\vec{x}') \int d\vec{x} \frac{e^{i\vec{q}\cdot\vec{x}}}{\dots} \\
 &= N \int d\vec{x}' e^{i\vec{q}\cdot\vec{x}'} f_p(\vec{x}') 2\pi \int_0^\infty r dr \int_0^{2\pi} d\phi e^{i\vec{q}\cdot\vec{x}} \\
 &= \underline{N \int d\vec{x}' e^{i\vec{q}\cdot\vec{x}'} f_p(\vec{x}') 2\pi \int_0^\infty r dr \int_0^{2\pi} d\phi e^{i\vec{q}\cdot\vec{x}}}
 \end{aligned}$$

$$= 2\pi \int_0^\infty r dr \frac{1}{iqr} \underbrace{(e^{iqr} - e^{-iqr})}_{2i \sin qr}$$

$$= \frac{4\pi}{q} \int_0^\infty dr \sin qr$$

$$= \frac{4\pi}{q^2} \int_0^\infty dz \sin z$$

$$= \frac{4\pi}{q^2} \left. -\cos z \right|_0^\infty$$

$$= \frac{4\pi}{q^2} (1 - \infty)$$

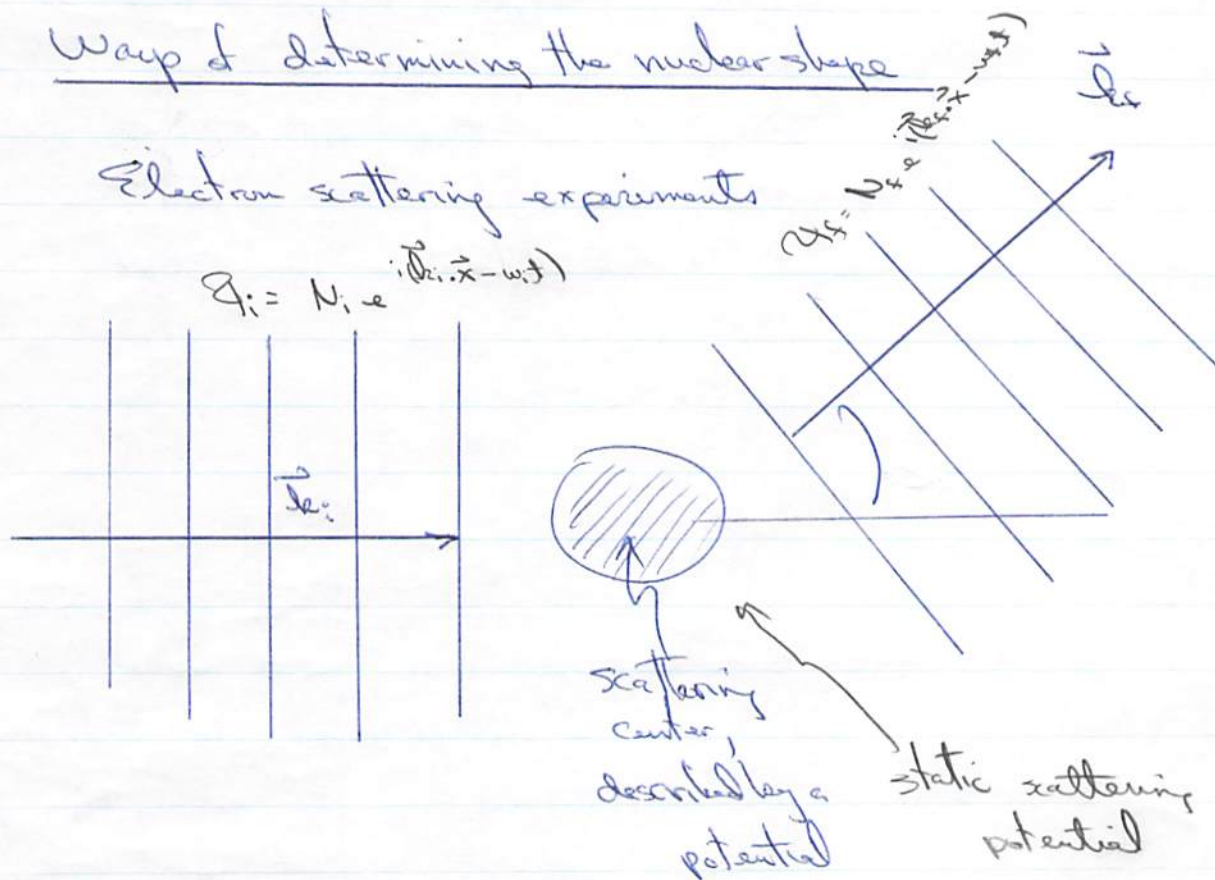
↑
"shore"!

Supplemental notes to lecture 2

(1)

Way of determining the nuclear shape

Electron scattering experiments



Recall Fermi's Golden rule (K. 2.79) (K. 2.80)

$$\lambda = \frac{2\pi}{\hbar} \left| \int d\vec{x} \psi_f^*(\vec{x}) V_{\text{pert}}(\vec{x}) \psi_i(\vec{x}) \right|^2 \rho(E)$$

↑
density of final states

In this case we regard free plane waves as the basis solutions of the SE, and a localized potential, in this case the Coulomb potential of the nucleus, as the perturbation. It's crude, but it works. (This is called the "Born approximation", treating ψ_f, ψ_i as

aside ... distorted wave BA, partial wave analysis

(2)

free waves.)

Generalization of 1D plane wave to 3D

$$\text{In 1D} \quad \psi_{\text{free}}(x) = N e^{i(kx - \omega t)}$$

norm factor

$$k = \sqrt{\frac{2mE}{\hbar^2}}; \hbar\omega = E$$

$$\text{In 3D} \quad \psi_{\text{free}}(\vec{x}) = N e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

Let's calculate this matrix element in detail

$$\psi_i(\vec{x}) = N_i e^{i(\vec{k}_i \cdot \vec{x} - \omega_i t)}$$

$$\psi_f(\vec{x}) = N_f e^{i(\vec{k}_f \cdot \vec{x} - \omega_f t)}$$

$$V_{fi} = \int d\vec{x} \psi_f^*(\vec{x}) V_p(\vec{x}) \psi_i(\vec{x})$$

$$= \int d\vec{x} N_f^* e^{-i(\vec{k}_f \cdot \vec{x} - \omega_f t)} V_p(\vec{x}) N_i e^{i(\vec{k}_i \cdot \vec{x} - \omega_i t)}$$

(1)

$$V_{fi} = N_f^* N_i \int d\vec{x} e^{i(\vec{k}_i - \vec{k}_f) \cdot \vec{x}} V_p(\vec{x}) e^{-i(\omega_i - \omega_f)t}$$

just as in Rutherford scattering, we expect $E_f = E_i$, elastic scattering, if the target is left unchanged.

$\vec{k}_i$

$\omega_i = \omega_f$

$|\vec{k}_i| = |\vec{k}_f|$

$|\omega_i| = |\omega_f|$

$$\vec{q} = \vec{r}_i - \vec{r}_f$$

(3)

$$V_f = N_f N_i \underbrace{\int d\vec{x} e^{i\vec{q} \cdot \vec{x}} V_p(\vec{x})}_{}$$

note: This is just the 3D Fourier integral. By Fourier inversion, if I measure $V_f(\vec{q})$, then I can obtain $V_p(\vec{x})$!

What is $V_p(\vec{x})$?

