

NERS 312

Elements of Nuclear Engineering and Radiological Sciences II

aka **Nuclear Physics for Nuclear Engineers**

Lecture Notes for Chapter 16: γ decay

Supplement to (Krane II: Chapter 10)

The lecture number corresponds directly to the chapter number in the online book.

The section numbers, and equation numbers correspond directly to those in the online book.

γ decay

α

β

γ

Chapter 16: In this Chapter you will learn

Chapter 16.0: Basic introduction to γ decay

- A comparison of α decay, β decay, and γ decay
- The foundations of Quantum Electrodynamics
- Why does γ decay take so long?

Chapter 16.1: Energetics of γ decay

- Obtaining Q from atomic mass tables

Chapter 16.2: Classical Electromagnetic Radiation

- Multipole expansions
- Electric dipoles and quadrupoles
- Magnetic dipoles
- Characteristics of multipolarity
- Parity
- Power radiated
- A general and more sophisticated treatment of classical multipole fields

Chapter 16.3: Transition to Quantum Mechanics

- The Weisskopf Estimates

Chapter 16.4: Angular momentum and parity selection rules

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Chapter 16.6: Internal Conversion

- Internal conversion, β decay and internal conversion energetics
- Internal conversion contribution to decay rate

Chapter 16: γ decay

So far we have discussed α decay and β decay modes of de-excitation of a nucleus. As seen below, most β -decay transitions are followed by γ decay.

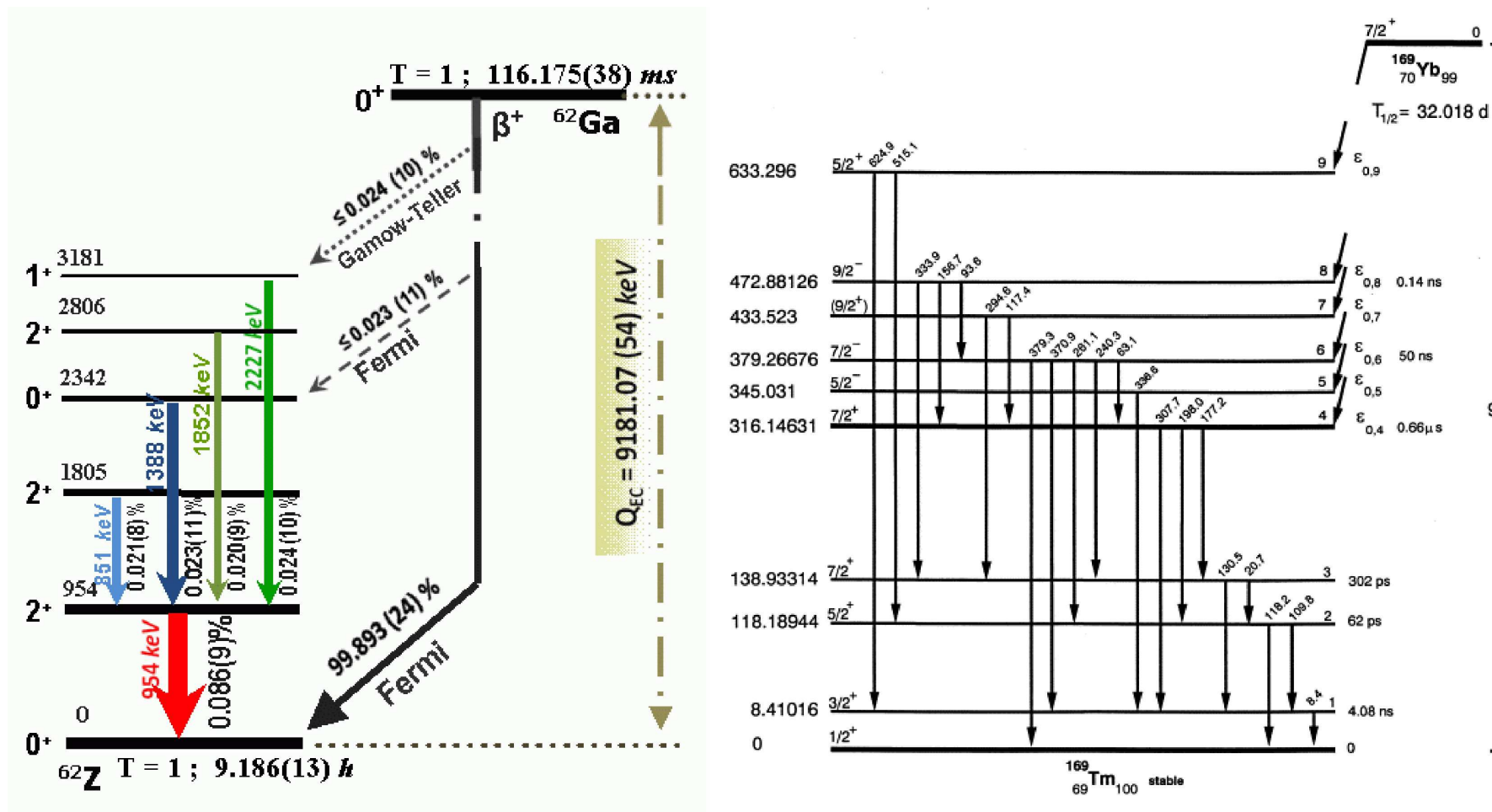


Figure 1: (Gallium) $^{62}\text{Ga} \rightarrow ^{62}\text{Zn}$ (Zinc) and (Ytterbium) $^{169}\text{Yb} \rightarrow ^{169}\text{Th}$ (Thulium) β^+ decays, followed by γ decay.

The transition equations in (1), indicate explicitly, that the daughter nucleus may be in an excited state.

This is, in fact, the usual case.

The most common form of nuclear de-excitation is via γ decay, the subject of this chapter.

$$\begin{array}{ll} {}^A_Z X_N \longrightarrow {}^{A-4}_{Z-2} X_{N-2}^* + \alpha & [\alpha - \text{decay}] \\ {}^A_Z X_N \longrightarrow {}^A_{Z\pm 1} X_{N\mp 1}^* + e^\mp + (\bar{\nu}_e/\nu_e) & [\beta^\mp \text{ decay}] \end{array} \quad (1)$$

A comparison of α decay, β decay, and γ decay

Now that we are discussing that last decay mode process, it makes sense to compare them.

Most naturally radioactive nuclei de-excite via an α decay.

The typical α -decay energy is 5 MeV, and the common range between 4 and 10 MeV.

The α -particle's "reduced" de Broglie wavelength is:

$$\lambda = \frac{\hbar}{p} = \frac{\hbar c}{pc} = \frac{\hbar c}{\sqrt{T(T + 2mc^2)}} \quad (2)$$

For the α -particle, the $\lambda \approx 1.02$ fm, with a range of about 0.72–1.14 fm.

This dimension is somewhat smaller than the nucleus, and this is why a semi-classical treatment of α decay is successful.

One can reasonably talk about α -particle formation at the edge of the nucleus, apply quantum tunneling methods, and predict the escape probability and half-life.

Another consequence is that, angular momentum carried off by the α -particle does not have a drastic effect on the decay rate, unlike β decay, or, as we shall see, γ decay.

β decay

β decay, on the other hand, involves any energy up to the reaction Q , typically 1 MeV, but ranging from a few keV to tens of MeV.

Thus, the typical λ is about 140 fm, and ranges from 10 fm up.

Thus, the typical β -particle has a large λ in comparison to the nuclear size, and a quantum mechanical approach is dictated, and wave analysis is called for.

Orbital angular momentum has a great effect on decay rate, as seen in the previous chapter.

The difference in decay rate from $L = 0$ (allowed transitions) to $L = 4$ (4th forbidden transition) spans 20 orders of magnitude.

γ decay

γ decay is now under our microscope. the typical γ decay is 1 MeV and ranges from about 0.1 – 10 MeV. The typical λ is about 40 fm, and ranges from about 20 – 2000 fm.

Clearly, only a quantum mechanical approach has a chance of success.

We expect, and shall confirm, that as $l \uparrow$, $\lambda \downarrow$, dramatically!

Fortunately, the development of Classical Electrodynamics was very mature by the time Quantum Mechanics was discovered. The “wave mechanics” of photons, meshed very easily with the wave mechanics of particles.

Non-relativistic and relativistic Quantum Electrodynamics started from a very solid classical wave-friendly basis.

The foundations of Quantum Electrodynamics

γ spectroscopy yields some of the most precise knowledge of nuclear structure, as spin, parity, and ΔE are all measurable.

Borrowing from our knowledge of atomic physics and working within the shell-model of the nucleus, we can write (in a formal fashion, at least) the nuclear wave function as a composite of single-particle nucleon wavefunctions, as follows:

$$\Psi_N = \prod_{a=1}^A C(l_a, s_a, t_a) \psi_a(l_a, s_a, t_a) , \quad (3)$$

where the composite wavefunction, Ψ_N , is the product of a combinatorial factor C , that combines the states as a collection of fermions. C combines the individual orbital (l) and spin (s) angular momenta, and isospin (t) quantum numbers.

Isospin?

Isospin is a quantum number analogous to intrinsic spin.

