

NERS 312

Elements of Nuclear Engineering and Radiological Sciences II

aka **Nuclear Physics for Nuclear Engineers**

Lecture Notes for Chapter 13: Radioactive decay

Supplement to (Krane II: Chapter 6)

*Note: The lecture number corresponds directly to the chapter number in the online book.
The section numbers, and equation numbers correspond directly to those in the online book.*

Chapter 13: In this Chapter you will learn

Chapter 13.1: The Radioactive Decay Law ...

1. The reason for exponential decay
2. What half-life is
3. Lifetime, or mean lifetime
4. Activity
5. Decay from one isotope, with two decay channels
6. Decay from one “parent”, to many stable “daughters”
7. Decay from two isotopes, to independent decay channels

Chapter 13.2: Quantum Theory of Radioactive Decay

1. Application to nuclear γ decay
2. The Lorentz distribution
3. Density of final states...
4. Density of final states for γ -decay
5. Derivation of Fermi's Golden Rule #2
6. The small perturbation approximation

Chapter 13.3: Production and Decay of Radioactivity

1. Secular equilibrium
2. Application to a real-life engineering application

Chapter 13.4: Growth of Daughter Activities

1. Parent $\Rightarrow$ Daughter $\Rightarrow$ Granddaughter
2. Very long-lived parent: $\lambda_1 \lll \lambda_2$ decays
3. Long-lived parent: $\lambda_1 \ll \lambda_2$ decays
4. Series of Decays — The Bateman equations

Chapter 13.5: Types of Decays

1. α decay
2. β decay
3. γ decay
4. Internal conversion
5. Nucleon emission
6. Spontaneous fission
7. Cluster decay

Chapter 13.8: Units for Measuring Radiation

1. Activity
2. Exposure
3. Absorbed Dose
4. Dose Equivalent

13.1—The Radioactive Decay Law

Exponential decay law

- Consider a system of particles, N_0 in number at time, $t = 0$.
- Each particle has an independent, = probability of decay per unit time, λ .
- How many particles are observed at a later time?
- Assume that N is large enough, use calculus.
- Particles are integral quantities, so this is really an approximation!

Thus, the change in N is given by:

$$\begin{aligned}dN &= -\lambda N dt; \quad N(0) = N_0 \\ \frac{dN}{N} &= -\lambda dt \\ d[\log N] &= -\lambda dt \\ \log N - \log N_0 &= -\lambda t \\ \log N &= \log N_0 - \lambda t \\ N &= N_0 \exp(-\lambda t)\end{aligned}\tag{13.1}$$

Voilà, the well-known *exponential decay law*, $N(t) = N_0 e^{-\lambda t}$.

Half-life

The *half-life*, $t_{1/2}$, is defined as follows:

$$\frac{N(t + t_{1/2})}{N(t)} \equiv \frac{1}{2} = \frac{N_0 \exp(-\lambda t - \lambda t_{1/2})}{N_0 \exp(-\lambda t)} = \exp(-\lambda t_{1/2}) ,$$

or,

$$t_{1/2} = \frac{\log 2}{\lambda} \approx \frac{0.693}{\lambda} . \quad (13.2)$$

Thus we see, a population of N radioactive particles at t would be reduced by half (on average) at time $t + t_{1/2}$.

Lifetime, or mean lifetime

The exponential law can also be interpreted as the *decay probability* for a single radioactive particle to decay in the interval dt , about t .. This probability, $p(t)$, properly normalized, is given by:

$$p(t)dt = \lambda e^{-\lambda t}dt \quad ; \quad \int_0^{\infty} p(t)dt = 1 . \quad (13.3)$$

The we see that the probability a particle decays within time t , $P(t)$ is given by,

$$P(t) = \int_0^t p(t')dt' = 1 - e^{-\lambda t}. \quad (13.4)$$

The *mean lifetime* or *lifetime* of a particle, τ , is evaluated by calculating $\langle t \rangle$, using the probability distribution (13.3),

$$\tau = \lambda \int_0^{\infty} t e^{-\lambda t} dt = \frac{1}{\lambda} . \quad (13.5)$$

Activity

The number of decays, ΔN , observed from t and $t + \Delta t$, obtained from (13.1)

$$N(t) = N_0 e^{-\lambda t}$$

is:

$$\Delta N = N(t) - N(t + \Delta t) = N_0 e^{-\lambda t} (1 - e^{-\lambda \Delta t}) .$$

If $\Delta t \ll \tau$, then, in the limit as $\Delta t \rightarrow 0$, we may rewrite the above as:

$$\lim_{\Delta t \rightarrow 0} \left| \frac{\Delta N}{\Delta t} \right| \equiv \left| \frac{dN}{dt} \right| = \lambda N_0 e^{-\lambda t} = \lambda N(t) \equiv \mathcal{A}(t) = \mathcal{A}_0 e^{-\lambda t} , \quad (13.6)$$

defining the *activity*, $\mathcal{A}(t)$, and its initial value, $\mathcal{A}_0$. Activity is usually what is measured, since N_0 and $N(t)$ are usually unknown, and not of particular interest in many applications.

What is generally of real interest is the actual activity of a real source, or the actual value of λ .

It must be emphasized that (13.6)

$$\mathcal{A}(t) = \mathcal{A}_0 e^{-\lambda t}$$

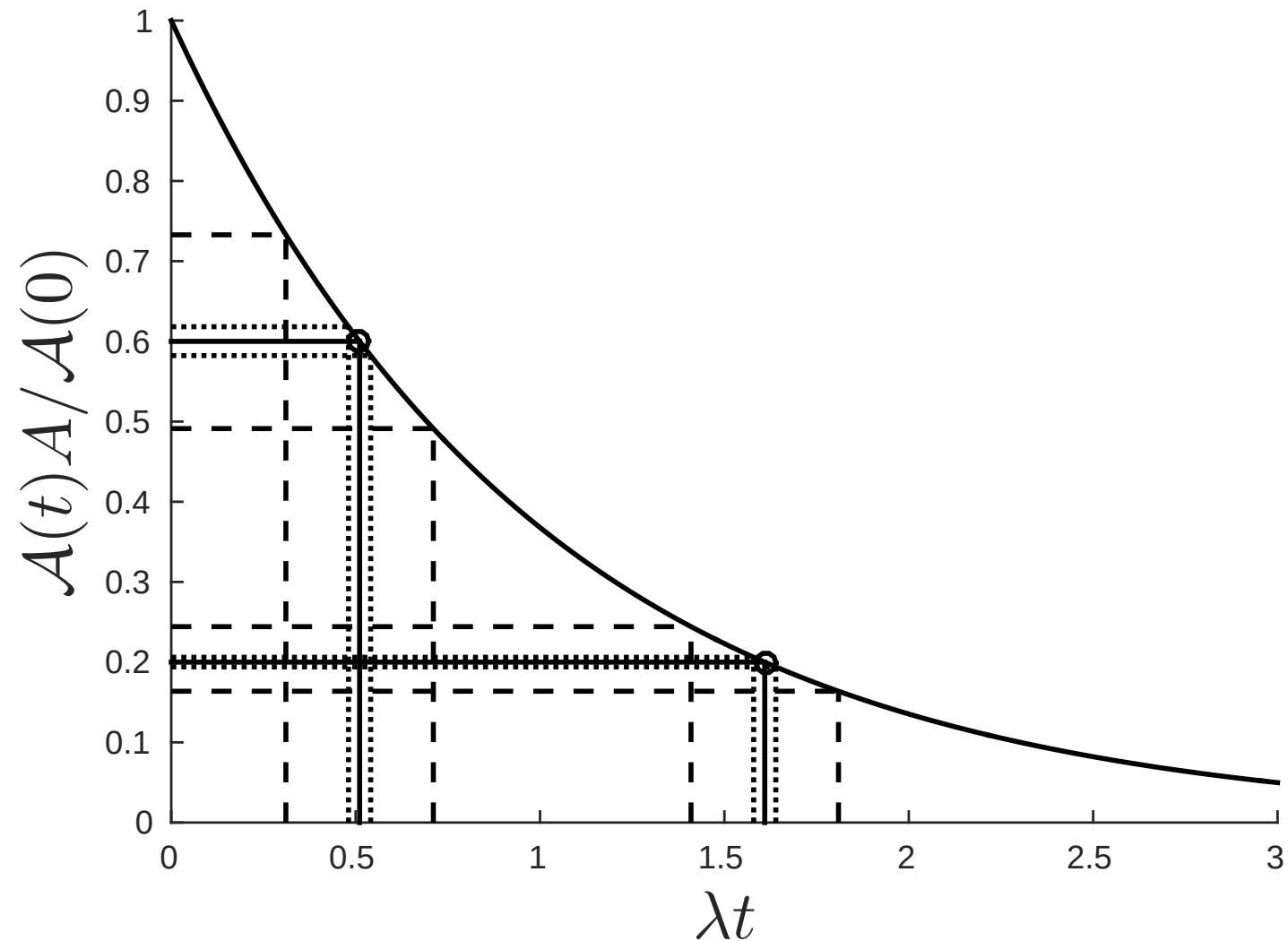
is an approximate relationship, based on $\Delta t \ll \tau$. Consequently, using it in any other expression, is subject to the same constraint.

In the two figures, this concept is illustrated, for a measurement of the activity at two different times, with two different Δt 's, and then with a plot of the correction factor:

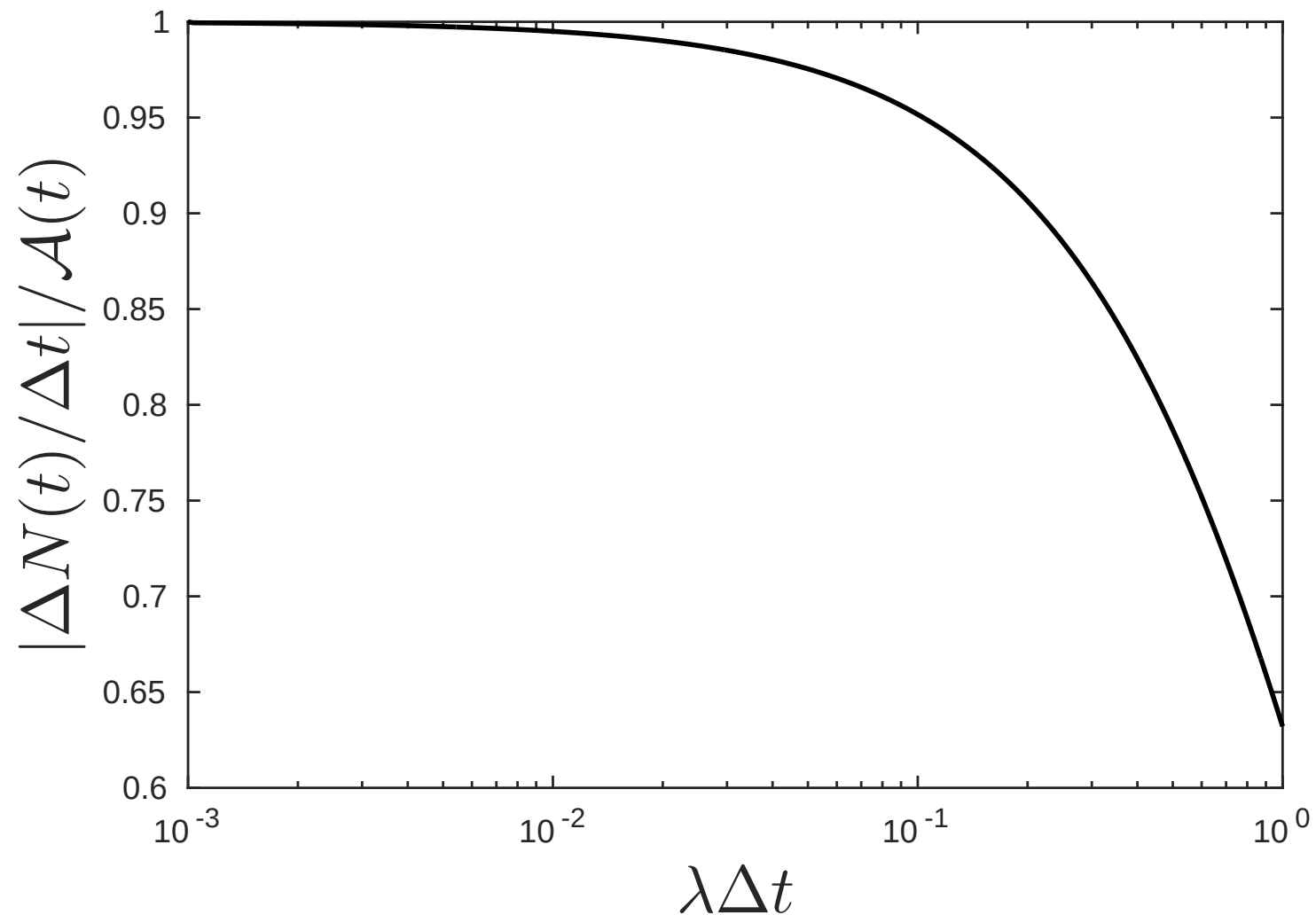
$$C(\lambda \Delta t) = \frac{\left| \frac{\Delta N}{\Delta t} \right|}{\mathcal{A}(t)} = \frac{1 - \exp(-\lambda \Delta t)}{\lambda \Delta t}.$$

Note that correction factor goes to unity as $\lambda \Delta t \rightarrow 0$. We also note that the measurement always underestimates $\mathcal{A}(t)$.