If we have a charge distribution $g(\vec{x}')$, arising from Z protons

$$V_p(\vec{x}) = \underbrace{-}_{\substack{\uparrow \\ \text{attractive}}} \underbrace{(Ze)(e)}_{\substack{\text{proton charge} \\ \text{electron charge}}} \underbrace{\int d\vec{x}' \frac{g(\vec{x}')}{|\vec{x} - \vec{x}'|}}_{\substack{\text{from electrostatics}}}$$

note: $Q = (Ze) \int d\vec{x}' g(\vec{x}') = Ze$

So, $\int d\vec{x}' g(\vec{x}') = 1$ proton probability distribution



Continuing

$$V_{fi} = N_f^* N_i \int d\vec{x} e^{i\vec{q} \cdot \vec{x}} \int d\vec{x}' \left(\frac{-Ze^2}{4\pi\epsilon_0} \right) \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$$= -\frac{Ze^2}{4\pi\epsilon_0} \int d\vec{x}' \rho(\vec{x}') \int d\vec{x} \frac{e^{i\vec{q} \cdot \vec{x}}}{|\vec{x} - \vec{x}'|}$$

$$e^{i\vec{q} \cdot \vec{x}} = e^{i\vec{q} \cdot (\vec{x} - \vec{x}') + i\vec{q} \cdot \vec{x}'} = e^{i\vec{q} \cdot (\vec{x} - \vec{x}')} e^{i\vec{q} \cdot \vec{x}'}$$

$$= -\frac{Ze^2}{4\pi\epsilon_0} \int d\vec{x}' e^{i\vec{q} \cdot \vec{x}'} \rho(\vec{x}') N_f^* N_i \int d\vec{x} \frac{e^{i\vec{q} \cdot (\vec{x} - \vec{x}')}}{|\vec{x} - \vec{x}'|}$$

↑
integrated
over the
charge
distribution
↑
integrated
over all
space

$$V_{fi} = \underbrace{\int d\vec{x}' e^{i\vec{q} \cdot \vec{x}'} \rho(\vec{x}')}_{\text{Fourier integral of the charge distribution}} \underbrace{\left[-\frac{Ze^2}{4\pi\epsilon_0} N_f^* N_i \int d\vec{u} \frac{e^{i\vec{q} \cdot \vec{u}}}{|\vec{u}|} \right]}_{\text{This is a function of } q \text{ only, and is related to the intensity of the } V_{fi} \text{ vs. angle}}$$

→ called the "nuclear form factor"

Contains all the info about the shape

shape factor

$$V_{fi} = F(\vec{q}) f(\vec{q})$$

experimentally measure

central
eqn
used to
infer
 $g(\vec{x})$ from
exp't

$$\left\{ \begin{array}{l} \text{measured} \\ \left[\frac{2\pi}{x} \frac{(V_{fi} V_{fi}^*)}{f^*(\vec{q}) f(\vec{q})} g_E \right] = F^*(\vec{q}) F(\vec{q}) \end{array} \right. \rightarrow \begin{array}{l} \text{charge distribution} \\ \downarrow \\ \text{infer } g(\vec{x}) \end{array}$$

from theory

Let's carry this a little further

$$F(\vec{q}) = \int d\vec{x}' e^{i\vec{q} \cdot \vec{x}'} g(\vec{x}')$$

For spherical nuclei $g(\vec{x}') = g(r')$

$$F(\vec{q}) = \int_0^{2\pi} d\phi' \int_0^\pi \sin\theta' d\theta' \int_0^\infty r'^2 dr' e^{i\vec{q} \cdot \vec{x}'} g(r')$$

without loss of generality (since spherical)
we can take $\vec{q} = q\hat{z}$, then

(6)

$$\begin{aligned}
 F(q) &= \int_0^{2\pi} d\phi' \int_0^\pi \sin\theta' d\theta' \int_0^\infty r'^2 dr' g(r') e^{i\mathbf{q} \cdot \mathbf{r}'} \\
 &= (2\pi) \int_0^\infty r'^2 dr' g(r') \int_{-\infty}^{\infty} d\mu e^{i\mathbf{q} \cdot \mathbf{r}'} \\
 &= 2\pi \int_0^\infty r'^2 dr' g(r') \frac{1}{iqr'} \underbrace{\left[e^{iqr'} - e^{-iqr'} \right]}_{2i \sin qr'}
 \end{aligned}$$

$$= \frac{4\pi}{q} \int_0^\infty r dr' g(r') \sin(qr')$$

$$F(q) = \frac{4\pi}{q} \int_0^\infty r dr' \sin(qr') g(r')$$

$$= \frac{4\pi R^3}{3} [\sin qR - qR \cos qR]$$

for a sphere with radius R
uniformly charged

is the special case of
spherical nuclei

What is $|\vec{q}|$?

$$\vec{q} = \vec{k}_i - \vec{k}_f$$

$$q^2 = (\vec{k}_i - \vec{k}_f) \cdot (\vec{k}_i - \vec{k}_f)$$

$$= k_i^2 - 2\vec{k}_i \cdot \vec{k}_f + k_f^2$$

$$q^2 = k_i^2 - 2k_i k_f \cos \theta + k_f^2$$

But $k_i = k_f$ (elastic) so

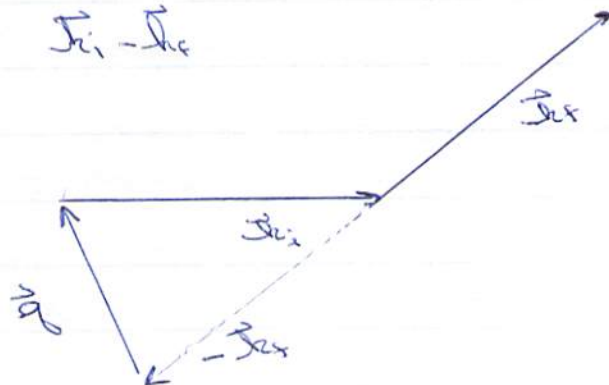
$$\begin{aligned} q^2 &= k^2 - 2k^2 \cos \theta + k^2 \\ &= 2k^2(1 - \cos \theta) \\ &= 2k^2(1 - \cos(\frac{\theta}{2} + \frac{\theta}{2})) \\ &= 2k^2(1 - (\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2})) \\ &= 2k^2(1 - \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}) \\ &= 2k^2(\sin^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}) \\ &= 4k^2 \sin^2 \frac{\theta}{2} \end{aligned}$$

$$\text{So, } q = 2k \sin \frac{\theta}{2}$$

$$\text{Since } \hbar k = p$$

$$q = \frac{2p \sin \frac{\theta}{2}}{\hbar}$$

$$\vec{q} = \vec{k}_i - \vec{k}_f$$



$\hbar \vec{q}$ is the momentum transfer

$\hbar q$ is its magnitude

What about this other factor $f(\vec{q})$ used in the analysis?

Recall,

$$f(\vec{q}) = \frac{-Ze^2}{4\pi\epsilon_0} N_f N_i \int d\vec{u} \frac{e^{i\vec{q}\cdot\vec{u}}}{|\vec{u}|}$$

$$\int d\vec{u} \frac{e^{i\vec{q}\cdot\vec{u}}}{|\vec{u}|} \underset{\substack{\uparrow \\ \text{all space}}}{=} \int_0^{2\pi} d\phi \int_{-1}^1 dp \int_0^\infty u^2 du \frac{e^{i q u p}}{u}$$

why assume $\vec{q} = q \hat{z}$

$$\begin{aligned} &= (2\pi) \int_0^\infty u du \underbrace{\int_{-1}^1 dp e^{i q u p}}_{\frac{1}{i q u} (e^{i q u} - e^{-i q u})} \\ &= \frac{2 \sin q u}{q u} \end{aligned}$$

(9)

$$= \frac{4\pi}{q} \lim_{\Lambda \rightarrow \infty} \int_0^{\Lambda} du \sin qu$$

$$= \frac{4\pi}{q} - \frac{1}{q} \lim_{\Lambda \rightarrow \infty} \cos qu \Big|_0^{\Lambda}$$

$$= \frac{4\pi}{q^2} \left(1 - \lim_{\Lambda \rightarrow \infty} \cos q\Lambda \right)$$

set to 0, why?

really wave packets, and the stuff at ∞ always dies out

$$f(q) = \frac{-Ze^2}{4\pi\epsilon_0} N_f N_i \frac{4\pi}{q^2}$$

$$f^*(q) f(q) = \left(\frac{Ze^2}{\epsilon_0} \right)^2 \frac{1}{q^4} \underbrace{|N_f|^2}_{=1} \underbrace{|N_i|^2}_{=1}$$

$$f^*(q) f(q) = \frac{(Ze^2/\epsilon_0)^2}{q^4}$$

$$f^*(q)f(q) = \left(\frac{Ze^2}{\epsilon_0} \right)^2 \left(\frac{\hbar}{2p \sin \frac{\theta}{2}} \right)^4$$

$$= \left(\frac{Ze^2 \hbar^2}{4\epsilon_0} \right)^2 \frac{1}{p^4 \sin^4 \frac{\theta}{2}}$$