Recognizing that the strong force is almost independent of nucleon type, the two nucleon states are “degenerate” much like spin states in atomic physics.

So, we can assign a conserved quantum number, called isospin. By convention, a proton has $t_p = \frac{1}{2}$, while a neutron has $t_n = -\frac{1}{2}$.

Transition rates between initial, Ψ_N^* and final, Ψ'_N , nuclear states, resulting from an electromagnetic decay producing a photon with energy, E_γ , can be described by Fermi's Golden Rule #2:

$$\lambda = \frac{2\pi}{\hbar} |\langle \Psi'_N \psi_\gamma | \mathcal{O}_{\text{em}} | \Psi_N^* \rangle|^2 \frac{dn_\gamma}{dE_\gamma}, \quad (4)$$

where $\mathcal{O}_{\text{em}}$ is the electromagnetic transition operator, and (dn_γ/dE_γ) is the density of final states factor.

$$\lambda = \frac{2\pi}{\hbar} |\langle \Psi'_N \psi_\gamma | \mathcal{O}_{\text{em}} | \Psi_N^* \rangle|^2 \frac{dn_\gamma}{dE_\gamma}$$

The photon wavefunction, ψ_γ and $\mathcal{O}_{\text{em}}$ are well known, therefore, measurements of λ provide detailed knowledge of nuclear structure. Theorists attempt to model Ψ_N .

The degree to which their predictions agree with experiments, indicates how good the models are, and this leads us to believe we understand something about nuclear structure.

A γ -decay lifetime is typically 10^{-12} seconds or so, and sometimes even as short as 10^{-19} seconds. However, this time span is an *eternity* in the life of an excited nucleon. It takes about 4×10^{-22} seconds for a nucleon to cross the nucleus.

Why does γ decay take so long?

There are several reasons:

1. We have already seen that the photon wavelength, from a nuclear transition, is many times the size of the nucleus. Therefore there is little overlap between the photon's wavefunction and the nucleon wavefunctions, the nucleon that is responsible for the emission of the photon. The reduction factor is approximately $(\lambda/R_N)^3$.
2. The photon carries off at least one unit of angular momentum. Therefore, the transition involves some degree of nucleon re-orientation. That is, Ψ'_N does not closely resemble Ψ_N^* , the disparity growing with increased j_γ . (The γ has one integral unit of intrinsic spin.)
3. The electromagnetic force is relatively weak compared to the strong force, about a factor of 137.

Chapter 16.1: Energetics of γ decay

A typical γ decay is depicted in Figure 2.

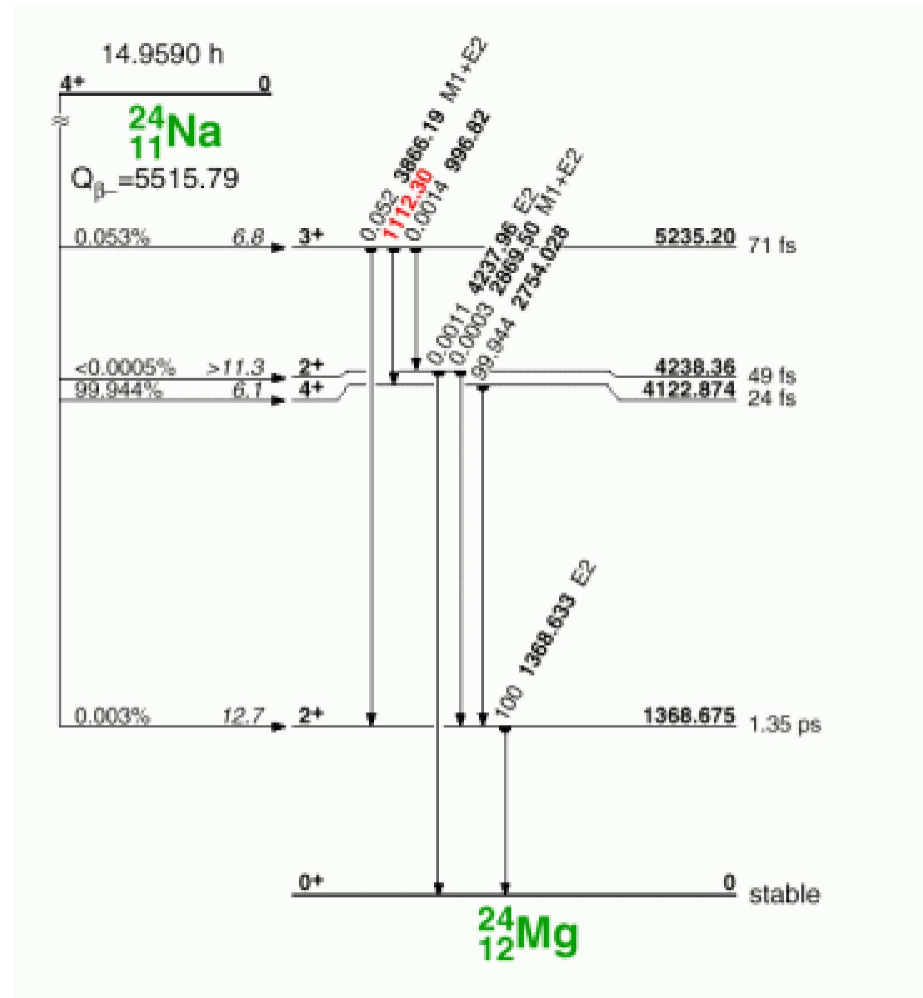
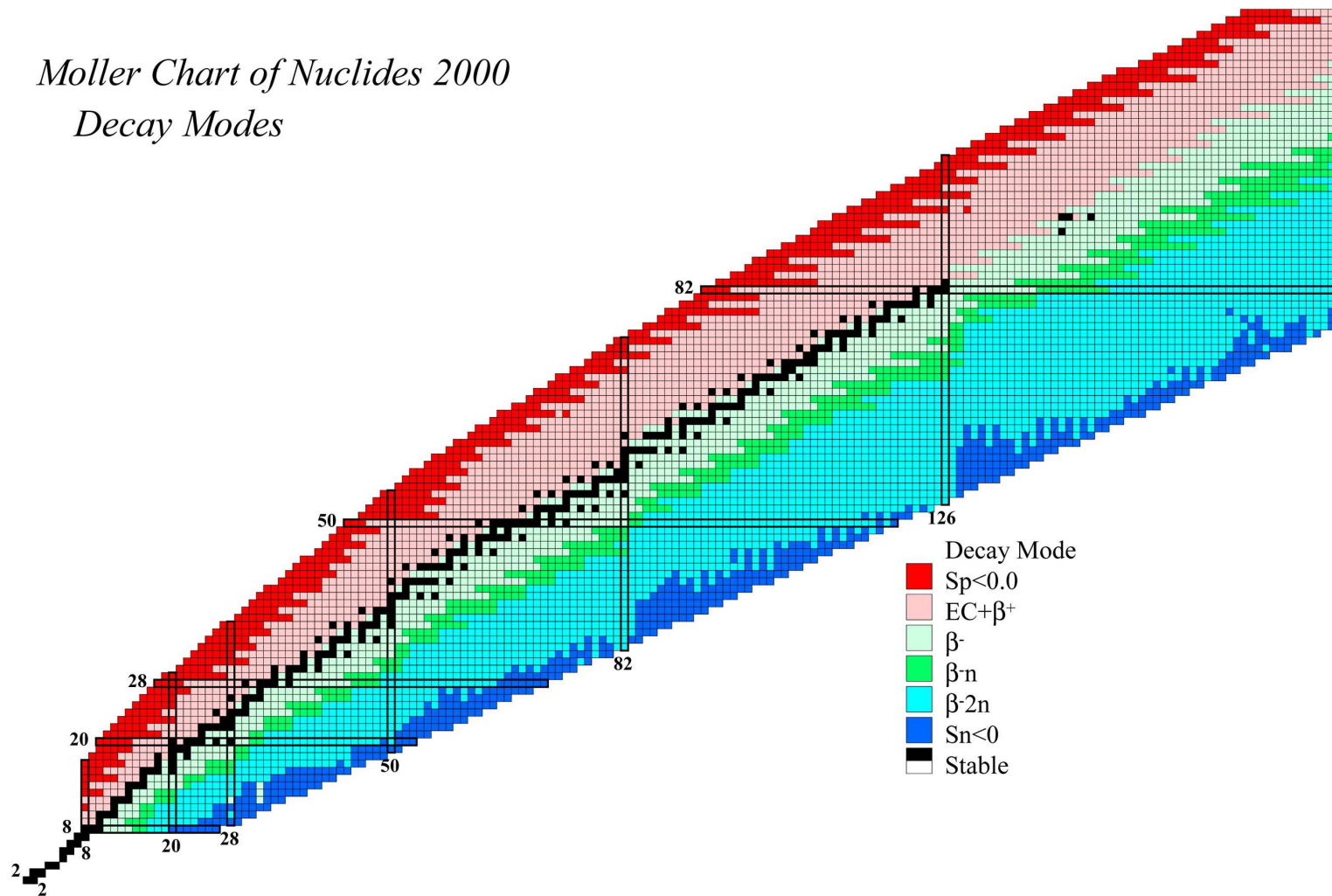


Figure 2: (Sodium) $^{24}\text{Na} \rightarrow ^{24}\text{Mg}$ (Zinc) β^- decay, followed by γ decay.

