

Solid angle

From the book "Fundamentals of Radiation Dosimetry and Radiological Physics"

Found at www.umich.edu/~nersa590/book.pdf

It's a hidden link, so type carefully.
(This course will be offered next term. 2cr Thu 6-9pm)

4.4.1 Solid angle subtended by a surface

We shall have need for this type of calculation shortly, and establish the notation now.

Consider a surface S , a surface element dS at $\vec{x}_S$, and its unit normal $\hat{n}(\vec{x}_S)$. The solid angle element of that surface element

$$d\Omega = dS \frac{|\hat{n}(\vec{x}_S) \cdot (\vec{x} - \vec{x}_S)|}{|\vec{x} - \vec{x}_S|^3}, \quad (4.32)$$

which is just the surface element, divided by the r^{-2} term, times the magnitude of the cosine of the angle between $\hat{n}$ and the vector $\vec{x} - \vec{x}_S$.

The total solid angle of the surface with respect to the point $\vec{x}$ is:

$$\Omega_S = \int_{S_v} dS \frac{|\hat{n}(\vec{x}_S) \cdot (\vec{x} - \vec{x}_S)|}{|\vec{x} - \vec{x}_S|^3}, \quad (4.33)$$

where the integral only includes surface elements that are visible to point $\vec{x}$.

Clearly, the solid angle of any closed surface enclosing $\vec{x}$ is 4π . Also, for closed surfaces, whether or not $\vec{x}$ is located inside the surface, we can relax the visibility criterion, and integrate over the entire surface, as follows.

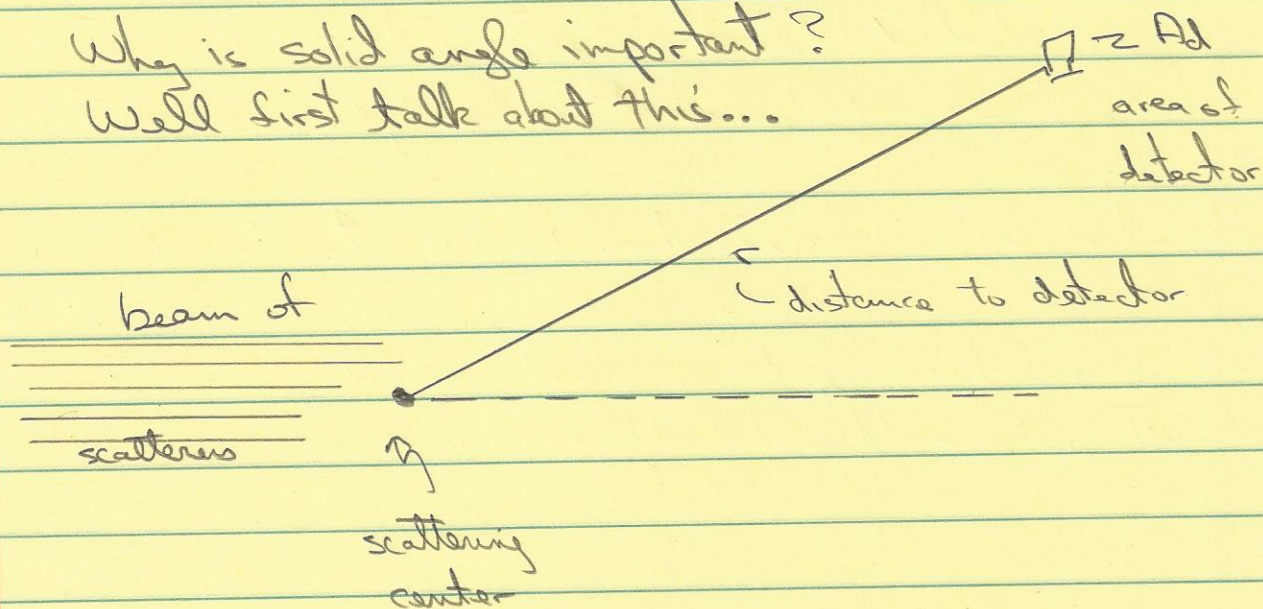
$$\Omega_S = \left| \int_S dS \frac{\hat{n}(\vec{x}_S) \cdot (\vec{x} - \vec{x}_S)}{|\vec{x} - \vec{x}_S|^3} \right|. \quad (4.34)$$

The absolute value is required because the direction of $\hat{n}$ is arbitrary.

That's the general expression, We shall get to (much) easier examples in a minute

Why is solid angle important?

We'll first talk about this...