Note: If the incoming charged particle was non-relativistic $p^2 = 2mK$

$$f^*(q)f(q) \stackrel{NR}{=} \left(\frac{Ze^2 \hbar^2}{16\epsilon_0 m} \right)^2 \frac{1}{K^2} \frac{1}{\sin^4 \frac{\theta}{2}}$$

Rutherford scattering

So,

$$|V_{fi}|^2 = \underbrace{F(\vec{r})F(\vec{r})}_{\uparrow \text{ modification of RS due to finite size of the nucleus}} \underbrace{f^*(q)f(q)}_{\text{Rutherford scattering (RS)}}$$

Recall,

$$F(\vec{q}) = \int d\vec{x} e^{i\vec{q}\cdot\vec{x}} g(\vec{x})$$

If $1q$

$$\text{if } \frac{1}{q} = \frac{\hbar}{2p \sin \frac{\theta}{2}} \gg R_{\text{nucleus}}$$

i.e. wavelength of incoming particle
 $\gg R_{\text{nucleus}}$.

$$e^{i\vec{q}\cdot\vec{x}} \approx 1$$

$$F(\vec{q}) \rightarrow 1$$

$$\therefore |V_{fi}|^2 = \text{RS formula}$$

As $p \uparrow$ wavelength approaches the
 nuclear size, and its structure can be resolved

As $p \uparrow \uparrow$ the internal structure of the
 nucleus can be resolved (quarks)

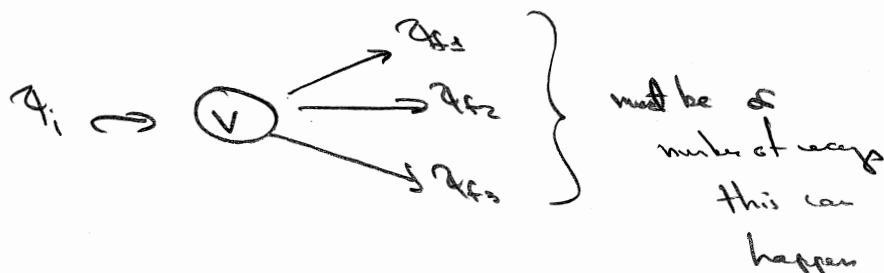
Show figure on p. 58

Transition rates

From Krane 2.8 p.40

When system is "perturbed" its transition rate is ^{2-magnitude}

$$\lambda\left[\frac{i}{f}\right] = \frac{2\pi}{\hbar[E-\tau]} \underbrace{\langle \psi_f |}_{\substack{\text{final} \\ \text{wf of} \\ \text{the} \\ \text{system}}} \underbrace{V_{\text{pert}}}_{\substack{\text{perturbing} \\ \text{potential}}} \underbrace{|\psi_i\rangle}_{\substack{\text{initial} \\ \text{wf of} \\ \text{the} \\ \text{system}}} \underbrace{[E^2]}_{\substack{\text{density} \\ \text{of final} \\ \text{states} \\ \text{of unique} \\ \text{quantum} \\ \text{states}}} \underbrace{\frac{dn}{dE_f [E]}}_{\substack{\text{from } E_f \rightarrow E_f + dE_f}}$$



The Rutherford scattering system

(see supplemental notes)

Coulomb self-energy

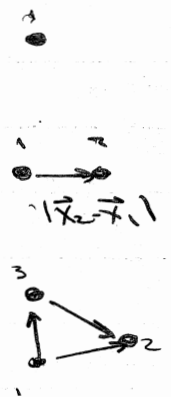
Assemble N protons

$$E_c \quad Z$$

$$\frac{e^2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_2 - \vec{r}_1|} \quad 2$$

$$\frac{e^2}{4\pi\epsilon_0} \left\{ \frac{1}{|\vec{r}_2 - \vec{r}_1|} + \frac{1}{|\vec{r}_3 - \vec{r}_1|} + \frac{1}{|\vec{r}_3 - \vec{r}_2|} \right\} 3$$

and so on



$$\frac{e^2}{4\pi\epsilon_0} \sum_{i=1}^Z \sum_{j=i+1}^Z \frac{1}{|\vec{r}_i - \vec{r}_j|} \quad Z$$

i	j	
1	2	
2		
i	j	
1	2, 3	(1,2) (1,3)
2	3	(2,3)
3		

another way

$$\frac{e^2}{4\pi\epsilon_0} \frac{1}{2} \sum_{i=1}^N \sum_{\substack{j=1 \\ i \neq j}}^N \frac{1}{|\vec{r}_i - \vec{r}_j|}$$

$N=2$

i	j	
1	2, 3	$\leftarrow (1,2) \quad (1,3) \leftarrow$
2	1, 3	$\leftarrow (2,1) \quad (2,3) \leftarrow$
3	1, 2	$\leftarrow (3,1) \quad (3,2) \leftarrow$

So,

$$E_c = \frac{1}{2} \frac{e^2}{4\pi\epsilon_0} \sum_{i=1}^N \sum_{\substack{j=1 \\ i \neq j}}^N \frac{1}{|\vec{r}_i - \vec{r}_j|}$$

Continuous Distribution

$$\sum_{i=1}^N \rightarrow Z \int d\vec{r}$$

$$E_c = \frac{1}{2} \frac{Z^2 e^2}{4\pi\epsilon_0} \int d\vec{r}_1 \rho(\vec{r}_1) \int d\vec{r}_2 \rho(\vec{r}_2) \frac{1}{|\vec{r}_1 - \vec{r}_2|}$$

Uniform charge dist'n

$$\rho = \frac{1}{\frac{4}{3}\pi R_0^3}$$

$$\int_{|\vec{x}| \leq R_0} d\vec{x} \rho = 1$$

$$\begin{aligned} E_c &= \frac{1}{2} \frac{Z^2 e^2}{4\pi\epsilon_0} \left(\frac{3}{4\pi R_0^3} \right)^2 \int_{|\vec{x}_1| < R_0} d\vec{x}_1 \int_{|\vec{x}_2| < R_0} d\vec{x}_2 \frac{1}{|\vec{x}_1 - \vec{x}_2|} \\ &= \underbrace{\frac{1}{2} \frac{Z^2 e^2}{4\pi\epsilon_0} \left(\frac{3}{4\pi} \right)^2 \frac{1}{R_0}}_{\text{constant}} \underbrace{\int_{|\vec{u}_1| \leq 1} d\vec{u}_1 \int_{|\vec{u}_2| \leq 1} d\vec{u}_2 \frac{1}{|\vec{u}_1 - \vec{u}_2|}}_{\text{I}} \end{aligned}$$

Let $\vec{u}_i$ lie along the $\hat{z}$ axis when we do the inner integrals

$$I = \underbrace{\int_0^{2\pi} d\phi_1 \int_0^\pi d\theta_1 \sin\theta_1}_{4\pi} \int_0^1 u_1^2 du_1 \underbrace{\int_0^{2\pi} d\phi_2 \int_0^\pi d\theta_2 \sin\theta_2}_{2\pi} \int_0^1 u_2^2 du_2$$

$$I = 8\pi^2 \int_0^1 u_1^2 du_1 \int_0^1 u_2^2 du_2 \int_{-1}^1 \frac{1}{[u_1^2 + u_2^2 - 2u_1 u_2 \mu]^{1/2}} d\mu$$

(26)

$$= 8\pi^2 \int_0^1 u_1^2 du_1 \int_0^1 u_2^2 du_2 - \frac{1}{u_1 u_2} [u_1^3 + u_2^3 - 2u_1 u_2 u_1^{1/2} u_2^{1/2}] \Big|_0^1$$

$$= 8\pi^2 \int_0^1 u_1 du_1 \int_0^1 u_2 du_2 \left[[(u_1 + u_2)^2]^{1/2} - [(u_1 - u_2)^2]^{1/2} \right]$$

$$= 8\pi^2 \int_0^1 u_1 du_1 \int_0^1 u_2 du_2 \left[\underbrace{(u_1 + u_2) - |u_1 - u_2|}_{\substack{2u_2 \quad u_2 < u_1 \\ 2u_1 \quad u_2 > u_1}} \right]$$

$$(4\pi)^2 \int_0^1 u_1 du_1 \left[\underbrace{\int_0^{u_1} u_2^2 du_2 + u_1 \int_{u_1}^1 u_2 du_2}_{\frac{u_1^3}{3} + \frac{u_1}{2}(1-u_1^2)} \right]$$

$$= \frac{u_1}{2} - \frac{u_1^3}{6}$$

$$I = (4\pi)^2 \int_0^1 du_1 \left(\frac{u_1^2}{2} - \frac{u_1^4}{6} \right)$$

$$= (4\pi)^2 \left(\frac{1}{6} - \frac{1}{6 \cdot 5} \right)$$

$$= (4\pi)^2 \left(\frac{5-1}{30} \right) = (4\pi)^2 \frac{2}{15}$$

$$E_c = \frac{1}{2} \frac{Z^2 e^2}{4\pi\epsilon_0} \frac{1}{R_N} \frac{1}{(\frac{4}{3}\pi)^2} \frac{Z}{155}$$

$$= \frac{3}{5} \frac{Z^2 e^2}{4\pi\epsilon_0 R_N}$$

K 3.16

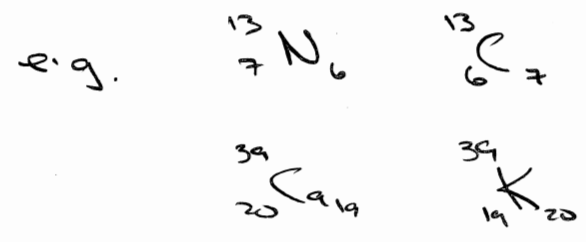
How to exploit this to measure the radius of the nucleus?