Activity measurements over finite time spans



Correction factor to idealized $\mathcal{A}(t)$



If it is the measured value of λ that is desired, we have a uniform expression:

$$\lambda = \frac{\log[\mathcal{A}(t_1)/\mathcal{A}(t_2)]}{t_2 - t_1} = \frac{\log[\mathcal{N}(t_1)/\mathcal{N}(t_2)]}{t_2 - t_1}$$

Finally, this discussion only applies for the decay of a single isotope, N_0 , to another (presumably stable) nucleus, N_1 . Any measurement involving very short measurement times, must be analyzed carefully.

Growth and decline of radioactivity generations

General considerations

The most general form for describing the growth and decline of the generations of radioisotopes is with the general differential form that described the growth and/or decline of isotope N_i :

$$dN_i = \sum_{j=1}^{N_i^p} N_j \lambda_{ji} dt - N_i \sum_{j=1}^{N_i^d} \lambda_{ij} dt ,$$

where,

N_i^p = Number of parents of isotope N_i

N_i^d = Number of daughters of isotope N_i

λ_{ji} = Rate of growth of isotope N_i from parent N_j

λ_{ij} = Rate of decay of isotope N_i into daughter N_j

If there are two subscripts on λ , the left one refers to that parent, and the right one refers to the daughter.

If there is only one subscript, it means that there is only one parent. That subscript refers to the daughter only. Most of the examples that follow fall into this category.

Mathematical considerations

From the previous expression, we see that we are faced with first-order differential equations, of varying complexity.

Fortunately, the most general solution is known.

Any first-order differential equation of the form:

$$df = g(t)dt - h(t)f(t)dt \quad ; \quad f(0) \text{ known}$$

has the solution:

$$f(t) = f(0)e^{-\int_0^t dt' h(t')} + e^{-\int_0^t dt' h(t')} \int_0^t dt' g(t') e^{\int_0^{t'} dt'' h(t'')}$$

Applying the general form to this solution, we obtain:

$$N_i(t) = N_i(0)e^{-\Lambda_i^d t} + \int_0^t dt' \sum_{j=1}^{N_i^p} N_j(t') \lambda_{ji} e^{-\Lambda_i^d (t-t')}$$

where

$$\Lambda_i^d = \sum_{j=1}^{N_i^d} \lambda_{ij}$$

Some important special cases

Only one parent: Index of i 's parent is p_i

$$N_i(t) = N_i(0)e^{-\Lambda_i^d t} + \int_0^t dt' N_{p_i}(t') \lambda_{p_i i} e^{-\Lambda_i^d (t-t')}$$

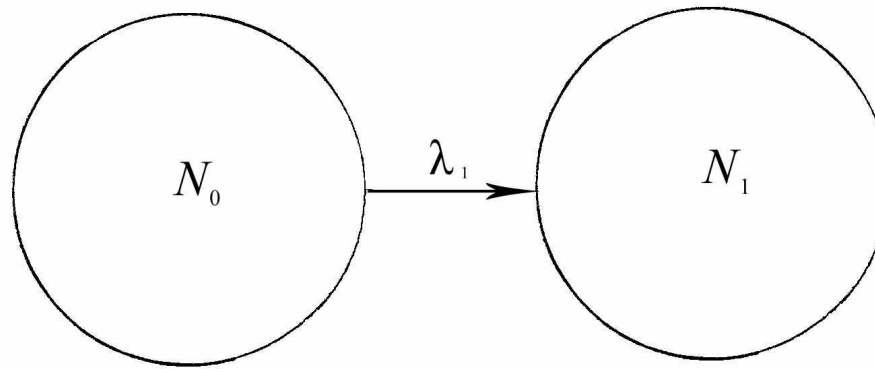
Only one daughter: Index of i 's daughter is d_i

$$N_i(t) = N_i(0)e^{-\lambda_{d_i} t} + \int_0^t dt' \sum_{j=1}^{N_i^p} N_j(t') \lambda_{ji} e^{-\lambda_{d_i} (t-t')}$$

One parent with index $i - 1$, one daughter with index $i + 1$

$$N_i(t) = N_i(0)e^{-\lambda_{i+1} t} + \int_0^t dt' N_{i-1}(t') \lambda_i e^{-\lambda_{i+1} (t-t')}$$

Example: One parent isotope, $N_0(t) \longrightarrow$ one stable daughter, $N_1(t)$



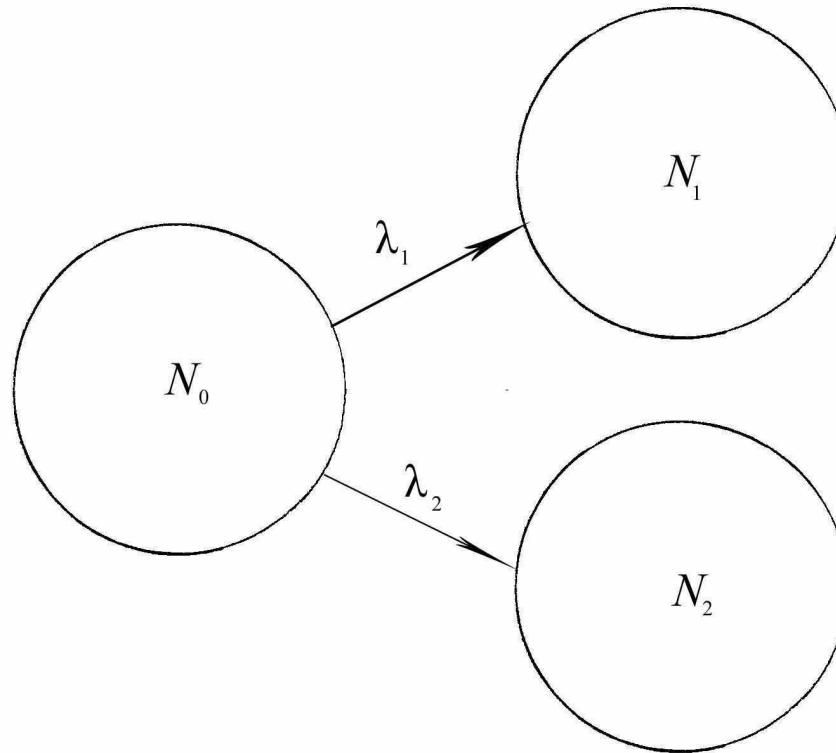
The differential equations describing the decline of N_0 and the growth of N_1 , with decay rate λ_1 are given as follows:

$$\begin{aligned} dN_0 &= -N_0\lambda_1 dt \\ dN_1 &= N_0\lambda_1 dt \\ d(N_0 + N_1) &= 0, \end{aligned} \tag{13.7}$$

with solutions:

$$\begin{aligned} N_0 &= N_0(0)e^{-\lambda_1 t} \\ N_1 &= N_1(0) + N_0(0)(1 - e^{-\lambda_1 t}) \\ N_0 + N_1 &= N_0(0) + N_1(0) \end{aligned} \tag{13.8}$$

Example: One parent isotope, $N_0(t) \longrightarrow$ two stable daughters, $N_1(t)$ & $N_2(t)$



N_0 decays with λ_1 into N_1 , or into N_2 with λ_2 . The total decay rate is $\lambda = \lambda_1 + \lambda_2$.

The differential equations are:

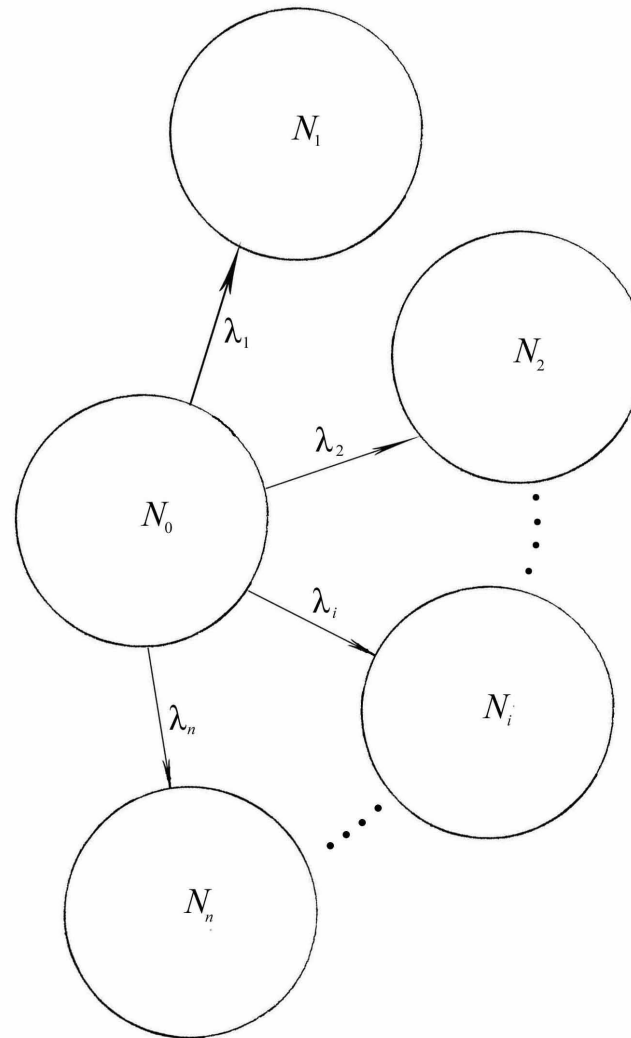
$$\begin{aligned}dN_0 &= -N_0\lambda dt \\dN_1 &= N_0\lambda_1 dt \\dN_2 &= N_0\lambda_2 dt \\d(N_0 + N_1 + N_2) &= 0 ,\end{aligned}\tag{13.9}$$

where $\lambda = \lambda_1 + \lambda_2$.

with solutions:

$$\begin{aligned}N_0 &= N_0(0)e^{-\lambda} \\N_1 &= N_1(0) + (\lambda_1/\lambda)N_0(0)(1 - e^{-\lambda t}) \\N_2 &= N_2(0) + (\lambda_2/\lambda)N_0(0)(1 - e^{-\lambda t}) \\N_0 + N_1 + N_2 &= N_0(0) + N_1(0) + N_2(0) .\end{aligned}\tag{13.10}$$

Example: One parent isotope, $N_0(t) \longrightarrow$ many stable daughters



The results of the one parent (nuclear isotope) $\longrightarrow$ 2 stable daughters), can easily be generalized to many daughters.

The differential equations are:

$$\begin{aligned}dN_0 &= -N_0\lambda dt \\dN_1 &= N_0\lambda_1 dt \\dN_2 &= N_0\lambda_2 dt \\&\vdots = \vdots \\dN_n &= N_0\lambda_n dt \\d\left(N_0 + \sum_{i=1}^n N_i\right) &= 0 ,\end{aligned}\tag{13.11}$$

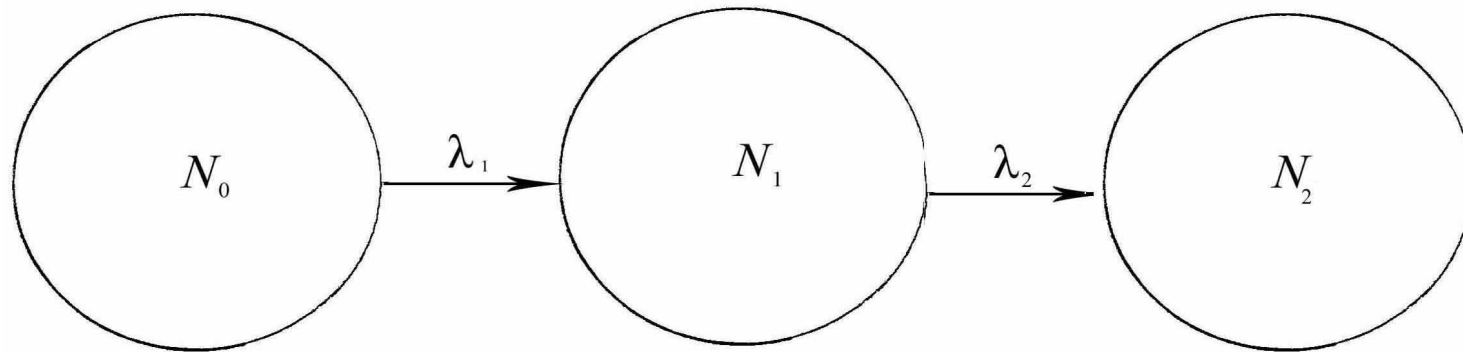
where $\lambda = \sum_{i=1}^n \lambda_{0i}$, or, $\sum_{i=1}^n \lambda_{0i}/\lambda = 1$.