Moller Chart of Nuclides 2000
Decay Modes



Note the large-mass stable nuclides.
Google “island of stability”, and go to the Wiki page.

A generic decay scheme and its associated energetics are described in (2). X^* represents the initial excited states, while X' is the final state of the transition, though usually, it is a lower energy excited state, that also decays by one of the known modes.

$$\begin{aligned} {}^A_ZX_N^* &\longrightarrow {}^A_ZX'_N + \gamma \\ [m_{X^*} - m_{X'}]c^2 &= T_{X'} + E_\gamma \\ Q &= T_{X'} + E_\gamma \end{aligned} \tag{5}$$

Thus we see that the reaction Q is distributed between the recoil energy of the daughter nucleus and the energy of the photon. Note that m_{X^*} and $m_{X'}$, are *nuclear* masses.

It turns out to be easier to use a relativistic formalism to determine the share of kinetic energy between the photon and the daughter nucleus, so we proceed that way. It can be shown that:

$$\begin{aligned} E_\gamma &= Q \left[\frac{1 + Q/2m_{X'}c^2}{1 + Q/m_{X'}c^2} \right] \\ T_{X'} &= \frac{Q}{2} \left[\frac{Q/m_{X'}c^2}{(1 + Q/m_{X'}c^2)} \right] . \end{aligned} \tag{6}$$

Since $Q \lll m_{X'}c^2$, or $(10^{-4}/A) < (Q/m_{X'}c^2) < (10^{-2}/A)$, we can approximate:

$$\begin{aligned} E_{\gamma} &\approx Q - \frac{Q^2}{2m_{X'}c^2} \\ T_{X'} &\approx \frac{Q^2}{2m_{X'}c^2} \end{aligned} \quad (7)$$

Although these recoil energies are small, they are in the range $(5 \rightarrow 50000 \text{ eV})/A$.

Except in the lower range, these recoil energies are strong enough to overwhelm atomic bonds and cause crystal structure fissures.

Finally, Krane uses a non-relativistic formalism to obtain the recoil energy. It can be shown that:

$$\frac{E_{\gamma}^{\text{rel}} - E_{\gamma}^{\text{non-rel}}}{Q} \approx 1.5 \times (10^{-13} \rightarrow 10^{-7})/A^3. \quad (8)$$

The relativistic correction can be ignored, but it is interesting that the relativistic calculation is easier! This is one place where correct intuition at the outset, would have caused you more work!

Obtaining Q from atomic mass tables

Finally, let us deal with a small subtlety regarding the reaction Q . The Q employed in (5) is different than what one would obtain from a difference of atomic masses from the mass tables.

The two are:

$$\begin{aligned} Q &= [m_{X^*} - m_{X'}]c^2 \\ Q' &= [m({}_Z^AX_N^*) - m({}_Z^AX'_N)]c^2 . \end{aligned} \tag{9}$$

Going back to the definition of atomic mass, we obtain:

$$Q = Q' + \sum_{i=1}^Z (B_i^* - B_i') , \tag{10}$$

where the B_i 's are the atomic binding energies of the i 'th atomic electron. About the only difference, insofar as the atomic electrons are concerned, is the different nuclear spin that these electrons see. This effect the hyperfine splitting energies of the atomic states, and can probably be ignored in (10). (Makes one wonder, though. Someone care to attempt some library research on this topic?)

Chapter 16.2: Classical Electromagnetic Radiation

The structure (and many of the conclusions) of the Quantum Mechanical description of electromagnetic radiation follows from the classical formulation.

Most important of these is the classical limit, through the correspondence principle.

Therefore, a detailed review of Classical Electromagnetic Radiation is well motivated.

Multipole expansions

Electric multipoles

The electric multipole expansion starts by considering the potential due to a static charge distribution:

$$V(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} . \quad (11)$$

We have encountered such an object before, in Chapter 10, but now we attempt to be a little more general than simply (!) describing the nuclear charge distribution.

Generally, we shall only consider charge conserving systems, where the time dependency may arise through a vibration or rotation of the charges in the distribution, but not through loss or gain of charge.

Assuming the charges to be localized, we expand (11) in $\vec{x}$.

That is, we are considering $\vec{x}$ to be outside of the charge distribution.

The result is:

$$V(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[\frac{Q_0}{|\vec{x}|} + \frac{Q_1}{|\vec{x}|^2} + \frac{Q_2}{|\vec{x}|^3} \cdots \right] \quad (12)$$

$$Q_0 = \int d\vec{x}' \rho(\vec{x}', t) \quad (13)$$

$$Q_1 = \int d\vec{x}' z' \rho(\vec{x}') \quad (14)$$

$$Q_2 = \frac{1}{2} \int d\vec{x}' (3z'^2 - r'^2) \rho(\vec{x}') \quad (15)$$

where Q_0 is the total charge (*aka* the monopole term), Q_1 is the dipole moment, and Q_2 is the quadrupole moment.

Higher moments would include the octupole moment, and the hexadecapole moment, and more.

In general,

$$V(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{x}|} \sum_n \frac{Q_n}{|\vec{x}|^n} \quad (16)$$

Q_0 is zero, for neutral charge distributions.

If the charge distribution contains charge of only one sign, the dipole moment could be made to disappear by choosing the coordinate system to be at the center of charge.

In classical E&M theory, the higher n 's diminish in influence as $|\vec{x}|$ grows.

In Quantum E&M theory, the higher n 's are associated with transitions that become weaker with increased n .

Electric dipoles and quadrupoles

To illustrate some features of dipoles, consider the situation described in Figure 3.

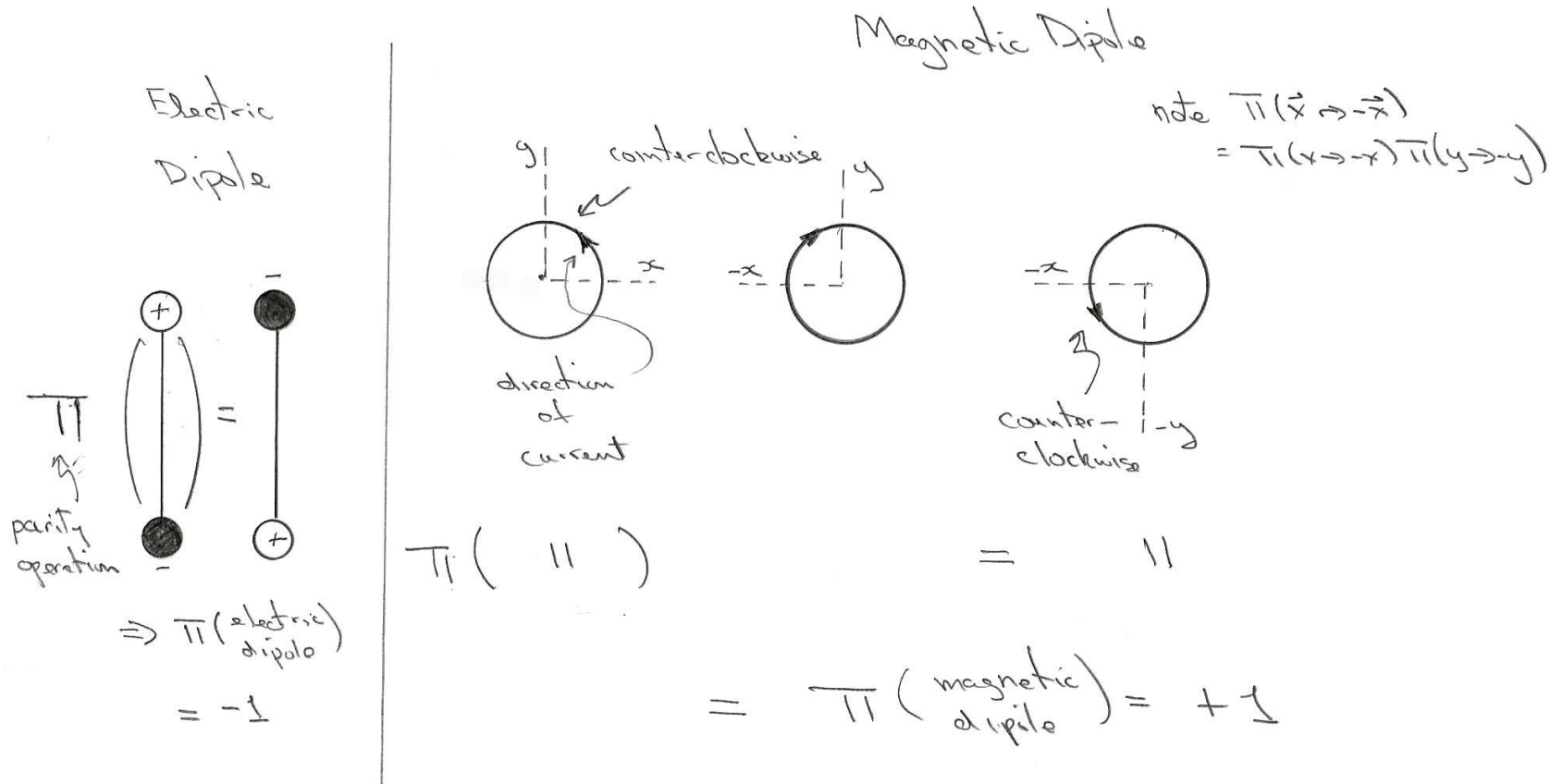


Figure 3: Parity illustration, for an electric dipole

Here, a charge q (assumed, without loss of generality, to be positive), is located on the z -axis at $z = l$. A charge of the opposite sign is on the z -axis, at $z = -l$. This is a pure electric dipole:

$$\begin{aligned}\rho(\vec{x}) &= q[\delta(x)\delta(y)\delta(z-l) - \delta(x)\delta(y)\delta(z+l)] \\ Q_0 &= 0 \\ Q_1 &= ql + (-q)(-l) = 2ql \\ Q_{n \geq 2}(t) &= 0\end{aligned}\tag{17}$$

Under a parity operation, $\vec{x} \rightarrow -\vec{x}$, the configuration in Figure 3, is opposite to its original configuration. Thus the parity of the electric dipole radiation is $\Pi(E^1) = -1$.