Use mirror nuclei

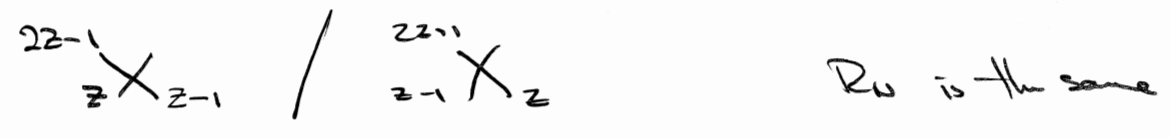
def $\rightarrow$



are mirrors of each other
A same $Z \leftrightarrow N$



Take the following mirror pair where Z's differ by 1



$$\Delta E_c = \frac{3}{5} \frac{e^2}{4\pi\epsilon_0} \frac{Z^2 - (Z-1)^2}{R_0 A^{1/3}}$$

$$= \frac{3}{5} \frac{e^2}{4\pi\epsilon_0} \frac{2Z-1}{R_0 A^{1/3}}$$

$$A = 2Z-1$$

$$= \frac{3}{5} \frac{e^2}{4\pi\epsilon_0} \frac{A^{2/3}}{R_0}$$

Plot ΔE_c vs $A^{2/3}$

Slope gives $\frac{3}{5} \frac{e^2}{4\pi\epsilon_0} \frac{1}{R_0}$

$$R_0 \sim 1.22 \text{ fm}$$

See graph on p. 56

Coulomb self energy (review)

$$E_c = \frac{1}{2} \frac{Z^2 e^2}{4\pi\epsilon_0} \int d\vec{x} \rho_p(\vec{x}) \int d\vec{x}' \frac{\rho_p(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

Assume uniform charge $\rho_p(\vec{x}) = \frac{1}{(4\pi/3 R_N^3)} \Theta(R_N - r)$

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \quad \left. \begin{array}{l} \text{step fn} \\ \text{Heaviside-Lorentz fn} \end{array} \right\}$$

$$E_c = \frac{1}{2} \frac{Z^2 e^2}{4\pi\epsilon_0} \frac{3^2}{(4\pi)^2} \frac{1}{R_N^6} \int_{|\vec{x}| \leq R_N} d\vec{x} \int_{|\vec{x}'| \leq R_N} d\vec{x}' \frac{1}{|\vec{x} - \vec{x}'|}$$

$$\text{Let } \frac{\vec{x}}{R_N} = \vec{u}$$

$$E_c = \frac{1}{2} \frac{Z^2 e^2}{4\pi\epsilon_0} \frac{1}{R_N} \frac{9}{(4\pi)^2} \int_{|\vec{u}| \leq 1} d\vec{u} \int_{|\vec{u}'| \leq 1} d\vec{u}' \frac{1}{|\vec{u} - \vec{u}'|}$$

integrals
over the
unit sphere

(30)

$$|\vec{r} - \vec{r}'| = [u^2 + u'^2 - 2uu'\mu]^{1/2}$$

$$I = \frac{q}{(4\pi)^2} \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \int_0^1 u^2 du \int_0^{2\pi} d\phi' \int_{-1}^1 d\mu' \int_0^1 u'^2 du' \frac{1}{[u^2 + u'^2 - 2uu'\mu]^{1/2}}$$

$\mu = \cos\theta \quad \mu' = \cos\theta'$

For purpose of inner integral $\vec{r}$ is fixed, let it be $\vec{r} = u\hat{z}$

$$I = \frac{q}{(4\pi)^2} (4\pi) \int_0^1 u^2 du (2\pi) \int_0^1 u'^2 du' \int_{-1}^1 d\mu' \frac{1}{[u^2 + u'^2 - 2uu'\mu]^{1/2}}$$

$$= \frac{q}{2} \int_0^1 u^2 du \int_0^1 u'^2 du' \left[-\frac{1}{uu'} (u^2 + u'^2 - 2uu'\mu)^{1/2} \right]_{-1}^1$$

$$= \frac{q}{2} \int_0^1 u du \int_0^1 u' du' \left[\sqrt{u^2 + u'^2 + 2uu'} - \sqrt{u^2 + u'^2 - 2uu'} \right]$$

$$= \frac{q}{2} \int_0^1 u du \int_0^1 u' du' [u + u' - |u - u'|]$$

$$= \frac{q}{2} \int_0^1 u du \left[2 \int_0^u u' du' u + 2 \int_u^1 u' du' u \right]$$

(31)

$$= 9 \int_0^1 u du \left[\frac{u^3}{3} + \frac{u}{2} (1 - u^2) \right]$$

$$= 9 \int_0^1 u du \left[\frac{u}{2} - \frac{u^3}{6} \right]$$

$$= \frac{9}{2} \int_0^1 du \left[u^2 - \frac{u^4}{3} \right]$$

$$= \frac{9}{2} \left[\frac{1}{3} - \frac{1}{15} \right]$$

$$= \frac{9}{2} \left[\frac{4}{15} \right] = \frac{3 \cdot 3}{2} \cdot \frac{2 \cdot 2}{3 \cdot 5} = \frac{6}{5}$$

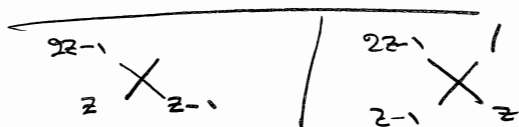
$$I = \frac{6}{5}$$

$$E_c = \frac{1}{2} \frac{Z^2 e^2}{4\pi\epsilon_0} \frac{1}{R_0} I$$

$$E_c = \frac{3}{5} \frac{Z^2 e^2}{4\pi\epsilon_0} \frac{1}{R_0}$$

(32)

for mirror nuclei

 ${}^A_Z X_N$ 

$$\Delta E_c = \frac{3}{5} \frac{e^2}{4\pi\epsilon_0} \left\{ \frac{Z^2}{R_0 A^{1/3}} - \frac{(Z-1)^2}{R_0 A^{1/3}} \right\}$$

$$= \frac{3}{5} \frac{e^2}{4\pi\epsilon_0 R_0 A^{1/3}} (Z^2 - (Z-1)^2)$$

$$\underbrace{\hspace{10em}}_{Z^2 - (Z^2 - 2Z + 1)} = \frac{2Z-1}{1} = A$$

$$\Delta E_c = \frac{3}{5} \frac{e^2}{4\pi\epsilon_0 R_0} A^{2/3}$$


 Measure these (beta-end point energies)

$$R_0 = 1.22 \text{ fm}$$

3.2 Mass; Abundance of Nuclei (read on own)

3.3 Nuclear Binding energy

$$(3.23) \quad m_{NC}c^2 = m_Ac^2 - \left(Zm_e c^2 - \sum_{i=1}^Z B_i \right)$$

$\uparrow$ rest mass energy of the nucleus (what we want)
 $\uparrow$ rest mass energy of the neutral atom (what we're given by nature)
 $\uparrow$ rest mass energy of all the electrons in the atom, bound
 $\uparrow$ binding energy of the i th electron