The solutions are given by:

$$\begin{aligned}N_0 &= N(0)e^{-\lambda t} \\N_i &= N_i(0) + (\lambda_{0i}/\lambda)N_0(1 - e^{-\lambda t}) \\N + \sum_{i=1}^n N_i &= N(0) + \sum_{i=1}^n N_i(0) .\end{aligned}\tag{13.12}$$

The quantity λ_{0i}/λ is called the *branching ratio* for decay into channel i .

Example: One parent isotope, two-generation decay chain



In this example, $N_1(0) = N_2(0) = 0$.

By definition, decays chains end in a stable nucleus.

The differential equations are:

$$dN_0 = -N_0\lambda_1 dt$$

$$dN_1 = N_0\lambda_1 dt - N_1\lambda_2 dt$$

$$dN_2 = N_1\lambda_2 dt$$

$$d(N_0 + N_1 + N_2) = 0 .$$

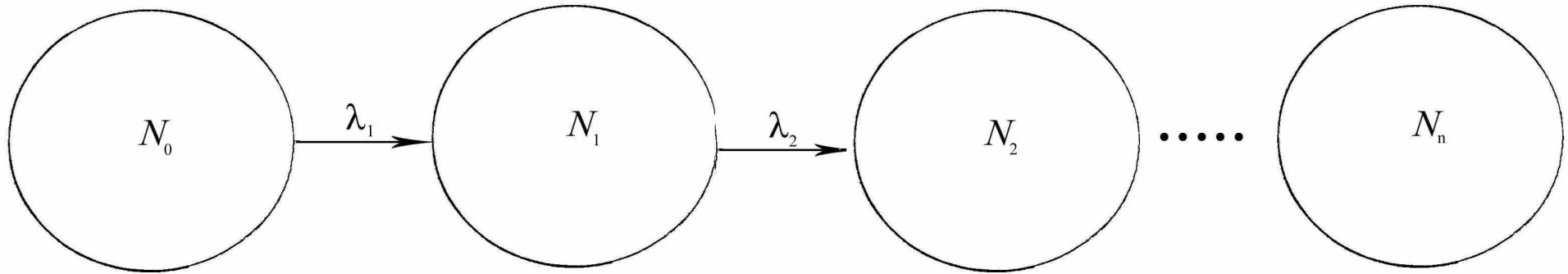
After some work, the solution is:

$$\begin{aligned}N_0 &= N(0)e^{-\lambda_1 t} \\N_1 &= \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}) \\N_2 &= N_0(0) - N_0(t) - N_1(t) \\N_0 + N_1 + N_2 &= N(0) .\end{aligned}\tag{13.13}$$

Example: One parent isotope, n-generation decay chain

In this example, $N_i(0) = 0$ for $i \geq 1$.

By definition, decays chains end in a stable nucleus.



These result in the Bateman equations, a special case of our general considerations...

$$\mathcal{A}_n = N_0(0) \sum_{i=0}^{n-1} c_i e^{-\lambda_{i+1}t}$$
$$c_m = \frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=0; i \neq m}^{n-1} (\lambda_{i+1} - \lambda_{m+1})}$$

Please note that Krane's equations single-index λ 's differ in that his λ refers to the parent, not the daughter.

Activity from two isotopes, with independent decay channels

In this case, the total activity measured is:

$$\mathcal{A}(t) = \mathcal{A}_0^a e^{-\lambda_a t} + \mathcal{A}_0^b e^{-\lambda_b t}, \quad (13.14)$$

where a and b label the two different isotopes.

The question arises:

How to determine $\mathcal{A}_0^a$, $\mathcal{A}_0^b$, λ_a and λ_b ?

Krane outlines a strategy portrayed in his Figure 6.2...

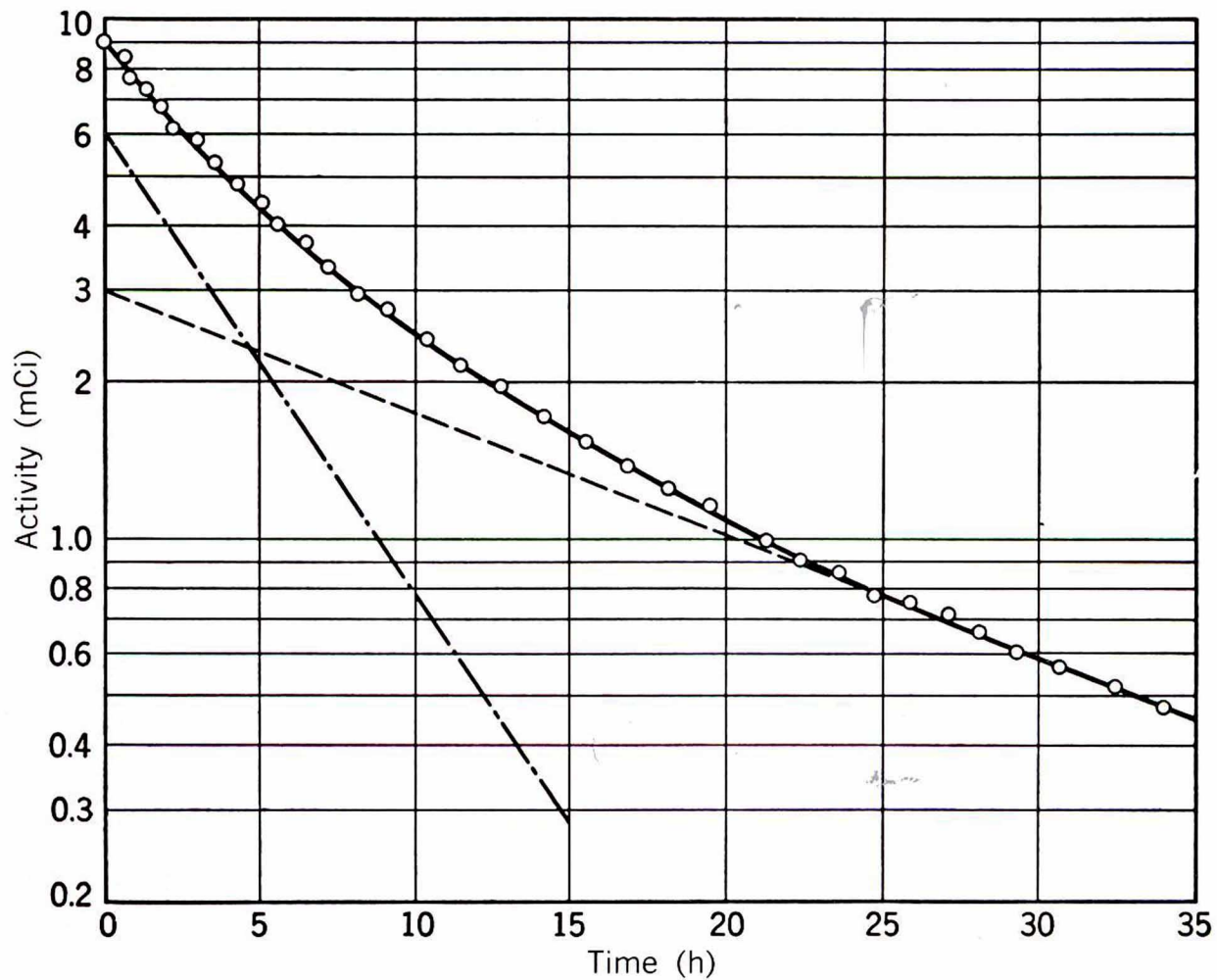


Figure 6.2 Decay curve for a sample containing a mixture of ^{64}Cu (12.7 h) and ^{61}Cu (3.4 h).

If, for example, isotope a has a considerably longer half-life, the linear tail of the log-linear plot of $\mathcal{A}$ can be extrapolated backward to isolate $\mathcal{A}_0^a$.

The slope of this line also yields λ_a . Then one plots $\log(\mathcal{A}(t) - \mathcal{A}_0^a e^{-\lambda_a t})$ that should give a straight line log-linear plot.

The values of $\mathcal{A}_0^b$ and λ_b can then be determined.

13.2—Quantum Theory of Radioactive Decay

The Quantum Theory of Radioactive Decay starts with a statement of Fermi's Golden Rule #2, the equation from which decays rates, and cross sections are obtained.

It is one of the central equations in Quantum Mechanics. Fermi's Golden Rule #2 for the transition rate (probability of transition per unit time), λ , is given by:

$$\lambda = \frac{2\pi}{\hbar} |\langle \psi_f | V_p | \psi_i \rangle|^2 \frac{dn_f}{dE_f}, \quad (13.15)$$

where ψ_i is the initial quantum state, operated on by a perturbation (transition) potential, V_p , resulting in the final quantum state, ψ_f .

The factor dn_f/dE_f is called the *density of final states* [sometimes given the notation $\rho(E_f)$].

The density of final states factor enumerates the number of possible final states (degeneracy) that can acquire the final energy E_f .

It is not possible to express a less generic form of this factor, without a specific application in mind.

It must be derived on a case-by-case basis, for a given application. We shall have opportunity to do for several examples before the conclusion of lecture.

Application to nuclear γ decay

In this case, we have the following situation:

$$\begin{aligned} N_i^* &\longrightarrow N_f + \gamma & (i, f) &\equiv (\text{initial, final}) & * &\equiv \text{excited} \\ \psi^{N_i^*} &\longrightarrow \psi^{N_f} \psi^\gamma & & (\text{wavefunctions}) \\ E_i &= E_f + E_\gamma + E_R & & (\text{CoE}) \end{aligned} \tag{13.16}$$

- N_i^* , has a γ -transition to N_f
- The final state may be an intermediate (but lower in energy) excited state.

If the energy of the excited state is uncertain, we know that its lifetime and its energy are connected through the Heisenberg Uncertainty relationship:

$$\Delta E \Delta t \geq \hbar/2 .$$

- This is a consequence of the wave description of matter.
- $\Delta E \uparrow \Rightarrow \Delta t \downarrow$
- $\Delta E \downarrow \Rightarrow \Delta t \uparrow$

What is the exact relationship between a state's lifetime, and the distribution of energies that are observed????

- The derivation is sketched in the following and the results presented.
- The details of the derivation are left to the optional section at the end of this section.

Derivation of the Lorentzian distribution

We start by assuming that the final nuclear state is given by:

$$\begin{aligned}\psi^{N_f}(\vec{x}, t) &= \psi^{N_f}(\vec{x}) e^{iE_f t/\hbar} e^{-t/(2\tau)} \\ |\psi^{N_f}(\vec{x}, t)|^2 &= |\psi^{N_f}(\vec{x})|^2 e^{-t/\tau},\end{aligned}\tag{13.17}$$

to agree with our discussion, in the last section, of the probability of decay of a single particle. Recall that τ is the “lifetime”.

The derivation in the next section reveals that the probability of observing decay energy E , $p(E)$, is given by:

$$p(E) = \frac{\Gamma}{2\pi} \frac{1}{(E - E_f)^2 + (\Gamma/2)^2},\tag{13.18}$$

where $\Gamma \equiv \hbar/\tau$. This probability distribution is normalized:

$$\int_{-\infty}^{\infty} dE p(E) = 1.$$

- The peak of this distribution is $p(E_f)$
- Γ width of the distribution at half-maximum
That is, $p(E_f \pm \Gamma/2) = \frac{1}{2}p(E_f)$
- $\langle E \rangle = E_f$.
- This distribution is called the *Lorentz distribution*, or simply, the *Lorentzian* function.
a.k.a. the *Cauchy-Lorentz*, *Cauchy*, or *Breit-Wigner* distribution
- Wikipedia has a useful page on this topic.
- The unnormalized Lorentzian is plotted in Figure 13.1
- The normalized Lorentzian is plotted in Figure 13.2.
- We note that this spread of energies is **intrinsic**
It has nothing to do with measurement uncertainties
Even with a perfect detector, we would observe this spread of detected energies
- It is a result of the finite lifetime, and Heisenberg Uncertainty

Lorentz distribution (unnormalized)