A similar argument for an electric quadrupole would lead us to conclude that $\Pi(E^2) = +1$. In general, the parity of an electric multipole is:

$$\Pi(EL) = (-1)^L .\tag{18}$$

Magnetic dipoles

Considerations for static magnetic multipole expansions are much more involved. Instead, we focus on the magnetic dipole moment:

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \left[\frac{3\vec{n}(\vec{n} \cdot \vec{m}) - \vec{m}}{|\vec{x}|^3} \right] , \quad (19)$$

where $\vec{n}$ is a unit vector in the direction of $\vec{x}$, $\vec{m}$ is the magnetic moment,

$$\vec{m} = \frac{1}{2} \int d\vec{x}' [\vec{x}' \times \vec{J}(\vec{x}')] . \quad (20)$$

Here, $\vec{J}$ is the current density.

Magnetic dipoles

To illustrate some features of magnetic dipoles, consider the situation described in Figure 4. Under a parity change, the magnetic dipole is unchanged. So, in this case, $\Pi(B^1) = +1$.

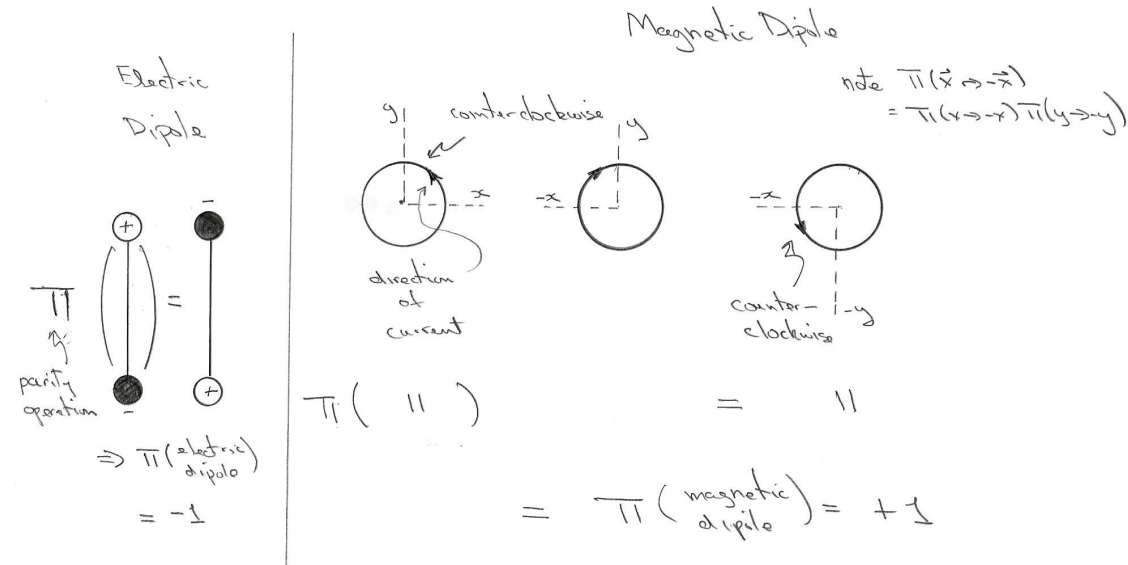


Figure 4: Parity illustration, for a magnetic dipole

A magnetic quadrupole, on the other hand, changes its sign.

In general,

$$\Pi(ML) = -(-1)^L. \quad (21)$$

Characteristics of multipolarity

L	multipolarity	$\Pi(EL ML)$	angular distribution
1	dipole	-1 / +1	Unique angular distribution, given later
2	quadrupole	+1 / -1	"
3	octupole	-1 / +1	"
4	hexadecapole	+1 / -1	"
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Parity Electric multipoles $\Pi(EL) = (-1)^L$, Magnetic multipoles $\Pi(EL) = (-1)^{L+1}$.

Power radiated: The power radiated is proportional to:

$$P(\sigma L) = \propto \frac{2(L+1)c}{\varepsilon_0 L [(2L+1)!!]^2} \left(\frac{\omega}{c}\right)^{2L+2} [m(\sigma L)]^2, \quad (22)$$

where, σ means either E or M , and

$m(\sigma L)$ is the E or M multipole moment of the appropriate kind.

A general and more sophisticated treatment of classical multipole fields

In order to make the transition to Quantum Mechanics a little more transparent, our jumping-off point from Classical Electrodynamics has to be a somewhat more advanced. Classical Electrodynamics can be a challenging topic on its own.

From advanced *E&M* (for example, J D Jackson's *Classical Electrodynamics*) [this textbook is a tour de force]...

If

$$\rho(\vec{x}, t) = \rho(\vec{x})e^{i\omega t} \quad (23)$$

$$\vec{J}(\vec{x}, t) = \vec{J}(\vec{x})e^{i\omega t} \quad (24)$$

$$\vec{m}(\vec{x}, t) = \vec{m}(\vec{x})e^{i\omega t}, \quad (25)$$

are the time-dependent charge density (protons density), current density (protons with orbital angular momentum), and magnetic moment density (proton and neutron intrinsic spins),

the radiation fields are characterized by the electric, Q_{lm} , and magnetic, M_{lm} , multipoles:

$$Q_{lm} = \int d\vec{x} |\vec{x}|^l Y_{lm}^*(\theta, \phi) \left[\rho(\vec{x}) - \frac{i\omega}{(l+1)c^2} \vec{\nabla} \cdot [\vec{x} \times \vec{m}(\vec{x})] \right] , \quad (26)$$

$$M_{lm} = - \int d\vec{x} |\vec{x}|^l Y_{lm}^*(\theta, \phi) \left[\nabla \cdot \vec{m}(\vec{x}) + \frac{1}{(l+1)} \vec{\nabla} \cdot [\vec{x} \times \vec{J}(\vec{x})] \right] . \quad (27)$$

The power, dP , radiated into solid angle $d\Omega$, by mode (l, m) is:

$$\frac{dP}{d\Omega} \left(l, m, \begin{bmatrix} E \\ M \end{bmatrix} \right) = \frac{2(l+1)c}{\varepsilon_0 l(2l+1)[(2l+1)!!]^2} \left(\frac{\omega}{c} \right)^{2l+2} \left| \frac{Q_{lm}}{M_{lm}} \right|^2 |X_{lm}(\theta, \phi)|^2 , \quad (28)$$

where (suppressing explicit dependence on θ and ϕ),

$$|X_{lm}|^2 = \frac{\frac{1}{2}(l-m)(l+m+1)|Y_{l,m+1}|^2 + \frac{1}{2}(l+m)(l-m+1)|Y_{l,m-1}|^2 + m^2|Y_{l,m}|^2}{l(l+1)} . \quad (29)$$

If all the m 's contribute equally,

$$\sum_{m=-l}^l |X_{lm}(\theta, \phi)|^2 = \frac{2l+1}{4\pi} . \quad (30)$$

In this case, the radiation is isotropic. You will also need (30) to get Krane's (10.8). For the non-isotropic distributions, first we recall the spherical harmonics

l (2^l -pole)	$ Y_{l,0} ^2$	$ Y_{l,\pm 1} ^2$	$ Y_{l,\pm 2} ^2$	$ Y_{l,\pm 3} ^2$
1 (dipole)	$\frac{3}{4\pi} \cos^2 \theta$	$\frac{3}{8\pi} \sin^2 \theta$	(n/a)	(n/a)
2 (quadrupole)	$\frac{5}{16\pi} (3 \cos^2 \theta - 1)^2$	$\frac{1}{8\pi} \sin^2 \theta \cos^2 \theta$	$\frac{15}{32\pi} \sin^4 \theta$	(n/a)
3 (octopole)	$\frac{49}{16\pi} (5 \cos^3 \theta - 3 \cos \theta)^2$	$\frac{21}{64\pi} \sin^2 \theta (5 \cos^2 \theta - 1)^2$	$\frac{105}{32\pi} \sin^4 \theta \cos^2 \theta$	$\frac{35}{64\pi} \sin^6 \theta$

l (2^l -pole)	$ X_{l,0} ^2$	$ X_{l,\pm 1} ^2$	$ X_{l,\pm 2} ^2$	$ X_{l,\pm 3} ^2$
1 (dipole)	$\frac{3}{8\pi} \sin^2 \theta$	$\frac{3}{16\pi} (1 + \cos^2 \theta)$	(n/a)	(n/a)
2 (quadrupole)	$\frac{15}{8\pi} \sin^2 \theta \cos^2 \theta$	$\frac{5}{16\pi} (1 - 3 \cos^2 \theta + 4 \cos^4 \theta)$	$\frac{5}{16\pi} (1 - \cos^4 \theta)$	(n/a)
3 (octopole)	?	?	?	?