Ignore last term:

$$m_{NC}c^2 \approx A \times 1000 \text{ MeV}$$

$$B_e \approx 10-100 \text{ keV} \ll \text{much smaller, so ignore}$$

(examine this) (next page)

by analogy

$$(3.24) \quad B_N = \left[Zm_p + Nm_n - (m(^A X) - Zm_e) \right] c^2$$

$\underbrace{\hspace{10em}}_{m_p + m_e \approx m(^1\text{H})}$

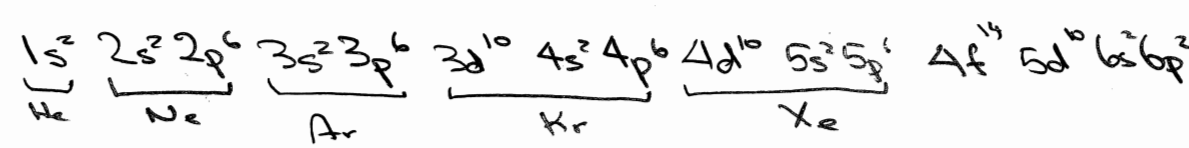
$$(3.25) \quad B_N = \left[Zm_p(^1\text{H}) + Nm_n - m(^A X) \right] c^2$$

$$c^2 = 931.50 \text{ MeV} / u$$

mass defect $\Delta = [m(^A X) - A] / c^2$ $\uparrow$ atomic mass units

2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50

Estimate the electronic BE of ^{209}Pb



n	
1	2
2	8
3	18
4	32
5	18
6	4

$$Z^2 (13.6 \text{ eV}) \left(2 + \frac{8}{2^2} + \frac{18}{3^2} + \frac{32}{4^2} + \frac{18}{5^2} + \frac{4}{6^2} \right)$$

$$= Z^2 (13.6 \text{ eV}) \left(2 + 2 + 2 + 2 + \overset{0.72}{\frac{18}{25}} + \overset{0.11}{\frac{4}{36}} \right)$$

$$8.83$$

$$(82)^2 (13.6) (8.83) = 0.986 \text{ MeV}$$

B_N gives us a sense of the total amount of assembly energy of a nucleus. We never measure this directly, but often look at differences between nearby nuclei

example

neutron separation energy " S_n "
 $\equiv$ energy required to liberate
 on neutron

$$(3.26) \quad S_n = B\left({}^A_Z X_N\right) - B\left({}^{A-1}_Z X_{N-1}\right)$$

dropping
 the subscript
 "N" from now
 on

from (3.25)

$$= \left[\cancel{Z}m({}^1_1\text{H}) + N m_n - m({}^A_Z X_N) \right] c^2 \\
- \left[\cancel{Z}m({}^1_1\text{H}) + (N-1)m_n - m({}^{A-1}_Z X_{N-1}) \right] c^2$$

$$(3.26) \quad S_n = \left[m({}^{A-1}_Z X_{N-1}) - m({}^A_Z X_N) + m_n \right] c^2$$

Similarly, the proton separation energy is

$$(3.27) \quad S_p = \left[m({}^{A-1}_{Z-1} X_N) - m({}^A_Z X_N) + m({}^1_1\text{H}) \right] c^2$$

	Nuclide	Δ (MeV)	S_n (MeV)		S_p (MeV)
stable 99.26%	$^{16}_8\text{O}$	-4.737	15.66	$\approx$	12.13
stable 0.038%	$^{17}_8\text{O}$	-0.910	4.14	$<$	13.78
64.5 s	$^{17}_9\text{F}$	+1.952	16.81	$>$	0.60

electron capture
 $p + e^- \rightarrow n$

Other usage

$\frac{B}{A}$, binding energy / nucleon

Semiripirical mass formula

$$B = a_v A$$



$$- a_s A^{2/3}$$

~~odd~~ $\rightarrow$ odd-even binding

Volume term, each nucleon is bound by the same amount (consequence of short range nature of the strong force)

those near the surface are missing neighbors to provide attraction

$$E_c = \frac{3}{5} \frac{Z(Z-1)}{4\pi\epsilon_0 R_0}$$

$$= \frac{3}{5} \frac{1}{4\pi\epsilon_0 R_0} \frac{Z(Z-1)}{A^{1/3}}$$

$$- a_c \frac{Z(Z-1)}{A^{1/3}}$$

$$- a_{\text{sym}} \frac{(A-2Z)^2}{A}$$

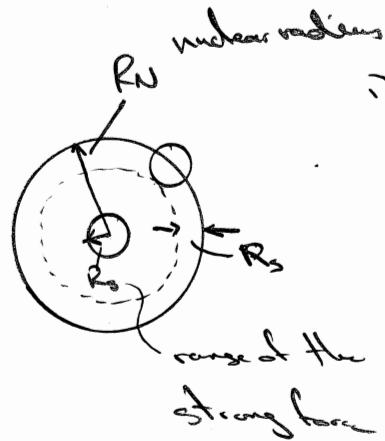
+ δ

$$\delta = a_p A^{-3/4} \begin{matrix} \text{even/even} \\ \text{even/odd} \\ \text{odd/even} \\ \text{odd/odd} \end{matrix}$$

favors small Z if N

a_v	15.5 MeV	
a_s	16.8 MeV	
a_c	0.72 MeV	$\frac{3}{5} \frac{e^2}{4\pi\epsilon_0 R_0} = 0.707$
a_{sym}	23 MeV	
a_p	34 MeV	

a_v & a_s can be used to estimate the range of the strong force



Assume strong force has a range R_s

$$R_0 = R_0 A^{1/3}$$

① $E_v = C \frac{4}{3} \pi R_N^3 = C \frac{4}{3} \pi R_0^3 A = a_v A$

② $E_s = C (4\pi R_N^2)^{1/2} R_s = 2\pi R_0^2 A^{2/3} R_s = a_s A^{2/3}$

① $C \frac{4}{3} \pi R_0^3 A = a_v A$

② $C 3\pi R_0^2 A^{2/3} R_s = a_s A^{2/3}$

①
② = $\frac{\frac{2}{3} R_0 A^{1/3}}{R_s} = \frac{a_v}{a_s} A^{1/3}$

$$R_s = \frac{2}{3} R_0 \frac{a_s}{a_v} = \frac{2}{3} (1.2 \text{ fm}) \left(\frac{16.8}{15.5} \right) = 0.982 \text{ fm}$$

$$N_{mn} = (A-Z)_{mn}$$

(20)

neutron excess

For fixed A , (3.2a) is a parabola in Z

Rewriting (3.2a)

$$M(Z, A) = \overset{\checkmark}{Z}_{m(1H)} + (A - \overset{\checkmark}{Z})_{mn}$$

$$- [a_v A - a_s A^{2/3} - a_c (\overset{\checkmark}{Z}^2 - \overset{\checkmark}{Z}) A^{-1/3}$$

$$- a_{sym} (A - 4\overset{\checkmark}{Z} + \frac{4\overset{\checkmark}{Z}^2}{A}) + \delta$$

$$\left(\underbrace{\frac{4a_{sym}}{A} + a_c A^{-1/3}}_a \right) \overset{\checkmark}{Z}^2$$

$$+ \left(\underbrace{m(1H) - m_n - a_c A^{1/3} - 4a_{sym}}_b \right) \overset{\checkmark}{Z}$$

$$+ \left(\underbrace{A m_n - a_v A + a_s A^{2/3} + a_{sym} A + \delta}_c \right)$$

$$a\overset{\checkmark}{Z}^2 + b\overset{\checkmark}{Z} + c$$

a is true sq so

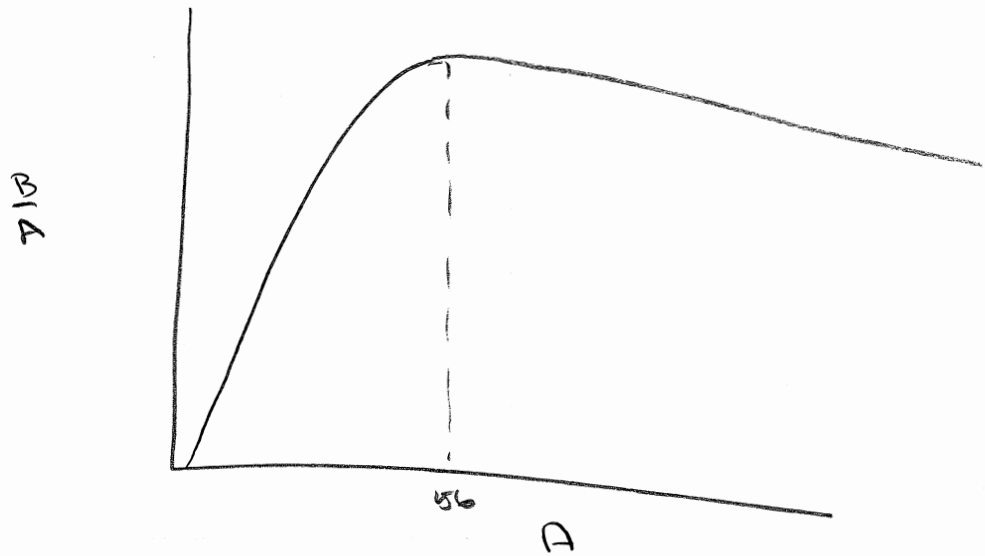
$$\overset{\checkmark}{Z}_{min} = \frac{-b}{2a} = \frac{a_c A^{1/3} + 4a_{sym} + m_n - m(1H)}{2a_{sym}/A + 2a_c A^{-1/3}}$$