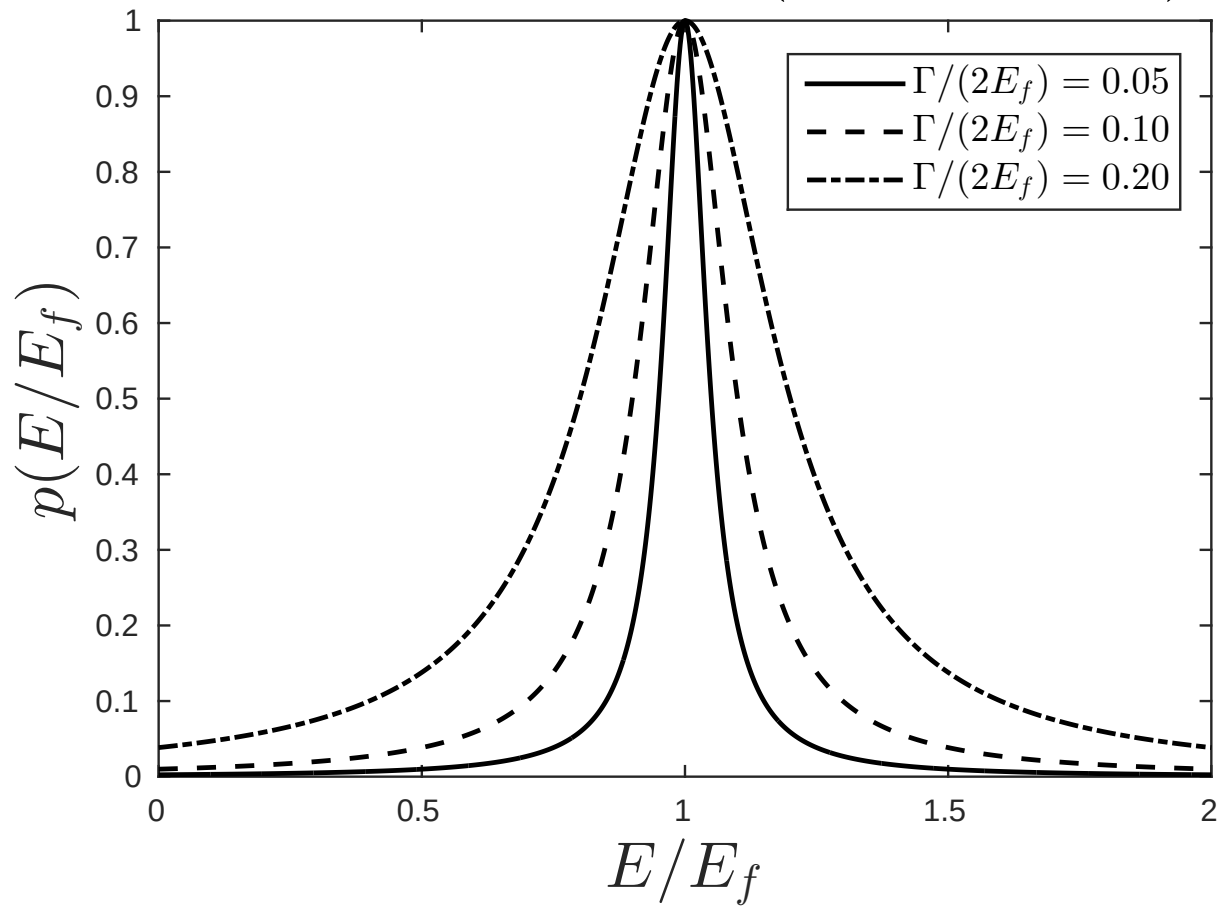


Figure 13.1: The unnormalized Lorentzian.

Lorentz distribution (normalized)

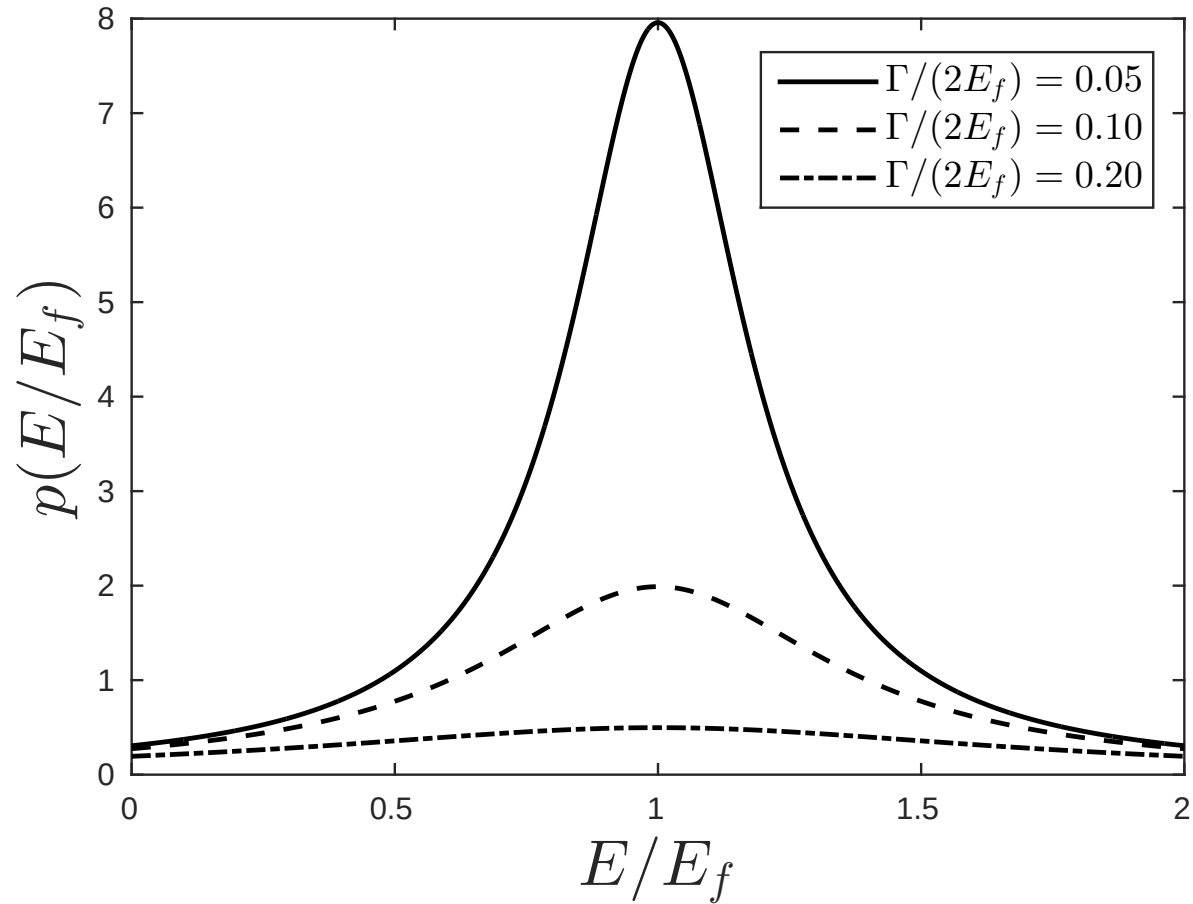


Figure 13.2: The normalized Lorentzian.

Consequences?

- For typical γ -transition lifetimes, $10^{-12} \text{ s} < \tau < \infty$, $0.00066 \text{ eV} > \Gamma > 0$
- γ spectroscopy can exquisitely isolate the individual energy levels
- This is drastically different for high-energy physics, where intrinsic widths can be of the order of 1 GeV or so. Nearby excitation can overlap with each other, and the identification of excited states (of hadrons) can be very difficult.

Density of final states

Finally, we calculate the density of states, after talking about it for so long!

- Given a final energy of a quantum system, how many states, dn_f enumerates the number of quantum states that fall in the range, $E_f \longrightarrow E_f + dE_f$?
- From this, we form the ratio dn_f/dE_f , *a.k.a.* the *density of final states*.
- This version assumes that there is just one free particle in the final state.
- This will cover most (but not all!) of the physics of interest in this course.

- Expressing the free particle wave function, as it would exist in a cubical L^3 box, with $V = 0$ inside, $V \rightarrow \infty$ outside, and one corner at the origin of the coordinate system
- As we discovered in NERS311, the wavefunction is given by:

$$\psi_{n_x, n_y, n_z}(x, y, z) = \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right) \quad (13.19)$$

where $1 \leq n_i < \infty$ are the 3 quantum numbers ($i = 1, 2, 3$) in the 3D system.

- The 3 momentum components are given by:

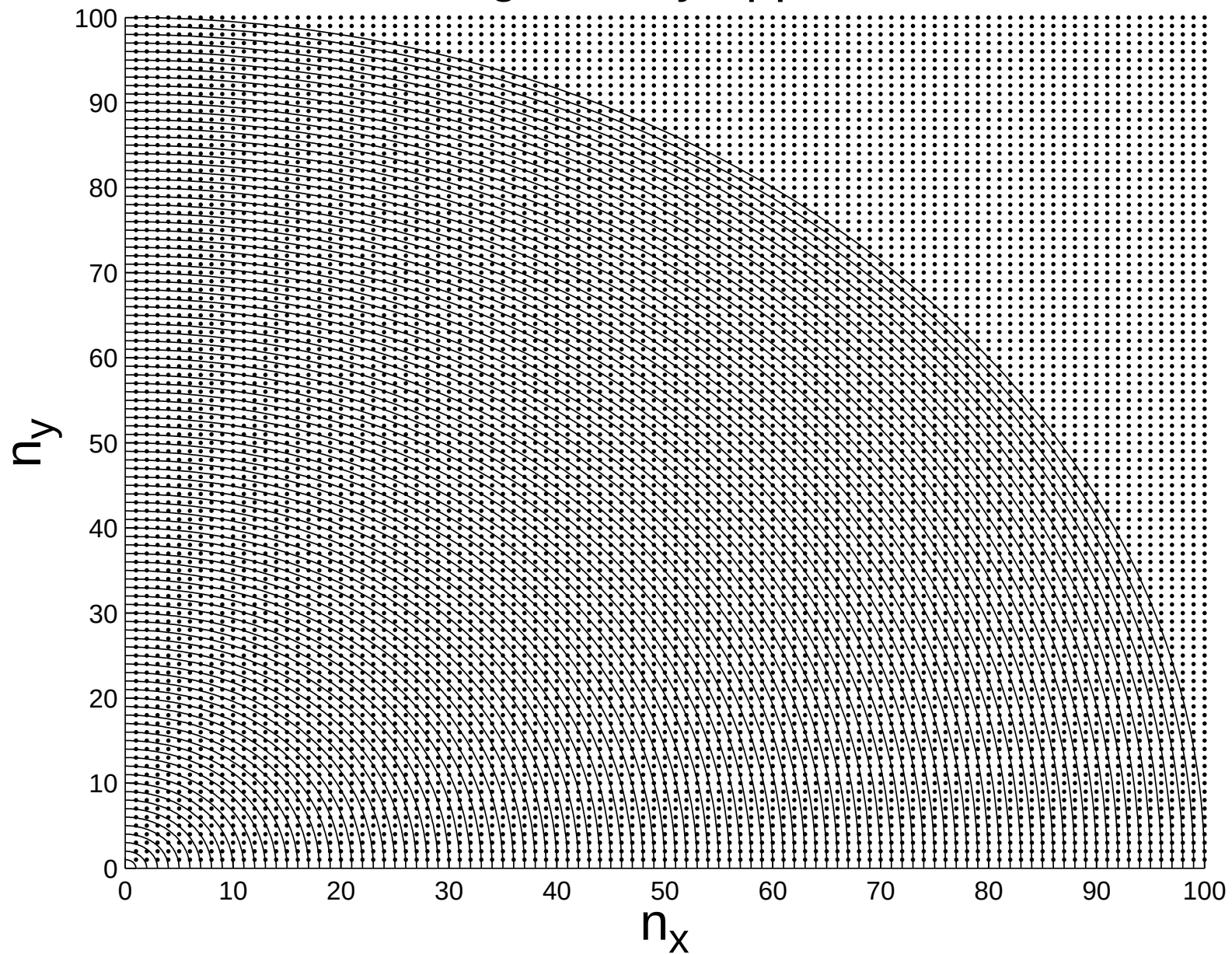
$$p_i = \frac{n_i \pi \hbar}{L} \quad (13.20)$$

- Form a 3D lattice of (n_x, n_y, n_z) points occupying the $(+, +, +)$ octant in space, since the n_i 's are positive
- L is very big (size of the universe), $\therefore$, treat the n_i as continuous
- Let $\vec{n} \equiv n_x \hat{x} + n_y \hat{y} + n_z \hat{z}$ and $n^2 = \vec{n} \cdot \vec{n}$
- The number of states, dn_f in a shell of thickness dn (all with the same momentum and energy) is given by:

$$\begin{aligned}
 dn_f &= \frac{1}{8} \frac{4\pi}{3} [(n + dn)^3 - n^3] \\
 &= \frac{\pi}{6} [n^3 + 3n^2 dn + 3n(dn)^2 + (dn)^3 - n^3] \\
 &= \frac{\pi}{6} [3n^2 dn + 3n(dn)^2 + (dn)^3] \\
 &= \frac{\pi n^2}{2} \left[1 + \frac{dn}{n} + \left(\frac{1}{3} \right) \left(\frac{dn}{n} \right)^2 \right] dn \\
 &\rightarrow \frac{\pi n^2}{2} dn \quad \text{in the large } L(n) \text{ limit} \\
 &= \frac{L^3 p_f^2}{2\pi^2 \hbar^3} dp_f \quad \text{adapting} \quad \left(p_i = \frac{n_i \pi \hbar}{L} \right) \longrightarrow \left(p_f = \frac{n_f \pi \hbar}{L} \right) \quad (13.21)
 \end{aligned}$$

The procedure is sketched on the next 6 pages.