These distributions are shown in Figure 5.

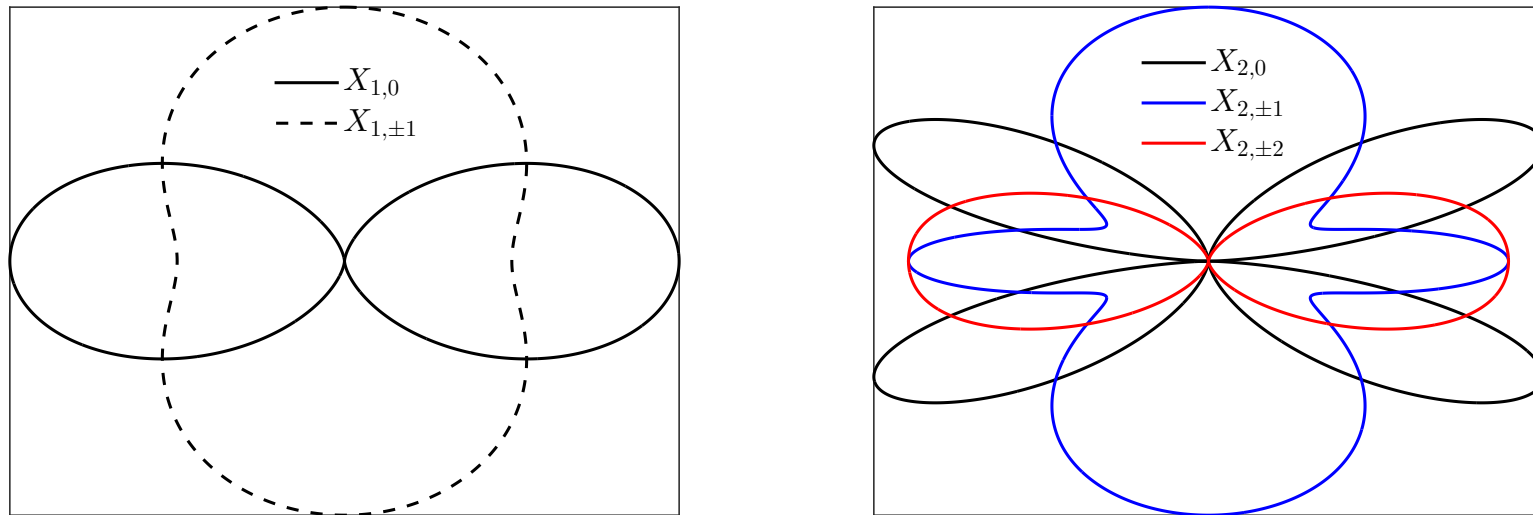


Figure 5: Angular distributions for $X_{1,m}$ and $X_{2,m}$

The angular distributions indicate how a measurement of the angular distribution map of the radiation field can identify the multipolarity of the radiation.

Measurement of the parity is accomplished by performing a secondary scattering experiment that is sensitive to the direction of, for example, the electric field vector.

The Compton interaction often used for this purpose.

Chapter 16.3: Transition to Quantum Mechanics

The transition to Quantum Mechanics is remarkably simple! Most of the hard work has been done in the classical analysis. In Quantum Mechanics, we have:

$$\frac{d\lambda}{d\Omega} \left(l, m, \begin{bmatrix} E \\ M \end{bmatrix} \right) = \frac{1}{\hbar\omega} \frac{dP}{d\Omega} \left(l, m, \begin{bmatrix} E \\ M \end{bmatrix} \right) . \quad (31)$$

that is, the transition rate (per photon) is taken from the expression for the power radiated, divided by the energy per photon, $\hbar\omega$.

The structure of the right hand side is identical to the classical expression, except that the charge density, current density and intrinsic magnetization density are replaced by the probability density, the probability current density, and the magnetic moment density.

$$Q_{lm}^{i \rightarrow f} = e \int d\vec{x} |\vec{x}|^l Y_{lm}^*(\theta, \phi) \left[\sum_{i=1}^Z (\psi_i^*)_f (\psi_i)_i \right] , \quad (32)$$

$$M_{lm}^{i \rightarrow f} = -\frac{1}{(l+1)} \frac{e\hbar}{m_p} \int d\vec{x} |\vec{x}|^l Y_{lm}^*(\theta, \phi) \left[\vec{\nabla} \cdot \left[\sum_{i=1}^Z (\psi_i^*)_f \vec{L} |(\psi_i)_i \right] \right] . \quad (33)$$

The Weisskopf Estimates

A detailed investigation of (32) and (33) requires detailed knowledge of the nuclear wavefunctions, in order to pin down the absolute decay rates of the various transition types. The angular distributions, however, and parities have been established already.

Detailed knowledge of the wavefunctions is not known. However, it would be nice to have a “ballpark” estimate, to get relative transition rates. Such an approximation has been done by Weisskopf, using the following strategy.

Let the radial part of both the initial and final wavefunctions be:

$$R(r) = \theta(R_N - r) / \sqrt{\int_0^{R_N} dr r^2},$$

where R_N is the nuclear edge, given by $R_N = R_0 A^{1/3}$. We note that these radial wavefunctions are properly normalized, since,

$$\int_0^\infty dr r^2 R(r) = 1.$$

The radial part of (32) is thus given by:

$$Q_{lm}^{i \rightarrow f} = e \int_0^\infty dr r^2 r^l |R(r)|^2 = e R_N^l \frac{3}{l+3} . \quad (34)$$

Substituting (34) and (33) results in:

$$\lambda(E l) = \frac{8\pi(l+1)}{l[(2l+1)!!]^2} \left\{ \frac{e^2}{4\pi\epsilon_0\hbar c} \right\} \left(\frac{E_\gamma R_N}{\hbar c} \right)^{2l+1} \left(\frac{3}{l+3} \right)^2 \frac{c}{R_N} , \quad (35)$$

for electric l -pole transitions.

All quantities enclosed in parentheses (of all kinds) are unitless.

Moreover, the quantity inside the $\{\}$'s is recognized as the fine structure constant, $\alpha = 1/137.036....$

The factor c/R is $1/(\text{the time for a photon to cross a nuclear diameter})$.

Similar considerations (I'm still looking for the source of this calculation) gives:

$$\lambda(Ml) = \frac{8\pi(l+1)}{l[(2l+1)!!]^2} \left(\mu_p - \frac{1}{l+1} \right)^2 \left(\frac{\hbar c}{m_p c^2 R_N} \right)^2 \left\{ \frac{e^2}{4\pi\epsilon_0 \hbar c} \right\} \left(\frac{E_\gamma R_N}{\hbar c} \right)^{2l+1} \left(\frac{3}{l+2} \right)^2 \frac{c}{R_N}, \quad (36)$$

where the second term in parentheses comes from the nuclear magnetron, rendered unitless by additional factors of c and R_N .

Krane never defines the factor μ_p , but according to my search (so far) is related to the gyromagnetic ration of the proton, divided by 2.

Krane also goes on to say that the factor $[\mu_p - 1/(l+1)]^2$ is often simply replaced by 10. (!)

Evaluating (35) and (36) leads to (according to Krane):

$$\begin{aligned}\lambda(E1) &= 1.0 \times 10^{14} A^{2/3} E_{\gamma}^3 \\ \lambda(E2) &= 7.3 \times 10^7 A^{4/3} E_{\gamma}^5 \\ \lambda(E3) &= 3.3 \times 10^1 A^2 E_{\gamma}^7 \\ \lambda(E4) &= 1.1 \times 10^{-5} A^{8/3} E_{\gamma}^9\end{aligned}\tag{37}$$

$$\begin{aligned}\lambda(M1) &= 5.6 \times 10^{13} E_{\gamma}^3 \\ \lambda(M2) &= 3.5 \times 10^7 A^{2/3} E_{\gamma}^5 \\ \lambda(M3) &= 1.6 \times 10^1 A^{4/3} E_{\gamma}^7 \\ \lambda(M4) &= 4.5 \times 10^{-6} A^2 E_{\gamma}^9\end{aligned}\tag{38}$$

Some conclusions about the Weisskopf estimates:

Same order, different type

$$\frac{\lambda(E l)}{\lambda(M l)} \approx 2 A^{2/3} .$$

Thus, for a given l , electric transition always dominates, with the difference getting large with increases A .