Extent of the nuclear force $\approx 1 \text{ fm}$

$$\frac{D}{B} = \frac{3 \text{ MeV}}{\text{nucleon}} \pm 10\% \text{ for}$$

most ^{stable} nuclei ($A > 10$)

$\frac{D}{B}$ peaks at ${}^{56}_{26}\text{Fe}$



SEMF

$$(3.29) \quad M(Z, A) = Zm({}^1_1\text{H}) + Nm_n - B(Z, A)/c^2$$

$$\text{or, } \frac{Z_{\text{ini}}}{A} = \frac{1}{2} \left[\frac{4a_{\text{sym}} + a_c A^{1/3} + m_n - m(^1\text{H})}{4a_{\text{sym}} + a_c A^{2/3}} \right]$$

$$\text{for small } A \quad \frac{Z_{\text{ini}}}{A} \approx \frac{1}{2}$$

for large A $\downarrow$ slowly

3.4 Nuclear angular momentum ($\vec{I}$) and parity (π)

Nuclear angular momentum

$$\vec{I} = \sum_{i=1}^A (\underbrace{\vec{l}_i + \vec{s}_i}_{\substack{\text{single particle} \\ \text{orbital and intrinsic} \\ \text{angular momentum}}}) \quad \left. \vphantom{\sum_{i=1}^A} \right\} \begin{array}{l} \text{momenta are added} \\ \text{in the QM sense} \end{array}$$

$$\langle \vec{I}^2 \rangle = \hbar^2 I(I+1)$$

$$\langle I_z \rangle = \hbar m_I \quad -I \leq m_I \leq I$$

integral jumps

constants of the motion in the QM sense

If the nuclear force were "central", i.e.

$$V_N(\vec{x}) = V_N(r)$$

we could write

$$\begin{aligned}\vec{I} &= \left(\sum_{i=1}^A \vec{l}_i \right) + \left(\sum_{i=1}^A \vec{s}_i \right) \\ &= \vec{J} + \vec{S}\end{aligned}$$

and J, M_J, S, M_S would all be constants of the motion

Recall that in atomic physics, the spin-orbit coupling part of the potential $V_{so}(r) \vec{l} \cdot \vec{s}$ was small and was treated as a perturbation. A modest magnetic field could split the degeneracies

For the nuclear force, the $\vec{l} \cdot \vec{s}$ interaction is very strong, and no magnetic field can split the levels.

So, for nuclei, there is only I , it is the only measurable thing.

Notation

 $\overline{I}$

any man of the nucleus

 $\vec{I}$

" " " one of its nucleons

Closed shells

$$I=0$$

Add one nucleon

$$I=j$$

to a closed shell

$$I = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$$

Other unsequences

Odd A nuclei spin $\frac{1}{2}$

Even A " integral

$$I = 0, 1, 2, \dots$$

Observational fact

All even-even nuclei are $I=0$
(effect of the pairing force)

Recall pairing energy

$$\text{even-even} \quad \delta = a_p A^{-3/4}$$

↑
34 MeV

Parity

$$\pi = \pm 1 \quad (\text{even/odd}) \text{ parity}$$

$$\psi(-\vec{x}) = \pm \psi(\vec{x})$$

just as before.

There is no known connection between

$$\underline{I} \text{ and } \pi$$

Quadrupole moments

Em and gravitational applications

Gravitational application

Q. Why do we always see the same face of the moon?

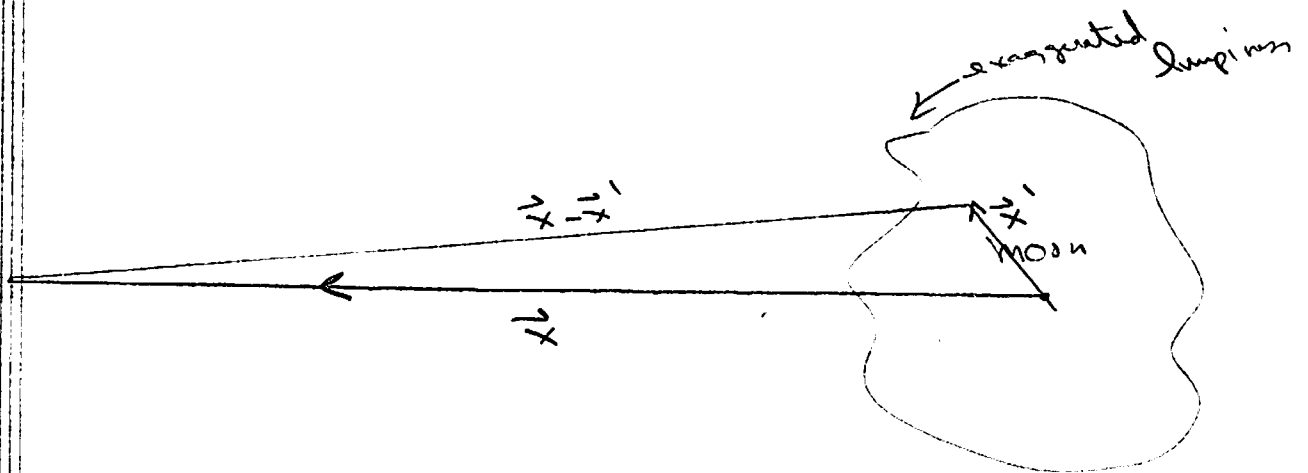
A. Quadrupole moments have locked it into place.

Q2. What is a quadrupole moment?

A?.

Consider the gravitational potential of the earth in the vicinity of the moon

$$V_{em}(\vec{x}) = -GM_E \int_{V_m} d\vec{x}' \frac{\rho_m(\vec{x}')}{|\vec{x} - \vec{x}'|} \quad (1)$$



note $|\vec{x}| \gg |\vec{x}'|$ So expand in a Taylor series expansion

(2)

$$\begin{aligned} \frac{1}{|\vec{x} - \vec{x}'|} &= [|\vec{x}|^2 - 2\vec{x} \cdot \vec{x}' + |\vec{x}'|^2]^{-1/2} \\ &= \frac{1}{|\vec{x}|} \left[1 - \underbrace{\frac{2\vec{x} \cdot \vec{x}'}{|\vec{x}|^2}}_{O\left(\frac{|\vec{x}'|}{|\vec{x}|}\right)} + \underbrace{\frac{|\vec{x}'|^2}{|\vec{x}|^2}}_{O\left(\frac{|\vec{x}'|^2}{|\vec{x}|^2}\right)} \right]^{-1/2} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{|\vec{x}|} \left\{ 1 - \frac{1}{2} \left[\frac{2\vec{x} \cdot \vec{x}'}{|\vec{x}|^2} + \frac{|\vec{x}'|^2}{|\vec{x}|^2} \right] \right. \\ &\quad \left. - \frac{\frac{1}{2} \left(-\frac{3}{2} \right)}{2} \left[\frac{4(\vec{x} \cdot \vec{x}')^2}{|\vec{x}|^4} \right] \right\} \\ &\quad + O\left(\left(\frac{|\vec{x}'|}{|\vec{x}|}\right)^3\right) \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad \frac{1}{|\vec{x} - \vec{x}'|} &= \frac{1}{|\vec{x}|} \left\{ 1 + \frac{\vec{x} \cdot \vec{x}'}{|\vec{x}|^2} + \frac{3}{2} \frac{(\vec{x} \cdot \vec{x}')^2}{|\vec{x}|^4} - \frac{1}{2} \frac{|\vec{x}'|^2}{|\vec{x}|^2} \right\} \\ &\quad + \text{higher order terms,} \end{aligned}$$

$$\begin{aligned} \textcircled{2} \rightarrow \textcircled{1} \Rightarrow V_{em}(\vec{x}) &= -GM_\epsilon \left\{ \frac{1}{|\vec{x}|} \int d\vec{x}' \rho_m(\vec{x}') \right. \\ &\quad + \frac{1}{|\vec{x}|^3} \vec{x} \cdot \int d\vec{x}' \vec{x}' \rho_m(\vec{x}') \\ &\quad \left. + \frac{1}{|\vec{x}|^5} \frac{1}{2} \int d\vec{x}' \rho_m(\vec{x}') [3(\vec{x} \cdot \vec{x}')^2 - |\vec{x}|^2 |\vec{x}'|^2] \right. \\ &\quad \left. + \text{higher order terms,} \right\} \end{aligned}$$