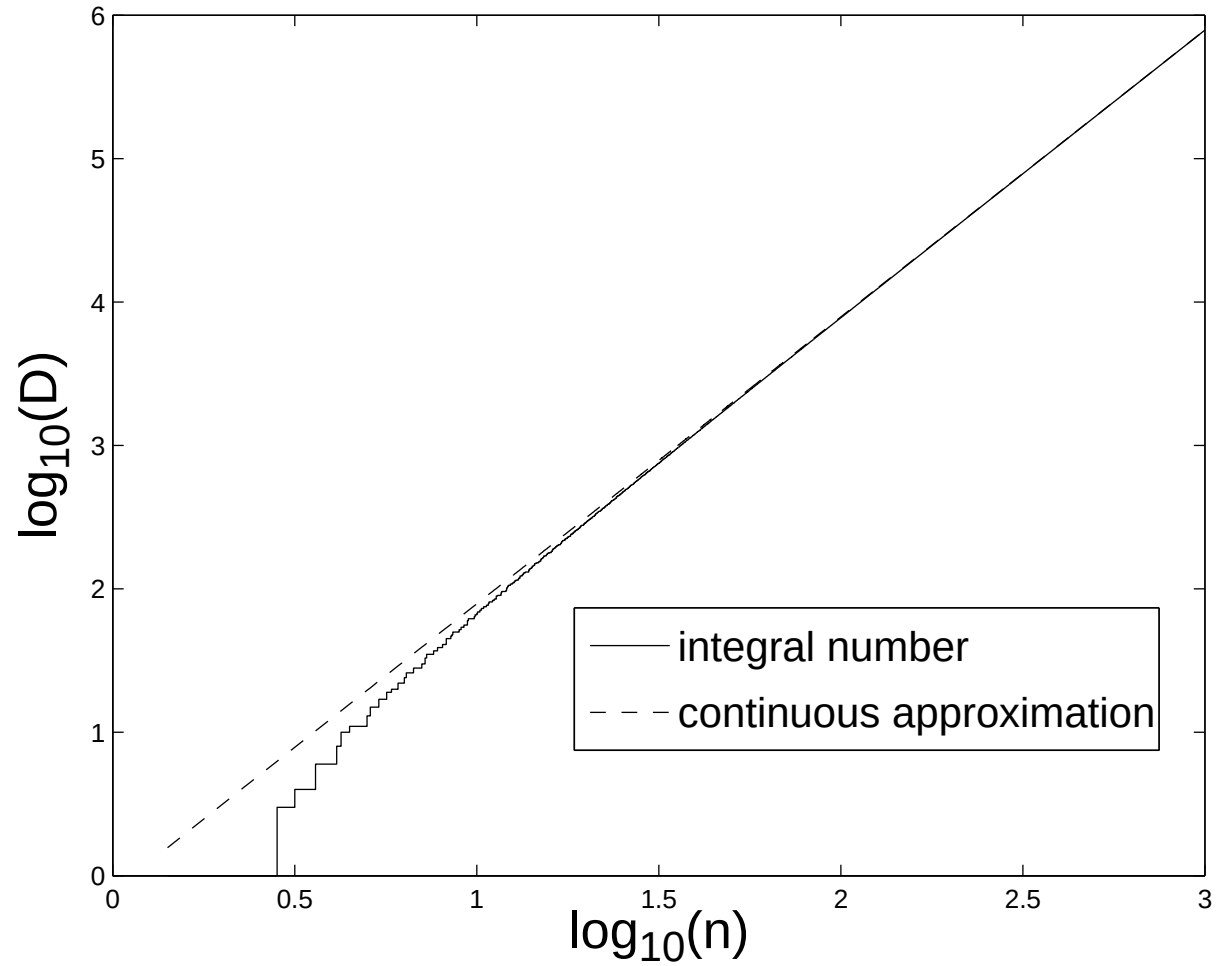
Continuous degeneracy approximation in 2D



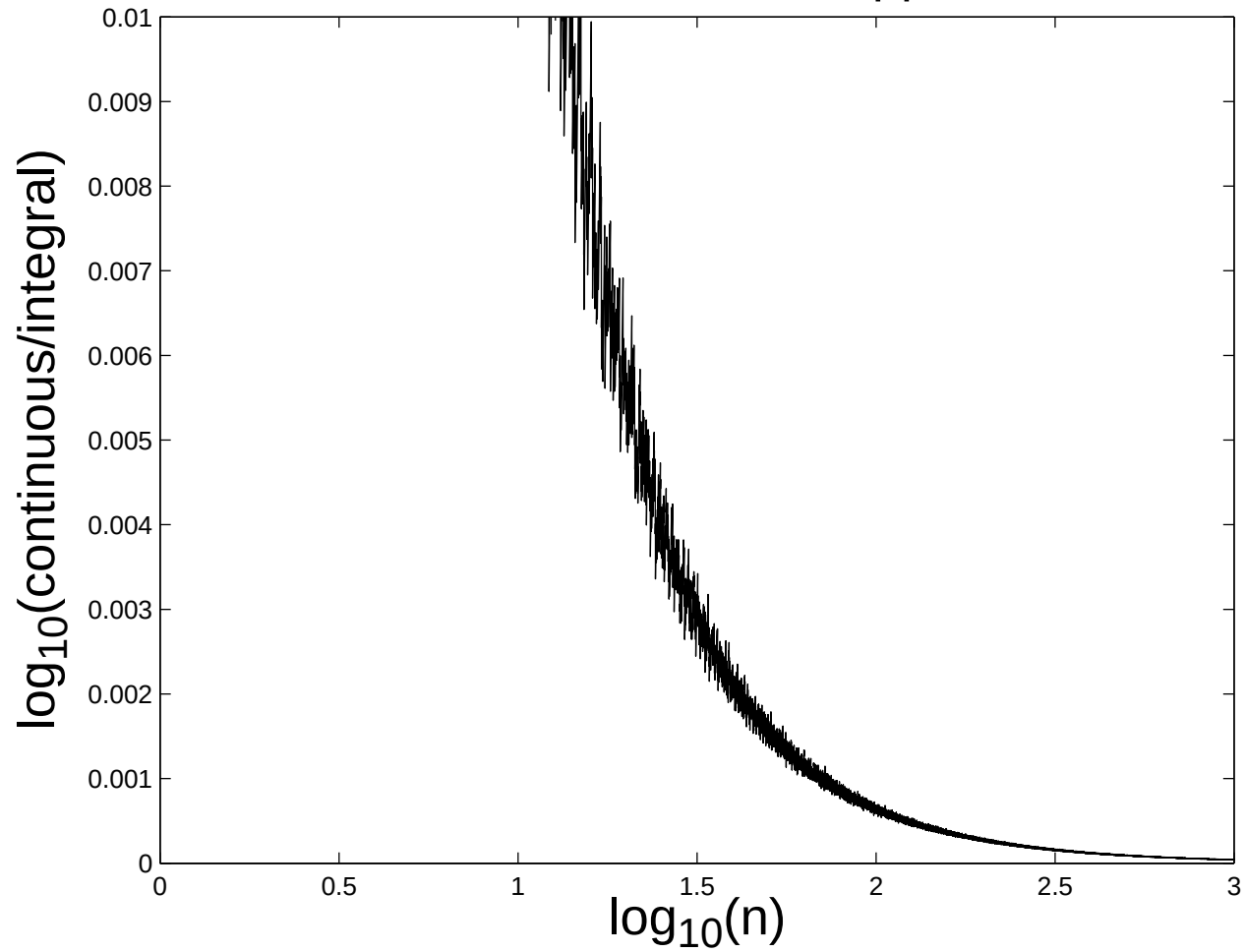
- n is a ginormous! number, of the order of 10^{15} if L is macroscopic
- n is as large as 10^{40} , if we use an L that is the size of the known universe (about 92 billion light years across)
- (Aside) The age of the universe is known to be 13.772(59) billion years, how did they measure that distance? [Wiki: “Universe”]

Regarding the plot on the previous page ...