Nearby order, same parity

$$\frac{\lambda(E(l+1))}{\lambda(M l)} \approx 10^{-6} A^{4/3} E^2 .$$

Thus, for large A and E , $E2$ can compete with $M1$, $E3$ can compete with $M2$ and so on.

Same type, different order

$$\frac{\lambda(\sigma(l+1))}{\lambda(\sigma l)} \approx 10^{-8} A^{2/3} E^2 ,$$

where σ is either E or M . The factor $A^{2/3} E^2 \leq 10^4$, so, as $l \uparrow$, $\lambda \downarrow$, dramatically!

Chapter 16.4: Angular momentum and parity selection rules

Conservation of total angular momentum require and parity dictates that:

$$\begin{aligned}\vec{I}_f &= \vec{I}_i + \vec{l} \\ \pi_f &= (-1)^l \pi_i && (E\text{-type}) \\ \pi_f &= (-1)^{l+1} \pi_i && (M\text{-type})\end{aligned}\tag{39}$$

Recalling the rules of quantized angular momentum addition,

$$|I_f - l| \leq I_i + l, \text{ or } \Delta I = |I_f - I_i| \leq l \leq I_f + I_i.$$

Note that the emission of an electromagnetic decay photon can not be associated with a $0 \rightarrow 0$ transition. (The can occur via internal conversion, discussed later.) The above parity selection rules can also be stated as follows:

$$\begin{aligned}\Delta\pi = \text{no} &\Rightarrow \text{even } E / \text{odd } M \\ \Delta\pi = \text{yes} &\Rightarrow \text{odd } E / \text{even } M\end{aligned}$$

Some examples

$$\frac{3\pi}{2} \rightarrow \frac{5\pi}{2}, \Delta\pi = \text{no}$$

$$1 \leq \Delta I \leq 4.$$

Therefore the only possibilities are: **M1**, $E2$, $M3$, $E4$. ($M1$ is the most likely.)

$$\frac{3\pi}{2} \rightarrow \frac{5\pi}{2}, \Delta\pi = \text{yes}$$

$$1 \leq \Delta I \leq 4.$$

Therefore, the only possibilities are: **E1**, $M2$, $E3$, $M4$. ($E1$ is the most likely.)

$$4\pi \rightarrow 0\pi, \Delta\pi = \text{unknown}$$

$\Delta I = 4$. Therefore, the only possibilities are: $E4$ if $\Delta\pi = \text{no}$, $M4$ otherwise.

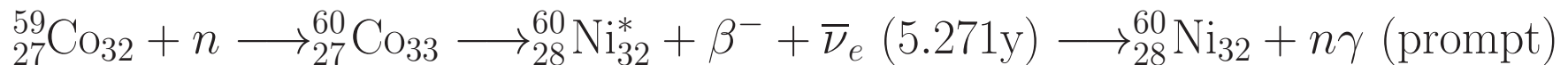
$$2^+ \rightarrow 0^+$$

The only possibility is an $E2$ transition.

Real-life example

One of the most important, and useful γ decays is the decay of ^{60}Co . It has been used extensively for radiotherapy purposes, and used today for industrial radiation processes, sterilization, food processing, material transformation, and other applications.

^{60}Co does not occur naturally, as it has a half-life of 5.27y. it is produced by neutron activation of stable ^{59}Co . The entire activation and decay sequences are given as follows, with a decay chart given in Figure 6. The decay chart also has information about spin and parity assignments that are needed to decipher the following table.



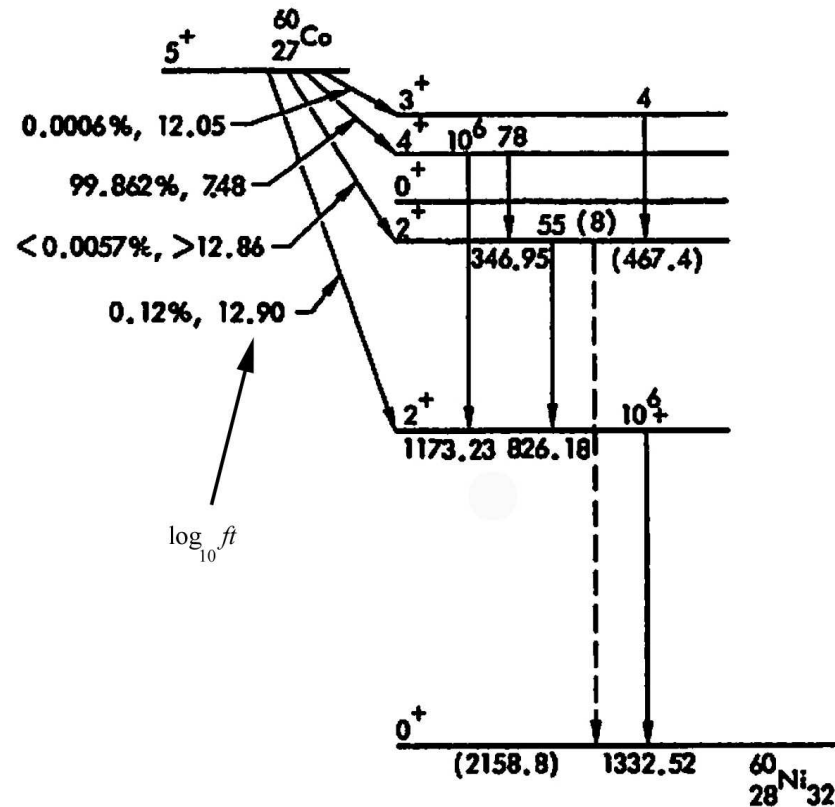


Figure 6: ^{60}Co decay scheme. Note the nuclear structure! The low-lying 2^+ state is a single quadrupole phonon excitation, the $(2^+, 0^+, 4^+)$ triplet is associated with a two-phonon excitation, and the 3^+ state is most probably the only visible state from a three-phonon excitation. The $\log_{10} ft = 7.48$ is most likely an allowed β decay, while the three $\log_{10} ft \approx 12$ are almost certainly 2^{nd} -forbidden β decays, unique 2^{nd} -forbidden in the case of $\Delta = 3$, with $\log_{10} ft = 12.05$. Note that the 0^+ state is not fed by β decay. An electromagnetic transition to this state would be accomplished via an *internal conversion*, a topic to be discussed in the next section. The resulting β would be visible as a sharp line in a measured β spectrum.

I_i^π	I_f^π	E_N (keV)	E_{γ_n} (keV)	exp	modes	τ (ps)	frac
4^+		2505.766				3.3	
	2_1^+		1173.237 (γ_4)	E2+M3	E2 ,M3,E4,M5,E6		$\equiv 1$
	2_2^+		346.93 (γ_5)	unk	E2 ,M3,E4,M5,E6		7.6×10^{-5}
	0_1^+		2505.766 (γ_6)	E4	E4 (pure)		2.0×10^{-8}
0_2^+		2284.87				> 1.5	
2_2^+		2158.64				0.59	
	2_1^+		826.06 (γ_2)	M1+E2	M1 ,E2,M3,E4		$\equiv 1$
	0_1^+		2158.57 (γ_3)	E2	E2 (pure)		0.176
2_1^+		1332.518				0.77	
	0_1^+		1332.518 (γ_1)	E2	E2 (pure)		

Table 1: The decay scheme of ^{60}Ni that is generated from the most probable β decay of ^{60}Co .

Comparison with Weisskopf estimates

Comparing	Details	theory	exp	ratio	Comments
γ_4, γ_1	E2(1173)/E2(1333)	0.53	0.23	2.3	
γ_4 modes	M3/E2 E4/E2			2.6×10^{-5} 5.1×10^{-13}	Mostly E2
γ_4, γ_5	E2(347)/E2(1173)	2.3×10^{-3}	7.6×10^{-5}	30	ψ_i/ψ_f mismatch? Collective effect?
γ_6, γ_4	E4(2505)/E2(1173)	6.2×10^{-8}	2.0×10^{-8}	3.1	
γ_6, γ_5	E4(2505)/E2(347)	2.7×10^{-5}	2.6×10^{-4}	0.10	
γ_2 modes	E2/M1			2.1×10^{-4}	Mostly M1
γ_2, γ_3	E2(2159)/M1(826)	0.026	0.176	0.15	

Table 2: Comparison of measurements of the decay rates for the γ transition rates for the decay photons of ^{60}Ni that are generated from the most probable β decay of ^{60}Co .

Chapter 16.6: Internal Conversion

$0^+ \rightarrow 0^+$ transitions can not occur via electromagnetic γ transitions. The electromagnetic operator does not have an $l = 0$ component, since the photon's intrinsic spin is $s = 1$.

However, there is an electromagnetic process that can cause a $0^+ \rightarrow 0^+$ transition, a process called *internal conversion*.

The classical visualization is that an electron (predominantly a K-shell electron) enters the nucleus, feels the electromagnetic force from a nucleon in an excited states, or a collection of nucleons in an excited states, and acquires enough energy to liberate the electron from the nucleus cause the nucleus to transition to a lower energy level. The nucleus, as a whole, recoils to conserve momentum.