(3)

$$\int d\vec{x}' \rho_m(\vec{x}) = M_m \quad \text{mass of the moon}$$

$$\int d\vec{x}' \vec{x}' \rho_m(\vec{x}) = \vec{Q} \quad \text{dipole moment}$$

Let $\vec{x} = r \hat{z}$ align along the z-axis

$$\textcircled{3} \quad V_{em} = -GM_e \left\{ \frac{M_e}{r} + \frac{1}{r^2} \hat{z} \cdot \vec{Q} + \frac{1}{r^3} \underbrace{\frac{1}{2} \int d\vec{x}' \rho(\vec{x}') (3z'^2 - r'^2)} \right\}$$

This is called the
gravitational
quadrupole moment

If the base of the $\vec{x}$ vector, i.e. the origin of our coordinate system, $\vec{Q} = 0$, by definition of the center of mass.

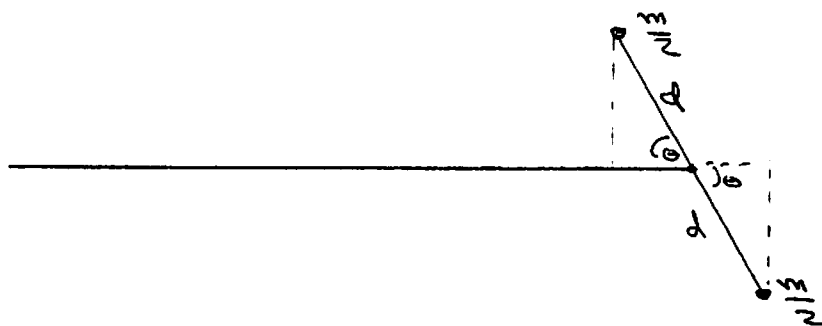
With an arbitrary origin

$$\vec{x}_{cm} = \frac{1}{M} \int d\vec{x} \vec{x} \rho(\vec{x})$$

$$\begin{aligned} \text{Put } \vec{x}_{cm} \text{ at } \vec{0}, \text{ i.e. } \vec{x}_{cm} - \vec{x}_{cm} = \vec{0} &= \frac{1}{M} \int d\vec{x} (\vec{x} - \vec{x}_{cm}) \rho(\vec{x}) \\ &= \frac{1}{M} \int d\vec{x}' \vec{x}' \rho(\vec{x}') = 0 \end{aligned}$$

Demonstration of why we always see the same face of the moon.

Consider a simpler model



$$V = -GM_E \frac{m_m}{2} \left\{ \frac{1}{\sqrt{(r-d\cos\theta)^2 + d^2\sin^2\theta}} + \frac{1}{\sqrt{(r+d\cos\theta)^2 + d^2\sin^2\theta}} \right\}$$

$$= -GM_E \frac{m_m}{2} \left\{ \frac{1}{[r^2 - 2rd\cos\theta + d^2]^{1/2}} + \frac{1}{[r^2 + 2rd\cos\theta + d^2]^{1/2}} \right\}$$

$$= -\frac{GM_E m_m}{2r} \left\{ \frac{1}{[1 - 2\epsilon\mu + \epsilon^2]^{1/2}} + \frac{1}{[1 + 2\epsilon\mu + \epsilon^2]^{1/2}} \right\}$$

$\mu = \cos\theta$
 $\epsilon = d/r$

$$= -\frac{GM_E m_m}{r} \left\{ 1 + \frac{\epsilon^2}{2} (3\mu^2 - 1) \right\}$$

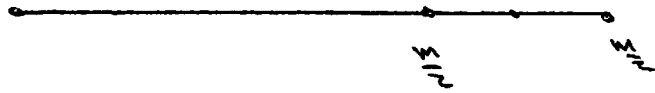
Taylor
expand

$$= \frac{1}{2r^2} \underbrace{d^2 (3\mu^2 - 1)}_{\text{quadrupole moment}}$$

$$(4) \quad V(\mu) = -\frac{GMm_e}{r} \left\{ 1 - \frac{e^2}{2} + \frac{3e^2}{2} \mu^2 \right\}$$

V is minimized when μ has its maximum value $\theta = 0, \pi$

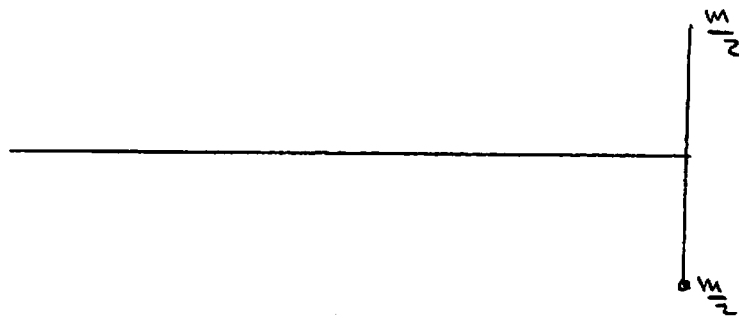
(A)



V is maximized when μ is minimized

$$\theta = \pm \frac{\pi}{2}$$

(B)



(4) is the equation of a harmonic oscillator

$$V(x) = V_0 - \frac{1}{2} kx^2$$

This case $k = 3e^2$

If there are any dissipative forces (tidal deformations), the system will come to rest in position A

(6)

Analogy with τ and u .

Go back to (3)

If the charged system contains charges of both signs $\vec{Q}$ can be non zero

The nucleus has charges of only 1 sign,
 $\therefore \vec{Q} = 0$

$$Q_2 = \int d\vec{x}' \rho_p(\vec{x}') (3z'^2 - r'^2) \quad \text{classical} \quad (5)$$

$$Q_2 = \int d\vec{x}' \rho^*(\vec{x}') (3z'^2 - r'^2) \rho(\vec{x}') \quad \text{quantum}$$

If ρ_p or $\rho^* \rho$ are spherically symmetric

$$Q_2 = 0$$

Demonstration

Quadrupole moment of an elliptic nucleus

relevant text KII p. 74, 142-143

Model the surface as $\leftarrow$ only truly ellipsoidal if β is small $\ll 1$

$$R(\Theta, \phi) = R_{av} [1 + \beta Y_{20}(\Theta, \phi)] \quad (5.14)$$

spherical $\rightarrow$ $Y_{20}(\Theta, \phi) = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\Theta - 1)$ (p27)
harmonic

note. does not depend on ϕ

$\beta \equiv$ deformation parameter

$\beta \approx 0.29$ for deformed nuclei
 $150 < A < 190, A > 220$

$$\frac{R_{min}}{R_{av}} (\Theta = \frac{\pi}{2}) = 1 - \beta \left(\frac{5}{16\pi}\right)^{1/2} \quad \text{if } \beta > 0$$

$$\frac{R_{max}}{R_{av}} (\Theta = 0, \pi) = 1 + 2\beta \left(\frac{5}{16\pi}\right)^{1/2}$$

For $\beta = 0.29$

$$0.91 \leq \frac{R}{R_{av}} \leq 1.18$$

(quite large)

(8)

Volume of this nucleus?