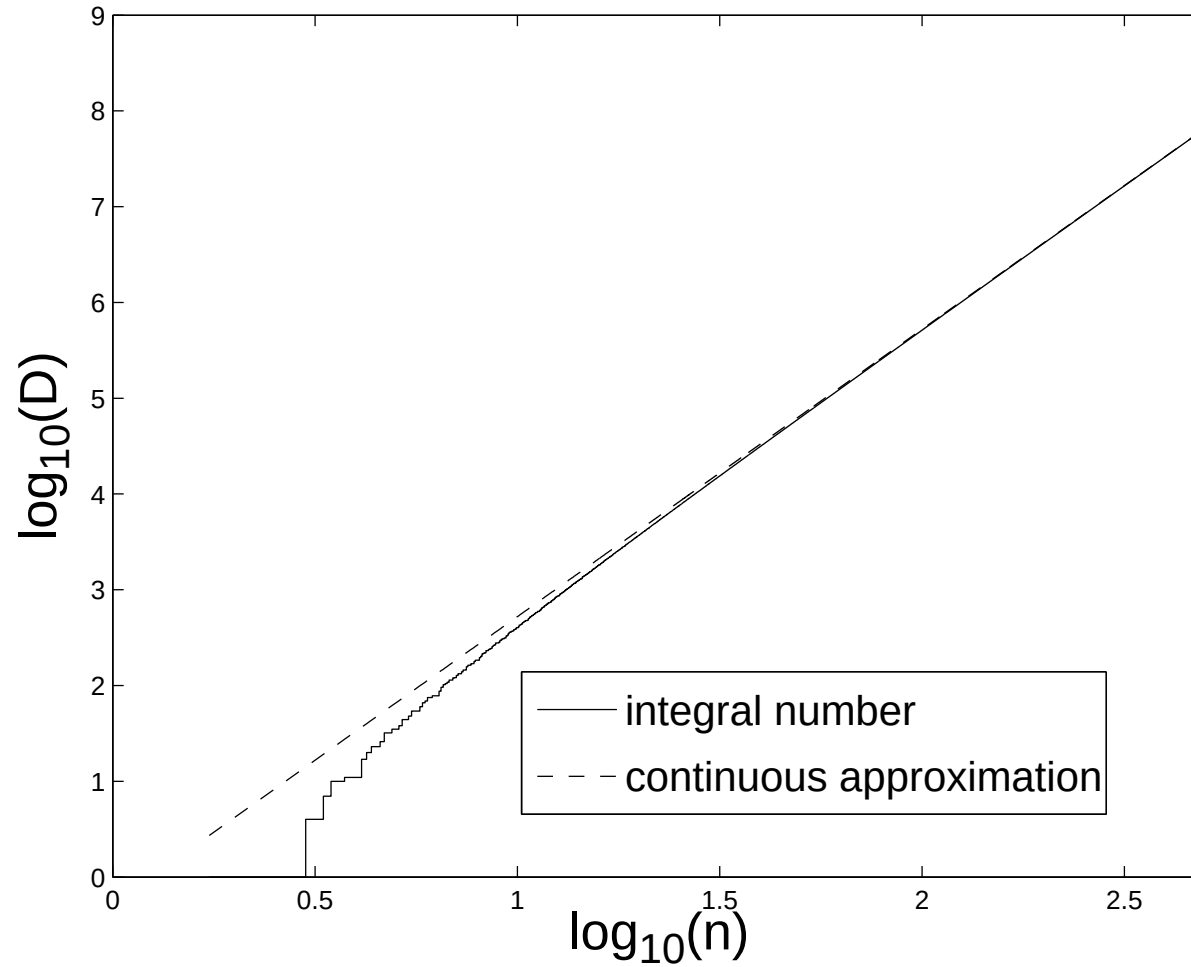
Degeneracy of energy states in a 2D Box



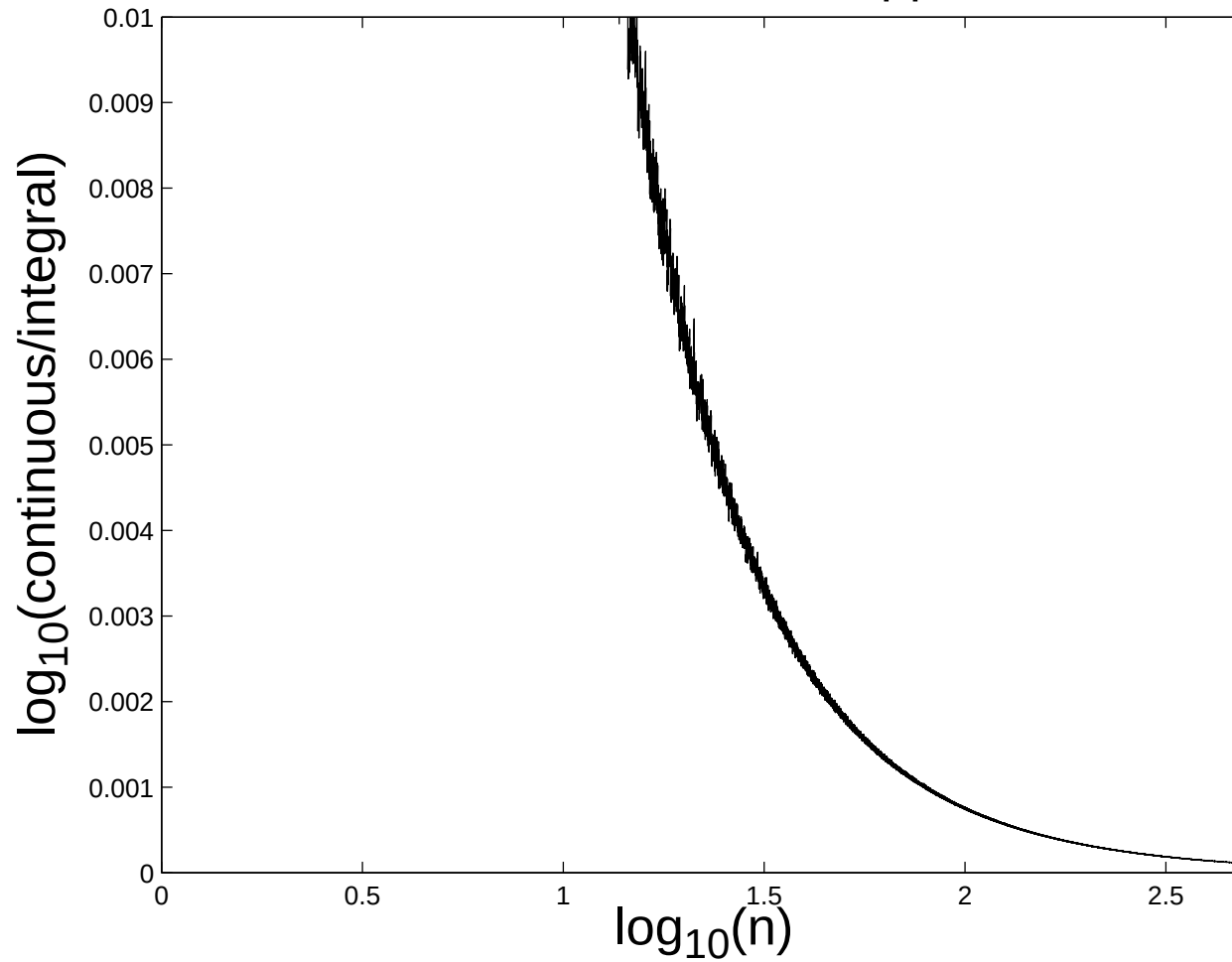
Error in the 2D continuous approximation



Degeneracy of energy states in a 3D Box



Error in the 3D continuous approximation



From (13.21)

$$dn_f = \frac{L^3 p_f^2}{2\pi^2 \hbar^3} dp_f$$

- This n_f is the degeneracy of p_f .
- The above expression is valid for quantum particles with or without mass, relativistic or not!

Finally, we write the density of states, as:

$$\therefore \frac{dn_f}{dE_f} = \left(\frac{dn_f}{dp_f} \right) \left(\frac{dp_f}{dE_f} \right) = \frac{L^3 p_f^2}{2\pi^2 \hbar^3} \left(\frac{dp_f}{dE_f} \right)$$

Using the familiar kinematic expressions,

$$E_f = \frac{p_f^2}{2m} \quad ; \quad E_f = p_f c \quad ; \quad E_f = \sqrt{(p_f c)^2 + (mc^2)^2}$$

for particles that are non-relativistic, massless, and relativistic with mass

Thus

$$\begin{aligned}\frac{dn_f}{dE_f} &= \frac{L^3}{2\pi^2\hbar^3} \sqrt{2m^3 E_f} \\ &= \frac{L^3}{2\pi^2\hbar^3} \frac{E_f^2}{c^3} \\ &= \frac{L^3}{2\pi^2\hbar^3} \frac{E_f \sqrt{E_f^2 - (mc^2)^2}}{c^3}\end{aligned}\tag{13.22}$$

for particles that are non-relativistic, massless, and relativistic with mass

γ -decay

For one photon in the final state, from (13.20)

$$E_\gamma = cp_\gamma = n\pi\hbar c/L . \quad (13.23)$$

Thus,

$$\begin{aligned} n &= \frac{E_\gamma L}{\pi\hbar c} \\ \frac{dn}{dE_\gamma} &= \frac{L}{\pi\hbar c} . \end{aligned} \quad (13.24)$$

Recognizing that $E_f = E_\gamma$, the results of (13.24) used in (13.22) gives:

$$\frac{dn_f}{dE_\gamma} = \frac{1}{2\pi^2} \frac{L^3}{(\hbar c)^3} E_\gamma^2 . \quad (13.25)$$

Specializing to γ decay, we adapt (13.15)

$$\lambda = \frac{2\pi}{\hbar} |\langle \psi_f | V_p | \psi_i \rangle|^2 \frac{dn_f}{dE_f} ,$$

using the notation of (13.16)

$$N_i^* \longrightarrow N_f + \gamma$$

$$\psi^{N_i^*} \longrightarrow \psi^{N_f} \psi^\gamma$$

$$E_i = E_f + E_\gamma + E_R ,$$

along with result of (13.25):

$$\frac{dn_f}{dE_\gamma} = \frac{1}{2\pi^2} \frac{L^3}{(\hbar c)^3} E_\gamma^2 .$$

to get:

$$\lambda_\gamma = \frac{L^3}{\pi \hbar (\hbar c)^3} |\langle \psi^{N_f} \psi^\gamma | V_p | \psi^{N_i} \rangle|^2 E_\gamma^2 .$$

- We would like to use the photon wavefunction, (13.19)

$$\psi_\gamma \equiv \psi_{n_x, n_y, n_z}(x, y, z) = \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right)$$

in our result so far

$$\lambda_\gamma = \frac{L^3}{\pi \hbar (\hbar c)^3} |\langle \psi^{N_f} \psi_\gamma | V_p | \psi^{N_i} \rangle|^2 E_\gamma^2 .$$

- We would like to preserve the norm of the $\langle \rangle$ expression above, that evaluates to the units (energy)².

Rewrite the above as:

$$\lambda_\gamma = \frac{\mathcal{O}(V_N)}{\pi \hbar (\hbar c)^3} \left| \langle \psi^{N_f} [L^{3/2} \psi^\gamma] | V_p | \psi^{N_i} \rangle \right|^2 E_\gamma^2 . \quad (13.26)$$

- $L^{3/2}$ denormalizes the photon wavefunction
- Call this denormalized wavefunction $L^{3/2} \psi^\gamma \equiv \tilde{\psi}^\gamma$
- $\tilde{\psi}^\gamma$ no units associated with it.
- $|\langle \psi^{N_f} \tilde{\psi}^\gamma | V_p | \psi^{N_i} \rangle|^2$ must have units (Energy²/Volume)
- V_N is the only volume-related metric in the calculation
- λ_γ has units s⁻¹, and is correctly dimensioned
- Most importantly, L has disappeared from the problem!

Interested students, please see the book for these advanced topics...

Derivation of Fermi's Golden Rule #2

Discussion of the “small perturbation” approximation

Derivation of the Lorentz distribution

13.3—Production and Decay of Radioactivity

Secular equilibrium

Consider a beam of radiation, with intensity I , that impinges upon a block of material containing nuclei of type “0”. There are N_0 nuclei in the target to start, and the external radiation activates the material, producing nuclear species “1”. We make two assumptions:

1. The target is “thin enough”, so that the external radiation is not attenuated.
2. The irradiation is weak enough so that N_0 does not decline.

With these assumptions, R , the rate of production of N_1 is given by:

$$R = I\sigma N_0 , \quad (13.55)$$

where σ is the radioactivity production cross section.

The differential equation describing the production of N_1 is given by:

$$dN_1 = Rdt - \lambda_1 N_1 dt , \quad (13.56)$$

where λ_1 is the decay rate of nucleus "1". Solving (13.56):

$$\begin{aligned} \dot{N}_1 + \lambda_1 N_1 &= R \\ (\dot{N}_1 + \lambda_1 N_1)e^{\lambda_1 t} &= Re^{\lambda_1 t} \\ \frac{d}{dt}(N_1 e^{\lambda_1 t}) &= Re^{\lambda_1 t} \\ N_1 e^{\lambda_1 t} - N_1(0) &= \frac{R}{\lambda_1}(e^{\lambda_1 t} - 1) \\ N_1 - N_1(0)e^{-\lambda_1 t} &= \frac{R}{\lambda_1}(1 - e^{-\lambda_1 t}) \\ N_1 &= N_1(0)e^{-\lambda_1 t} + \frac{R}{\lambda_1}(1 - e^{-\lambda_1 t}) . \end{aligned} \quad (13.57)$$

Usually the target contains no activity at $t = 0$. If $N_1(0) = 0$:

$$\begin{aligned} N_1 &= \frac{R}{\lambda_1}(1 - e^{-\lambda_1 t}) \\ \mathcal{A}_1 &= R(1 - e^{-\lambda_1 t}) . \end{aligned} \quad (13.58)$$

Condition	Beam	$\mathcal{A}_1(t)$	Description
$\lambda_1 t \ll 1$ i.e. $t \ll \tau$	ON	$R\lambda_1 t$	initial production
$\lambda_1 t \gg 1$ i.e. $t \gg \tau$	ON	R	secular equilibrium
$t \geq t_0$, t_0 is arbitrary	OFF (at $t = t_0$)	$N_1(t_0)\lambda_1 e^{-\lambda_1(t-t_0)}$	decay of activity
$t \geq t_0 \gg \tau$	OFF (at $t = t_0$)	$R e^{-\lambda_1(t-t_0)}$	decay of activity (from secular equilibrium)

During the condition of *secular equilibrium*, the rate of production is the same as the rate of decay, producing an unchanging number of radioactive daughter nuclei. (There will be some statistic fluctuation of this number.)

A depiction of a nuclide reaching secular equilibrium is shown in Figure 13.3.

Rise to secular equilibrium/early beam shut-off

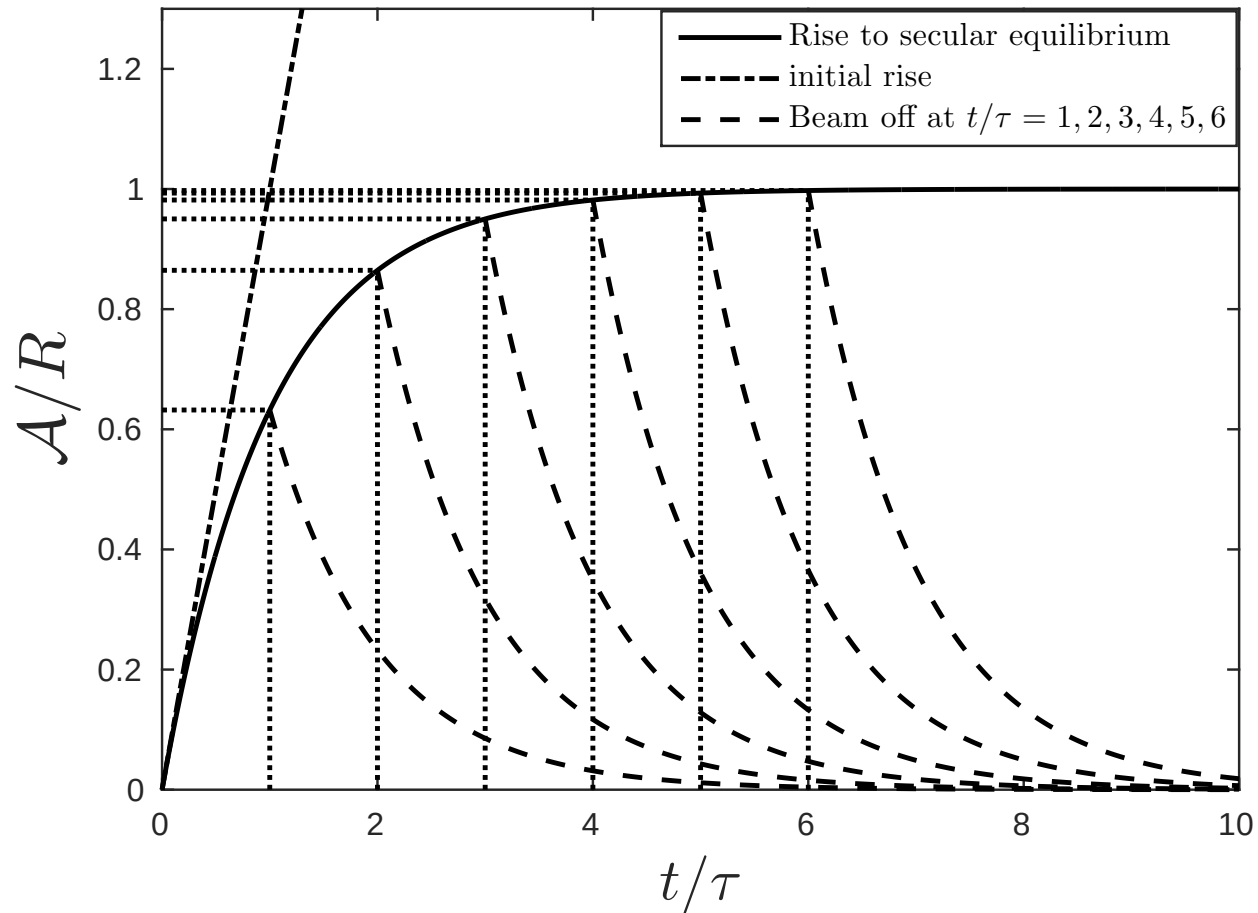


Figure 13.3: A nuclide reaching secular equilibrium is depicted. Also depicted are the decays of the source once the beam is shot off at $t/\tau = 1, 2, 3, 4, 5$.