The quantum mechanical picture is similar, and is depicted graphically in Figure 7.

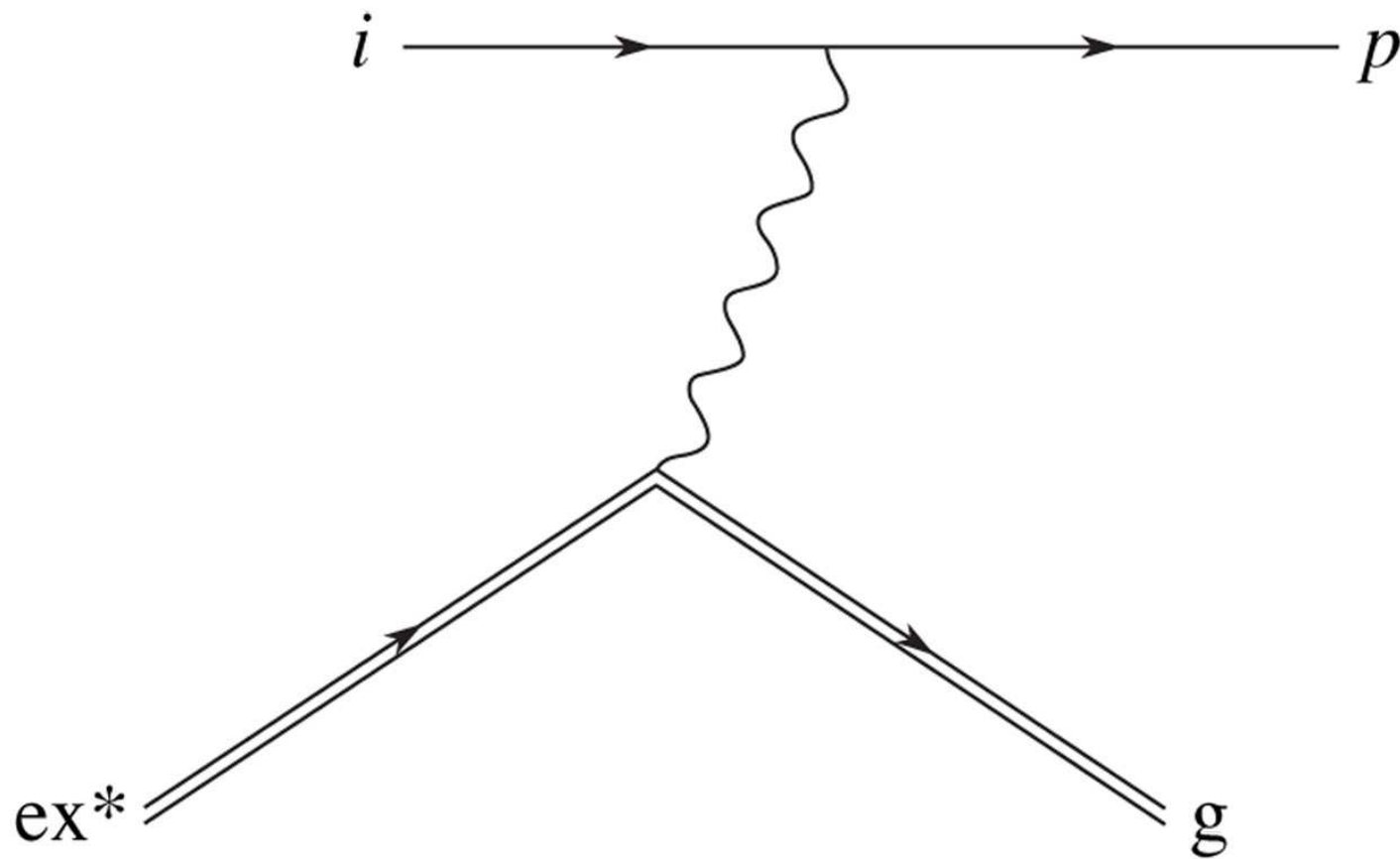


Figure 7: The Quantum Mechanical description of internal conversion in terms of a Feynman diagram. The double line denotes a nuclear transition from the excited (ex^*) to the ground (g) state. The electron transitions its initial state in the atom (i) to the unbound state (p).

We've learned that, whether we're adapting classical electrodynamics to quantum electrodynamics, or if we are starting with a more rigorous Fermi Golden Rule #2 approach, that the most important part, and the least known part, is the calculation of the transition matrix element, $\mathcal{M}^{i \rightarrow f}$.

For internal conversion, the matrix element is:

$$\mathcal{M}_{ic}^{i \rightarrow f} = \langle \psi_N^f \psi_e^{\text{free}} | \mathcal{O}_{em} | \psi_e^{\text{bound}} \psi_N^i \rangle . \quad (40)$$

This is to be compared with the matrix element for γ transitions:

$$\mathcal{M}_{\gamma}^{i \rightarrow f} = \langle \psi_N^f \psi_{\gamma}^{\text{free}} | \mathcal{O}_{em} | \psi_N^i \rangle . \quad (41)$$

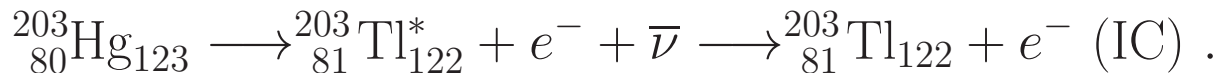
Note the similarity between the matrix elements. They contain many physical similarities, that are accompanied by mathematical similarity. In fact, *every* γ decay is in competition with internal conversion (more on this later). The only one that stands alone is a $0^+ \rightarrow 0^+$ internal conversion, that has no γ counterpart.

Internal conversion, β decay and internal conversion energetics

When a nucleus in an excited state decays, it does so via α decay, β decay, or γ decay.

Often a decay can proceed via more than one decay channel, and the daughter, and her progeny, can decay as well.

A particular interesting case, from the standpoint of measurement, is when β decay and internal conversion happen in close temporal proximity. For example:



Thus, a measurement of the electrons being emitted by radioactive ${}^{203}_{80}\text{Hg}_{123}$, sees both β decay electrons, as well as IC electrons.

However, the signatures of both of these processes are very distinct. The β decay electrons give a continuous energy spectrum, while the IC electrons are seen as sharp lines on the β spectrum, at the energies of the nuclear transition (less recoil).

The following figure shows this.

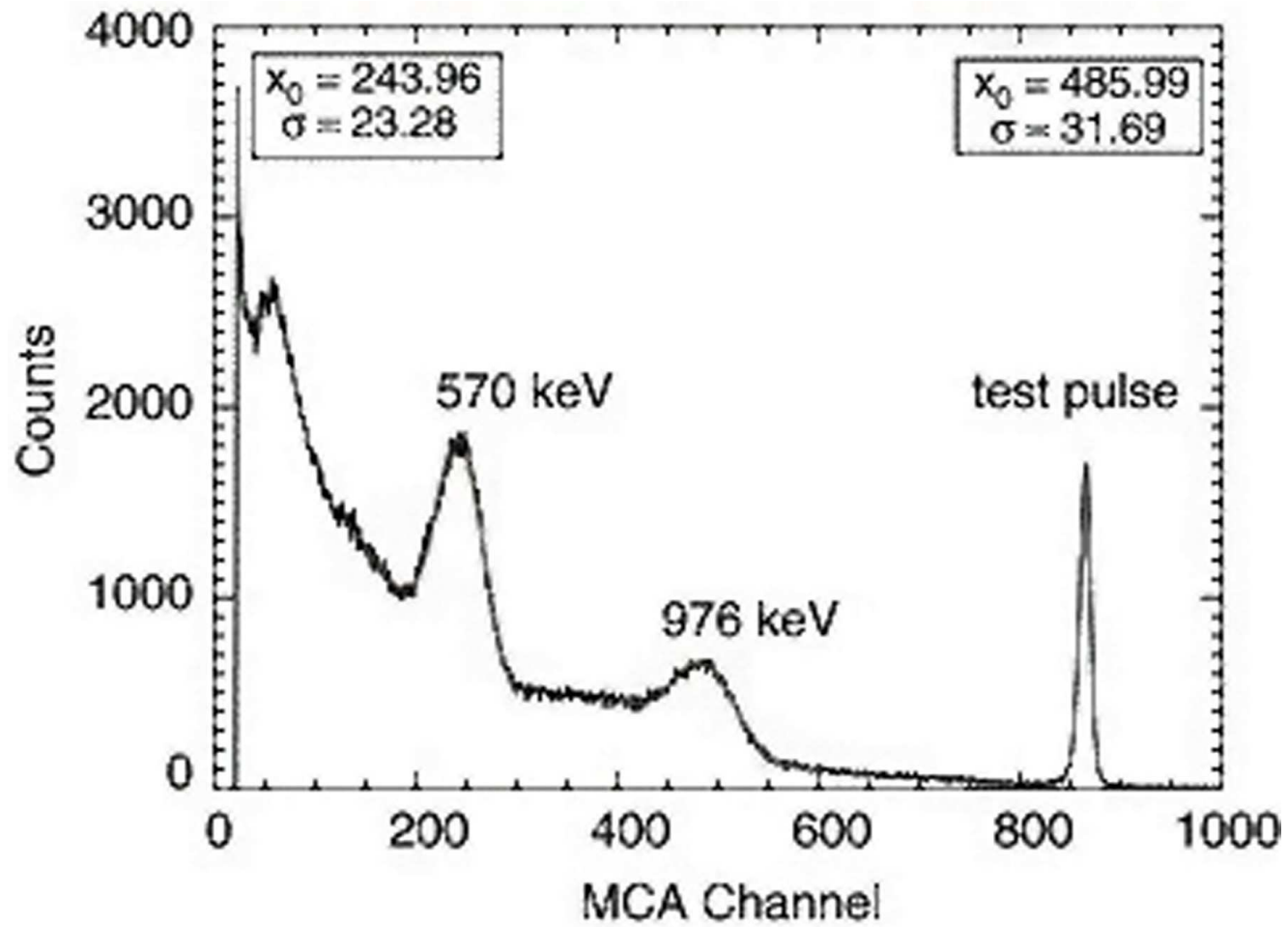


Figure 8: Internal conversion electrons superimposed upon a β spectrum.