$$\begin{aligned}
 V &= \int_{r < R(\theta, \phi)} d\vec{x} \\
 &= \int_{4\pi} d\Omega \int_0^{1+\beta Y_{20}} r^2 dr \\
 &= \int_{4\pi} d\Omega \left. \frac{r^3}{3} \right|_0^{R(1+\beta Y_{20}(\theta, \phi))}
 \end{aligned}$$

$$= \frac{R^3}{3} \int_{4\pi} d\Omega (1 + \beta Y_{20})^3$$

$$\begin{aligned}
 &= \frac{R^3}{3} \left[\int_{4\pi} d\Omega + 3\beta \int_{4\pi} d\Omega Y_{20} \right. \\
 &\quad \left. + 3\beta^2 \int_{4\pi} d\Omega Y_{20}^2 + \beta^3 \int_{4\pi} d\Omega Y_{20}^3 \right]
 \end{aligned}$$

$$I_n = \int_{4\pi} d\Omega Y_{20}^n$$

$$I_0 = 4\pi$$

$$I_1 = 0$$

$$I_2 = 1$$

$$I_3 = \frac{1}{7} \sqrt{\frac{5}{\pi}}$$

} ortho
normality

$$\text{note } I_n = \frac{1}{2^{n+1}} \left[2^n \left(\frac{5}{16\pi} \right)^{n/2} 4\pi - 2n \sqrt{\frac{5}{16\pi}} I_{n-1} \right]$$

$$\text{note } I_3 = \int_{4\pi} d\Omega \left[\left(\frac{5}{16\pi} \right)^{3/2} (3\cos^2\theta - 1) \right]^3$$

$$= 2\pi \left(\frac{5}{16\pi} \right)^{3/2} \int_{-1}^1 d\mu (3\mu^2 - 1)^3$$

(9)

$$V = \frac{4\pi R_{av}^3}{3} \left\{ 1 + \frac{3}{4\pi} \beta^2 + \frac{1}{28} \sqrt{\frac{5}{\pi}} \beta^3 \right\}$$

for $\beta = 0.29$ ↑
~ 2% correction ↑
~ 0.1% correction

note $O(\beta^2)$

$O(\beta^3)$ correction to results will be ignored.

$$Q_2 = Z \int_{r < R(\theta)} d\vec{x} g_r(3z^2 - r^2) \quad g_r = \frac{1}{\frac{4\pi}{3} R_{av}^3}$$

$$= \frac{Z 2\pi}{\left(\frac{4\pi}{3} R_{av}^3\right)} \int_{-1}^1 d\mu (3\mu^2 - 1) \int_0^{R_{av}(1+\beta\gamma_{20})} r^4 dr$$

Ignoring β^2 term

$$= \frac{Z 2\pi}{\left(\frac{4\pi}{3} R_{av}^3\right)} \int_{-1}^1 d\mu (3\mu^2 - 1) \frac{(1 + \beta\gamma_{20})^5}{5}$$

$$= \frac{3}{\sqrt{5\pi}} Z R_{av}^2 \beta \left\{ 1 + \underbrace{\frac{2}{7} \left(\frac{5}{\pi}\right)^{1/2}}_{0.36} \beta + \frac{9}{28\pi} \beta^2 + \dots \right\}$$

~ 10% correction ~ 1% correction
(need correction to V as well)

So, ignoring the β^2 term

$$Q_2 = \frac{3\beta^2}{\sqrt{5\pi}} \left\{ 1 + \underbrace{0.36\beta}_{\sim 1} \right\}$$

K (5.16) gives the
constant as 0.16
(it is wrong)

make correction in your book

3.3 Nuclear Binding Energy (cont.)

From last time

← binding energy of a nucleus

$$(3.26) \quad B(Z, A) = a_v A - a_s A^{2/3} - a_c Z(Z-1) A^{-1/3} - a_{\text{sym}} \frac{(A-2Z)^2}{A} + \delta$$

$$\begin{aligned} \delta = \text{pairing term} &= 0 \quad \text{odd } A \\ &= a_p A^{-3/4} &= +1 \quad \begin{matrix} Z & N \\ \text{even} & \text{even} \end{matrix} \\ & &= -1 \quad \begin{matrix} Z & N \\ \text{odd} & \text{odd} \end{matrix} \end{aligned}$$

Consequently, the mass of a nucleus is

$$(3.29) \quad M(Z, A) = Zm(^1\text{H}) + (A-Z)m_n - B(Z, A)/c^2$$

For fixed A , (3.29) is a parabola with a minimum Z_{min} , that corresponds (approximately) to the most stable nucleus of a given A .

$$(3.30) \quad Z_{\text{min}} = \frac{[m_n - m(^1\text{H})]c^2 + a_c A^{-1/3} + 4a_{\text{sym}}}{2a_c A^{-1/3} + 3a_{\text{sym}} A^{-1}}$$

← missing in frame

Show $M(Z, A)$ parabolas p.69

3.5 Nuclear Electromagnetic moments (cont.)

Electric Quadrupole $Q_2 = \int d^3x \rho_p(\vec{x}) (3z^2 - r^2)$
already discussed.

Magnetic dipoles

A circular loop of current (moving charge)
has a magnetic dipole term

Classically, a particle with mass m & charge e
undergoing a circular orbit (radius r) has a
magnetic dipole moment

$$|\vec{\mu}| = \left(\frac{e}{2\pi r / v} \right) \pi r^2 = \frac{e v r}{2} = \frac{e}{2m} m v r = \frac{e}{2m} |\vec{L}|$$

classical
angular
momentum

Its z -projection is

$$\mu_z = \frac{e}{2m} L_z$$

In QM we define the magnetic moment
as (3.33)

$$\mu = \frac{e\hbar}{2m} (L_z)_{\max} = \frac{e\hbar}{2m} l \quad \text{since } m_{l, \max} = +l$$

$\frac{e\hbar}{2m}$ is a fundamental property of the particle.

It is called the "magneton"

For electrons, it is called the "Bohr magneton"

$$\mu_B = \frac{e\hbar}{2m_e} = 5.7884 \times 10^{-5} \text{ eV/T}$$

For the proton

$$\underbrace{\mu_N = \frac{e\hbar}{2m_p}}_{\text{nuclear magneton}} = 3.1525 \times 10^{-8} \text{ eV/T}$$

Note $\mu_N \ll \mu_B$ 2000 times smaller

Most of the magnetic effects in materials is due to electrons, not protons

The conventional way of writing (3.33) is

$$(3.34) \quad \left\{ \begin{array}{ll} \mu = g_e \mu_B & \text{for electrons} \\ \mu = g_N \mu_N & \text{for nucleons} \end{array} \right.$$

$$g_e = \begin{matrix} 1 & \text{for electrons and protons} \\ 0 & \text{for neutrons (neutral)} \end{matrix}$$

What about spin?

That's a moving charge as well, so it should have a magnetic moment

Relativistic QM for point like particles

For electrons

$$\mu = g_s \mu_B S$$

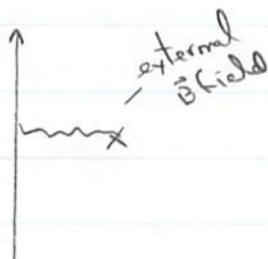
$$g_s = 2 \left(1 + \frac{\alpha}{2\pi} + \text{higher order terms} \right)$$

$$= 2.00232$$

Scientific Cultural

Aside:

$$g_s = 2 \left\{ \begin{matrix} 1 \\ \frac{\alpha}{2\pi} \end{matrix} \right.$$



higher order terms

~ 900 diagrams have to be analyzed

$$g_p^2 = 2 \left(1 + \frac{\alpha}{2\pi} - 0.328478455 \left(\frac{\alpha}{\pi} \right)^2 + 1.183(11) \left(\frac{\alpha}{\pi} \right)^3 \right)$$

$$= \begin{array}{ll} 1159652359(282) \times 10^{-12} & \text{theory} \\ 1159652410(200) \times 10^{-12} & \text{exp't} \end{array}$$

probably the finest agreement in
all of physics

If, the proton were a point particle

$$g_{s, \text{proton}} = 2 \left(1 + \underbrace{\text{small correction}}_{\text{could be as much as } 200\% \text{ the } \alpha\text{-correction, but still } \lesssim 1} \right)$$

experimentally

$$\left\{ \begin{array}{l} g_{s, \text{proton}} = 5.5856912(22) \\ g_{s, \text{neutron}} = -3.8260937(10) \end{array} \right. !!$$

best evidence of internal structure to n's & p's