A real-life engineering application

A typical engineering challenge, in the area of creating radioactive sources, is to minimize the cost of producing a given amount of activity. We model this as follows:

The cost, C , per unit activity, factoring start-up costs, S_0 , (manufacture of the inactive source, delivery costs, operator start-up and take-down), and the cost of running the accelerator or reactor, per unit meanlife, R_0 , is given as follows:

$$C = \frac{S_0 + R_0 x_0}{1 - e^{-x_0}}, \quad (13.59)$$

where x_0 is the number of meanlives that the target is irradiated.

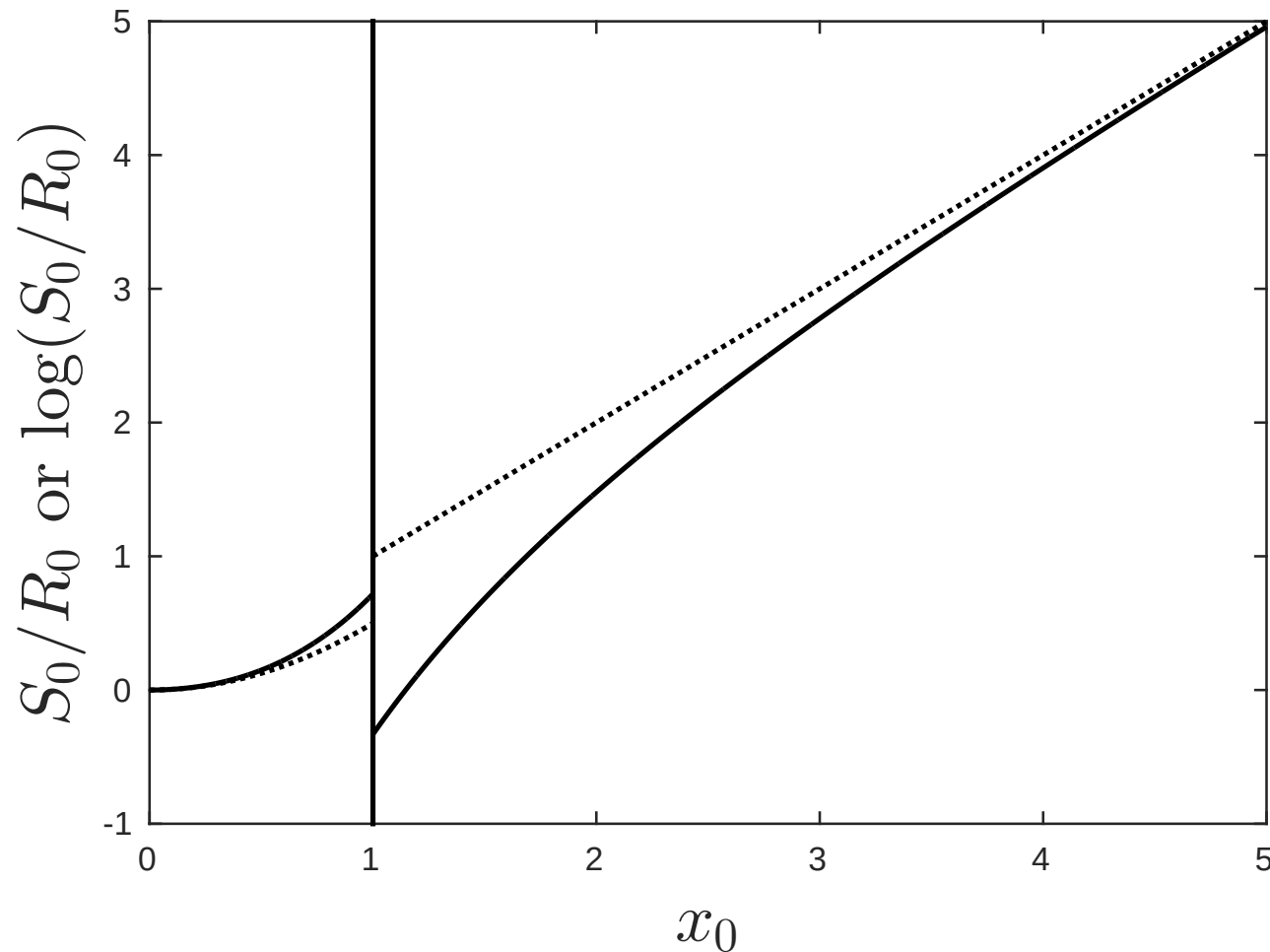
The optimization condition is given by:

$$S_0/R_0 = e^{x_0} - (1 + x_0). \quad (13.60)$$

For small S_0/R_0 , the optimum $x_0 \approx \sqrt{2S_0/R_0}$. For large S_0/R_0 , the optimum $x_0 \approx \log(S_0/R_0)$.

The solutions for the optimum value of x_0 can be obtained from the figure shown on the next page.

Optimum irradiation time



Optimization curves for determining x_0 , as a function of $\alpha = S_0/R_0$.

The figure has been split into two parts, $x_0 < 1$ for which S_0/R_0 is shown, and $x_0 > 1$ for which $\log(S_0/R_0)$ is shown. The approximations discussed above, are plotted as dotted lines.

13.4—Growth of Daughter Activities

Parent $\Rightarrow$ Daughter $\Rightarrow$ Granddaughter (stable)

We now describe the simplest decay *chain*, whereby a parent decays to an unstable daughter, that decay to a stable granddaughter. We'll use Krane's notation:

Differential equation	At $t = 0$	Description
$\dot{N}_1 = -\lambda_1 N_1$	$N_1(0) = N_0$	N_1 : Parent—rate constant λ_1 (decay only)
$\dot{N}_2 = \lambda_1 N_1 - \lambda_2 N_2$	$N_2(0) = 0$	N_2 : Daughter—growth, decay rate constant λ_2
$\dot{N}_3 = \lambda_2 N_2$	$N_3(0) = 0$	N_3 : Granddaughter—growth only
$\dot{N}_1 + \dot{N}_2 + \dot{N}_3 = 0$		“Conservation of particles”

The integrals are elementary, giving, for the N_i 's:

$$\begin{aligned} N_1 &= N_0 e^{-\lambda_1 t} \\ N_2 &= N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \\ N_3 &= N_0 - N_1 - N_2 \end{aligned} \tag{13.61}$$

In this discussion, we concern ourselves mostly with the activity of the daughter, in relation to the parent, namely:

$$\mathcal{A}_2 = \mathcal{A}_1 \frac{\lambda_2}{\lambda_2 - \lambda_1} (1 - e^{-(\lambda_2 - \lambda_1)t}) , \quad (13.62)$$

and consider some special cases.

Very long-lived parent: $\lambda_1 \lll \lambda_2$

In this case, the parent's meanlife is considered to be much longer than that of the daughter, essentially infinite within the time span of any measurement of interest. In other words, the activity of the parent is constant. In this case, (13.62) becomes:

$$\mathcal{A}_2 = \mathcal{A}_1(0)(1 - e^{-\lambda_2 t}) . \quad (13.63)$$

Comparing with (13.58), we see that (13.63) describes $\mathcal{A}_2$'s rise to secular equilibrium, with effective rate constant, $\mathcal{A}_1(0)$.

Long-lived parent: $\lambda_1 \ll \lambda_2$

In this case, the parent lives much longer than the daughter, but the parent does have a measurable decline, within the time span of the measurement.

We rewrite (13.62)

$$\mathcal{A}_2 = \mathcal{A}_1 \frac{\lambda_2}{\lambda_2 - \lambda_1} (1 - e^{-(\lambda_2 - \lambda_1)t}) ,$$

slightly as:

$$\mathcal{A}_2(t) = \mathcal{A}_1(t) \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} (1 - e^{-(\lambda_2 - \lambda_1)t}) \right) . \quad (13.64)$$

(13.64) describes $\mathcal{A}_2(t)$ a modulation of $\mathcal{A}_1(t)$.

An example is shown in figure 13.4, on the next page.

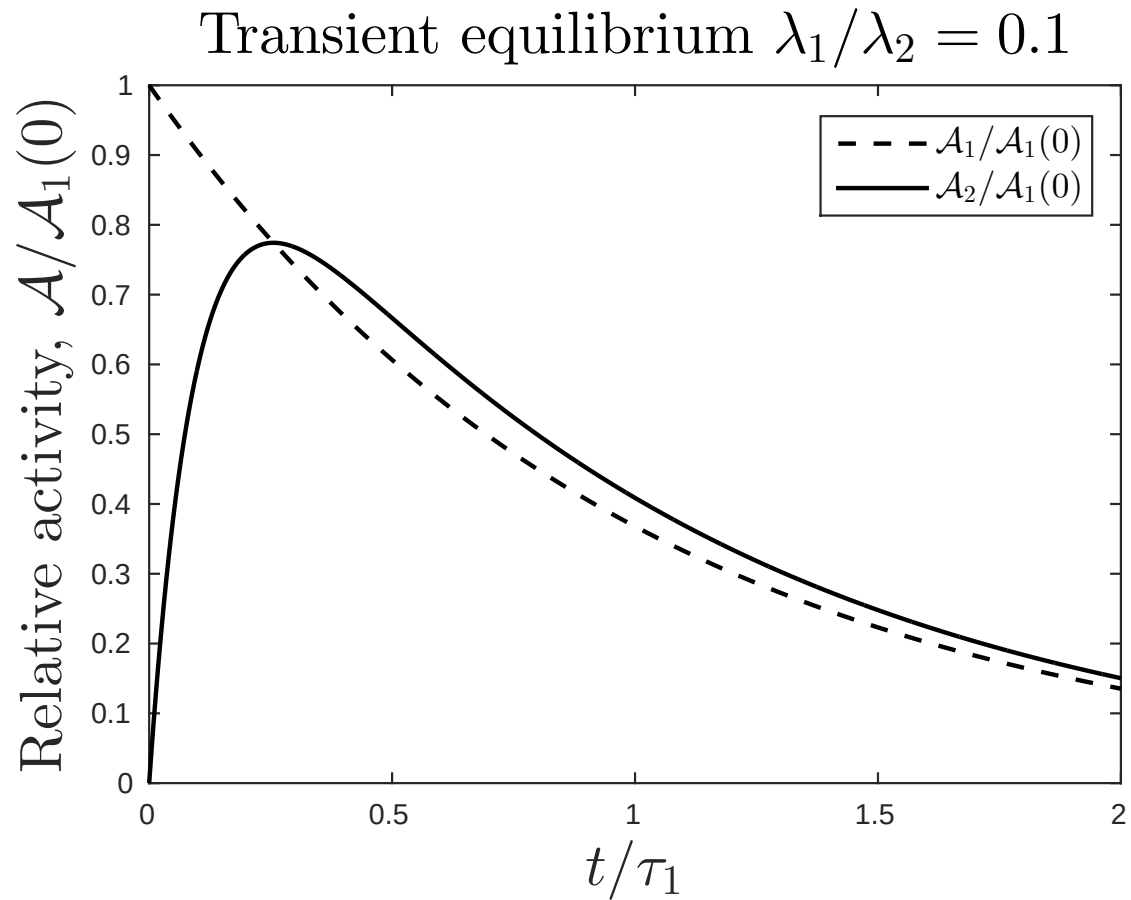


Figure 13.4: The relative activity of the daughter, $\mathcal{A}_2$ and the parent, $\mathcal{A}_1$, for $\lambda_2 = 10\lambda_1$.

We see, from the figure, that the activity of the daughter, $\mathcal{A}_2$, rises quickly to match the parent, and then follows the parent activity closely.

This latter condition, beyond a few meanlives of the daughter, is called the region of *transient secular equilibrium*, or more commonly, *transient equilibrium*.

In the region of transient equilibrium, the activity of the daughter is always slightly greater than that of the parent, with a temporal offset of about τ_2 .