Using relativistic kinematics, dictated by the kinetic energies observed, the kinetic energy of the kinetic electron is given by:

$$T_e = \frac{Q - B}{1 + m_e/m_{X'}} \left[\frac{1 + \frac{Q-B}{2m_{X'}c^2}}{1 + \frac{Q-B}{(m_{X'}+m_e)c^2}} \right] . \quad (42)$$

The first denominator term is usually ignored. (It is about a $0.0005/A$) correction.)

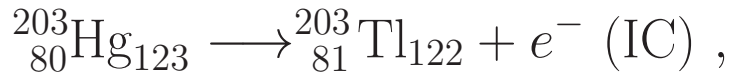
The term inside the large []s is usually expressed as unity. (The magnitude of this term represents a $(0.005 \rightarrow 0.00005/A)$ correction.)

These corrections may be needed for lighter elements with large Q . The worst case I have found is that of ^{16}O decay, a $0^+ \rightarrow 0^+$ transition, that is clearly identified as internal conversion.

The error in the case is about 0.5% if the corrections are ignored.

The B term is the binding energy of the converted electron. This can not be ignored, and depends upon the atomic shell from which the electron was converted.

For the specific example,



$Q = 279.910$, and:

X-ray notation	spectroscopic notation	B (keV)	T_e (keV)
K	$1s_{1/2}$	85.529	193.661
L _I	$2s_{1/2}$	15.347	263.843
L _{II}	$2p_{1/2}$	14.698	264.492
L _{III}	$2p_{3/2}$	12.657	266.533
M _I	$3s_{1/2}$	3.704	275.486

Internal conversion contribution to decay rate

Since γ decay and internal conversion both contribute to the electromagnetic decay rate, we can write the total decay rate as a sum of the processes, in increasing specificity:

$$\lambda_{em} = \lambda_{\gamma} + \lambda_e , \quad (43)$$

where λ_{γ} is the γ transition rate, and λ_e is the internal conversion transition rate.

In terms of the ratio, $\alpha \equiv \lambda_e/\lambda_{\gamma}$,

$$\begin{aligned} \lambda_{em} &= \lambda_{\gamma}(1 + \alpha) \\ \lambda_{em} &= \lambda_{\gamma} + \lambda_{e_K} + \lambda_{e_L} + \lambda_{e_M} \dots \\ &\text{or } \lambda_{\gamma}(1 + \alpha_K + \alpha_L + \alpha_M \dots) \\ \lambda_{em} &= \lambda_{\gamma} + \lambda_{e_K} + \lambda_{e_{L_I}} + \lambda_{e_{L_{II}}} + \lambda_{e_{L_{III}}} + \lambda_{e_{M_I}} \dots \\ &\text{or } \lambda_{\gamma}(1 + \alpha_K + \alpha_{L_I} + \alpha_{L_{II}} + \alpha_{L_{III}} + \alpha_{M_I} \dots) \end{aligned} \quad (44)$$

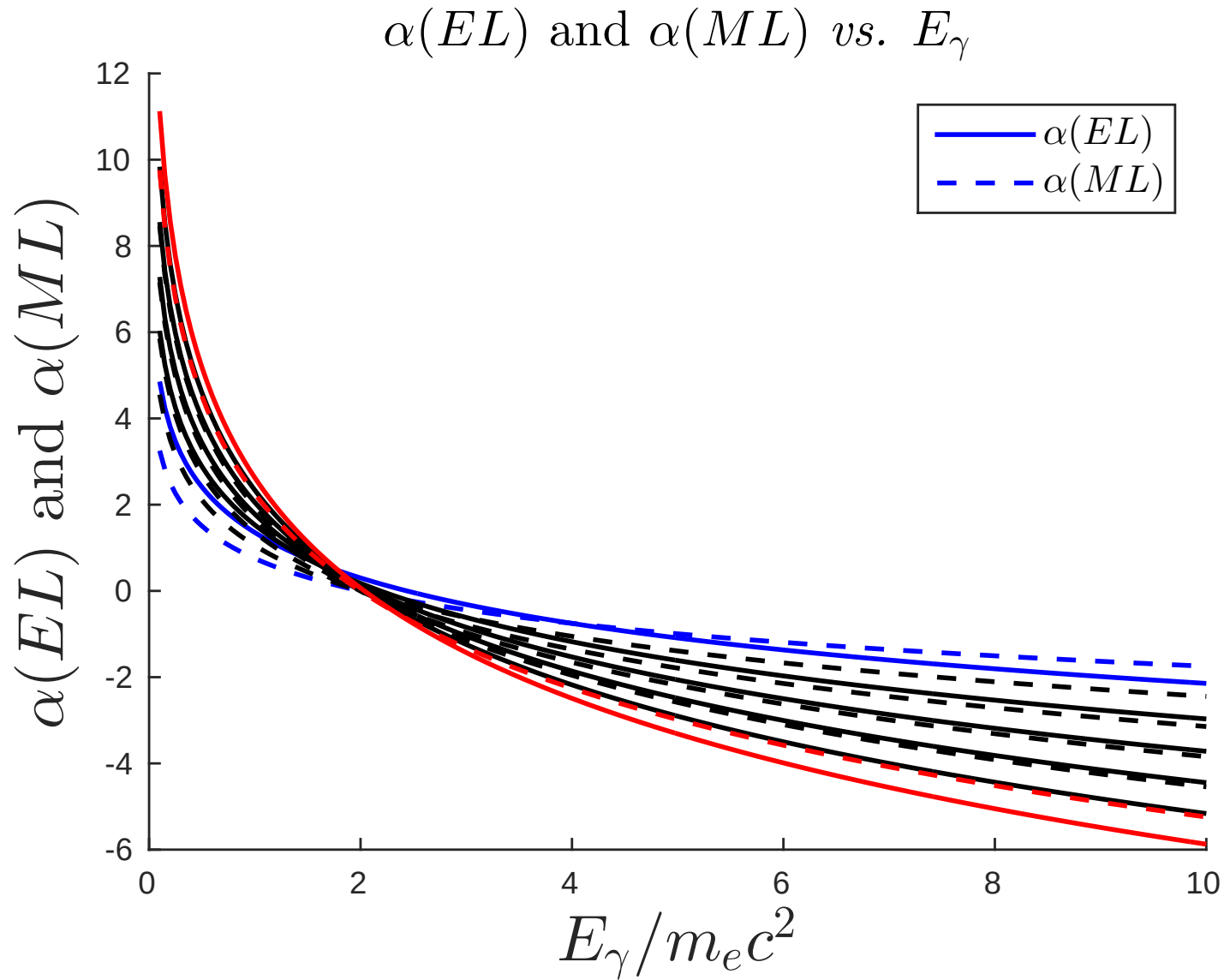
Using hydrogenic wavefunctions, we may estimate:

$$\begin{aligned}\alpha(EL) &\cong \frac{Z^3}{n^3} \left(\frac{L}{L+1} \right) \alpha^4 \left(\frac{2m_e c^2}{E_\gamma} \right)^{L+5/2} \\ \alpha(ML) &\cong \frac{Z^3}{n^3} \alpha^4 \left(\frac{2m_e c^2}{E_\gamma} \right)^{L+3/2}\end{aligned}\tag{45}$$

The trends that are predicted, all verified by experiment, are:

$\alpha(\sigma L)$	$\propto Z^3$
$E_\gamma \uparrow$	$\alpha(\sigma L) \downarrow$
$L \uparrow$	$\alpha(ML) \uparrow$ for $E_\gamma < 2 \text{ MeV}$, else $\downarrow$
$L \uparrow$	$\alpha(EL) \uparrow$ for $E_\gamma < \text{value given below}$, else $\downarrow$
$n \uparrow$	$\alpha(\sigma L) \downarrow$

$\alpha(ML)$ and $\alpha(EL)$ are plotted on the next page.



$L = 1$ lines are rendered in blue, while the $L = 6$ lines are in red.

$L = [2, 3, 4, 5]$ lines are in black.