Special case: $\lambda = \lambda_1 = \lambda_2$

In this case, parent and daughter decay with the same meanlife. This is more of a mathematical curiosity, but should it occur, the result can be derived from (13.64) by assuming $\lambda_1 = \lambda$, $\lambda_1 = \lambda + \epsilon$, performing a series expansion in ϵ , and taking the $\epsilon \rightarrow 0$ at the end. The result is:

$$\mathcal{A}_2(t) = \mathcal{A}_1(t)\lambda t . \quad (13.65)$$

Series of Decays

In this case, we consider a series of descendants, $N_1 \Rightarrow N_2 \Rightarrow N_2 \cdots \Rightarrow N_n$, with rate constants, $\lambda_1, \lambda_2 \cdots \lambda_n$. That is, there are n generations, starting with the parent, N_1 , and ending with a final stable (grand) ^{$n-2$} daughter, for $n \geq 2$.

The result of the solution of the differential equations:

Differential equation	At $t = 0$	Description
$\dot{N}_1 = -\lambda_1 N_1$	$N_1(0) = N_0$	N_1 : Parent—rate constant λ_1 (decay only)
$\dot{N}_i = \lambda_{i-1} N_{i-1} - \lambda_i N_i$	$N_i(0) = 0$	N_i : (grand) ^{$n-2$} , decay rate constant λ_i
$\dot{N}_n = \lambda_n N_n$	$N_n(0) = 0$	N_n : Stable end of chain (growth only)
$\sum_{i=1}^n \dot{N}_i = 0$		“Conservation of particles”

The result is given by the *Bateman equations* (note the exclusions in the product terms.):

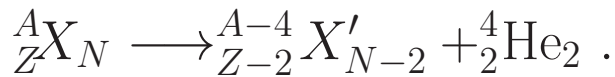
$$\begin{aligned}
 N_1 &= N_0 e^{-\lambda_1 t} \\
 N_{1 < i < n} &= N_0 \sum_{j=1}^i \left(\frac{\prod_{k=1; k \neq i}^i \lambda_k}{\prod_{k=1; k \neq j}^i (\lambda_k - \lambda_j)} \right) e^{-\lambda_j t} \\
 N_n &= N_0 - \sum_{i=2}^{n-1} N_i .
 \end{aligned} \tag{13.66}$$

13.5—Types of Decays

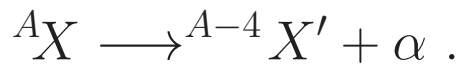
Details are covered elsewhere in this course. Here we just give a list of relevant decays for Nuclear Engineering and Radiological Sciences.

α decay

This is the form employed when an accurate mass calculation is to be performed. Electron masses cancel, and the small differences in electron binding energy are ignored.



Usually the following shorthand is employed:



Applications of α -emitters:

- “Gadget” initiators (α -decay followed by n -emission.)
- Search for superheavy elements. See
!http://t2.lanl.gov/tour/shn.html!
- Smoke detectors.
- Power generation in space probes and artificial hearts.
- Unsealed source radiotherapy.
- Can be used to reduce static cling.
Really! I’m not kidding! The real question is “How?”

β decay

$$n \longrightarrow p + e^{-} + \bar{\nu}_e \quad \beta^{-} \text{ decay}$$

$${}^A_Z X_N \longrightarrow {}^A_{Z+1} X'_{N-1} + e^{-} + \bar{\nu}_e$$

$$p \longrightarrow n + e^{+} + \nu_e \quad \beta^{+} \text{ decay}$$

$${}^A_Z X_N \longrightarrow {}^A_{Z-1} X'_{N+1} + e^{+} + \nu_e$$

$$p + e^{-} \longrightarrow n + \nu_e \quad \text{electron capture } (\varepsilon)$$

$${}^A_Z X_N \longrightarrow {}^A_{Z-1} X'_{N+1} + \nu_e$$

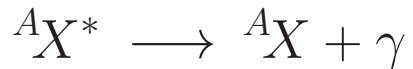
- A free neutron will β^{-} decay with a meanlife of 886.7(8) s. A neutron in a nucleus will β^{-} decay, but only when that process is favorable energetically.
- Free protons do not decay, that is, it has never been observed. Proton decay is predicted by Grand Unified Theories (GUTs). However, the predicted probability of decay is exceedingly small. A lower bound for proton decay has been established experimentally, setting the half-life at greater than 6.6×10^{33} years. This is interesting, but of little consequence for Nuclear Engineering. Protons in a nucleus, if favored energetically, do β^{+} decay.

- Finally, there is a process called *electron capture*, (ε), or *K-capture*, whereby a proton in a nucleus captures an orbital electron (usually from a $1s$ atomic orbital, and converts itself to a neutron.
- All these process result in a *electron neutrino*, ν_e , or an *electron antineutrino*, $\bar{\nu}_e$. By convention, antiparticles, like the antiproton, $\bar{p}$, and the antineutron, $\bar{n}$, are written with an overline $\bar{}$ or an “overtilde” $\sim$. There are exceptions to this rule. The positron, e^+ is the e^- ’s antiparticle. However, it is never written as $\overline{e^-}$.

Some applications of β -emitters:

- Betavoltaics (non-thermal) (long-life, low-power batteries).
- Radiotherapy (brachytherapy).
- PET (Positron Emission Tomography.)
- Radiopharmaceuticals.
- Quality assurance in large-scale paper production.
- Irradiation of domestic ruminant (cattle, goats, sheep, bison, deer, camels, alpacas, llamas) behinds to cure the effects of “fly strike”. (I’m not kidding about this one either.) (And I’d really rather not delve into the details of “fly strike”. Kindly google this one on your own.)

γ decay

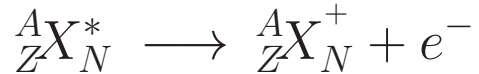


Here, a nucleus in an excited state, denoted by the asterisk, decays via the γ process, to a lower excited state, or the ground state. All nuclei that are observed to have excited states, ($A > 5$), have γ transitions.

Applications of γ -emitters:

- Basic physics: Nuclear structure, astrophysics.
- Radiotherapy, ${}^{60}\text{Co}$, ${}^{137}\text{Cs}$, brachytherapy..
- Sterilization of pharmaceutical products, food.
- Imaging vehicles for National Security Administration purposes.
- Industrial quality assurance.
- Discovery of oil. (Oil-well logging.)

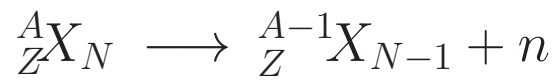
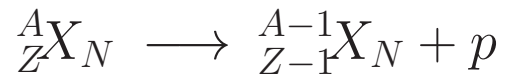
Internal conversion



Here, a nucleus in an excited state (denoted by the $*$, de-excites by exchanging a virtual photon with a K -shell electron in a “close encounter” with the nucleus. The electron acquires the de-excitation energy and exits the nucleus, leaving the atom in a singly-charged state.

Since it is usually a K -shell electron that is usually ejected, the atom is also in an excited state, followed by fluorescence and Auger electron emission, to de-excite the atom.

Nucleon emission



Applications of nucleon emitters:

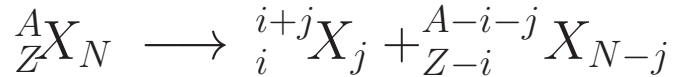
- Fission.

Spontaneous fission

$${}^A_ZX_N \longrightarrow \sum_{i,j} c_{i,j} [{}^{i+j}_iX_j + {}^{A-i-j}_{Z-i}X_{N-j}]$$

Here a nucleus fractures into 2 (equation shown) or more (equation not shown) nuclei. This is similar to “normal” fission, except that it occurs spontaneously. Generally a spectrum of nuclei result, with the probabilities given by the c coefficients.

Cluster decay



where $i, j > 2$.

From:

http://en.wikipedia.org/wiki/Cluster_decay!...

Cluster decay is a type of nuclear decay in which a radioactive atom emits a cluster of neutrons and protons heavier than an alpha particle. This type of decay happens only in nuclides which decay predominantly by alpha decay, and occurs only a small percentage of the time in all cases. Cluster decay is limited to heavy atoms which have enough nuclear energy to expel a portion of its nucleus.

13.8—Units for Measuring Radiation

Some useful radiometric quantities are listed in Table 13.1, along their traditional (outdated but still in some use), along with their new, almost universally adopted, SI.

Quantity	Measures...	Old Units	New (SI) Units
Activity ($\mathcal{A}$)	decay rate	curie (Ci)	becquerel (Bq)
Exposure (X)	ionization in air	röntgen (R)	C/kg
Absorbed Dose (D)	Energy absorption	rad	gray ($\text{Gy} = \text{J/kg}$)
Dose Equivalent (DE)	Radiological effectiveness	rem	sievert ($\text{Sv} = \text{J/kg}$)

Table 13.1: Units for the measurement of radiation

SI units	Definition	Notes:
Bq	1 “decay” /s	derived
Gy	1 J/kg	derived
Sv	1 J/kg	derived
Traditional units	Conversion	Notes:
Ci	3.7×10^{10} Bq (exactly)	Charge/(1g) of dry, STP air
R	$1 \text{ esu}/(0.001293 \text{ g}) = 2.58 \times 10^{-4} \text{ C/kg}$	
rad	$1 \text{ ergs/g} = 10^{-2} \text{ Gy}$	
rem	$1 \text{ rem} = 10^{-2} \text{ Sv}$	

Table 13.2: Conversion factors

A derived unit is one that is based upon the several base units in SI, namely: length ([m], meter), mass ([kg], kilogram), time([s], seconds) current([A], ampere), temperature ([K], degrees in Kelvin), amount of substance ([mol],mole), luminosity ([cd], candela).

Activity

Activity ($\mathcal{A}$) has been covered already. However, the units of measurement were not discussed. The traditional unit, the *curie*, (Ci), was named in honor of Marie Curie, and the modern unit in honor of Henri Bequerel.

Exposure

Exposure, given the symbol X , is defined as

$$X = \lim_{\Delta m \rightarrow 0} \frac{\Delta Q}{\Delta m}, \quad (13.67)$$

where ΔQ is the amount of charge of one sign produced in, dry, STP air. (NIST definition: STP at STP: 20°C (293.15 K, 68°F, and an absolute pressure of 101.325 kPa (14.696 psi, 1 atm).) There are no accepted derived SI unit for exposure. Exposure measurements are probably the most accurately measured radiometric quantity.

Absorbed Dose

Absorbed dose measures the energy absorbed in matter, due to radiation. The traditional unit, the rad, has been supplanted by the Gy.

Dose Equivalent

The tradition unit, the rem (for röntgen equivalent man) and its modern counterpart, the Sv, attempts to accounts for the radiological damage from different “qualities” (species of particle imparting the dose) of radiation. Quality factors dependent on the energy of the radiation and are given in the Table below.

Radiation	Energy	QF
X -rays, γ , $e^\pm$, $\mu^\pm$	all	1
p (non-recoil)	> 1 MeV	5
n	< 10 keV	5
	10–100 keV	10
	10 keV – 2 MeV	20
	2–20 MeV	10
	> 20 MeV	5
α , fission fragments, heavy nuclei	all	20

Units for the measurement of radiation

The conversion from D to DE is given by:

$$DE = \sum_i \int dE D_i(E) \times QF_i(E) , \quad (13.68)$$

where we sum over radiation types, and integrate over the energy of the radiation that imparts dose.

Things to think about ...

1. Radioactive Decay

- (a) Given a decay rate λ , derive the “exponential decay law”.
- (b) What is “lifetime”, or “mean lifetime”? Derive an expression for it.
- (c) What is “Activity”?
- (a) Show diagrams and differential equations for:
 - i. Decay from one isotope, with two decay channels with stable daughters.
 - ii. Decay from two isotopes, to independent decay channels

2. What is secular equilibrium? Derive the relation that describes it.

3. What do the Bateman equations describe?

4. Consider the case of a radioactive parent, decay rate λ_1 that decays to a radioactive daughter, decay rate λ_2 .

- (a) Solve the differential equation.
- (b) Discuss the limit for a very long-lived parent: $\lambda_1 \lll \lambda_2$ and show that it contains the secular equilibrium condition.
- (c) Discuss the limit for a long-lived parent: $\lambda_1 \ll \lambda_2$ and plot the activity of the parent and the daughter.

...Things to think about

5. Quantum decay theory

- (a) What is the Lorentz distribution? What does it mean?
- (b) State Fermi's Golden Rule #2
- (c) What does the density of final states describe?
- (d) **extra credit** Derive the density of final states for relativistic particles with and without mass, and non-relativistic particles.

6. Types of decays ... Describe and give at least one application of:

- (a) α decay
- (b) β decay
- (c) γ decay
- (d) Internal conversion
- (e) Nucleon emission
- (f) Spontaneous fission
- (g) Cluster decay

7. Radiological physics. What is:

- (a) Exposure
- (b) Absorbed Dose
- (c) Dose Equivalent