If the detector dimensions are $\ll r$ the fraction of "angular" space occupied by the detector is

$$\frac{A_d}{4\pi r^2} \leftarrow \begin{array}{l} \text{the detector "covers" the scattering} \\ \text{center} \end{array}$$

Area of a sphere centered at the scattering center and containing A_d

This is called the "solid" angle of the detector,

Therefore, the volume element in spherical - polar coordinates is

$$dV = r^2 dr \sin\theta d\theta d\phi$$

The surface area of a sphere can be expressed as

$$\begin{aligned} A_{sph} &= \int d\vec{x} \delta(r - R_{sph}) \\ &= \underbrace{\int_0^{R_{sph}} r^2 dr \delta(r - R_{sph})}_{= R_{sph}^2} \underbrace{\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi}_{= 4\pi} \\ &= 4\pi R_{sph}^2 \end{aligned}$$

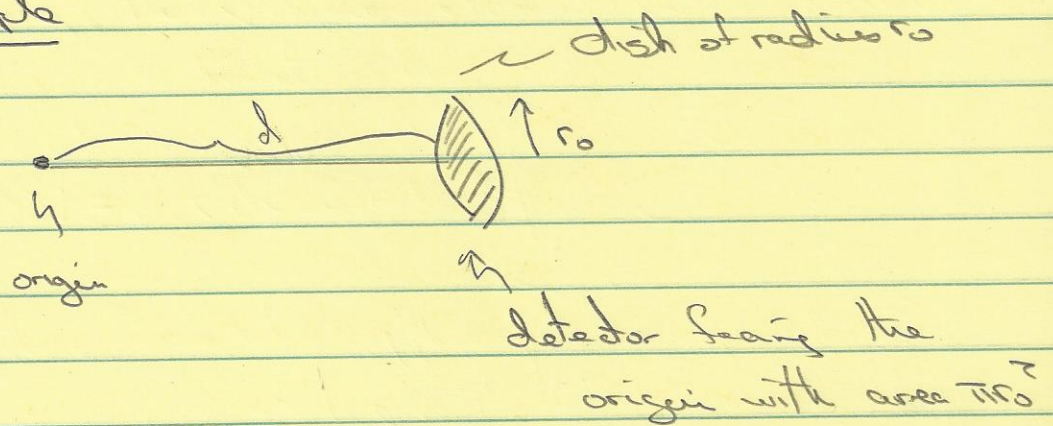
Consequently, a differential element of area on a sphere is

$$dA_{sph} = R_{sph}^2 \sin\theta d\theta d\phi$$

$$\textcircled{1} \quad \left[\frac{dA_{sph}}{R_{sph}^2} = d\Omega \right] \quad \text{which defines differential solid angle}$$

$$d\Omega = \sin\theta d\theta d\phi$$

(5)

ExampleWhat is ΔR_{det} ?Ans. Aligning d along the z -axis

$$\Delta R_{det} = \int_0^{\cos^{-1}[\frac{d}{[d^2 + r_0^2]^{1/2}}]} \sin \theta \, d\theta \int_0^{2\pi} d\phi$$

$$\text{Let } \mu = \cos \theta \quad d\mu = -\sin \theta \, d\theta$$

$$\Delta R_{det} = 2\pi \int_{\left[\frac{d}{[d^2 + r_0^2]^{1/2}}\right]}^1 d\mu$$

$$\boxed{\Delta R_{det} = 2\pi \left[1 - \frac{d}{[d^2 + r_0^2]^{1/2}} \right]}$$

Makes sense?

(6)

try the extremes! (Theorists always do this)

$$\lim_{d \rightarrow \infty} \Delta R_{\text{det}} = 2\pi \quad \checkmark$$

(covers half of full space, 4π)

$$\lim_{d \rightarrow \infty} \Delta R_{\text{det}} = \lim_{d \rightarrow \infty} 2\pi \left[1 - \left[1 + \frac{r_0^2}{d^2} \right]^{-1/2} \right]$$

$$= \lim_{d \rightarrow \infty} 2\pi \left[1 - \left[1 - \frac{1}{2} \frac{r_0^2}{d^2} + O\left(\frac{r_0^4}{d^4}\right) \right] \right]$$

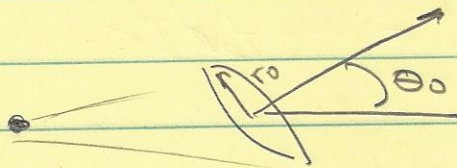
$$= \lim_{d \rightarrow \infty} 2\pi \left[\frac{1}{2} \frac{r_0^2}{d^2} + O\left(\frac{r_0^4}{d^4}\right) \right]$$

$$\Rightarrow \frac{\pi r_0^2}{d^2}$$

$$\lim_{d \rightarrow \infty} \Delta R_{\text{det}} = \frac{\pi r_0^2}{d^2} \quad \text{e.g. cf. } \frac{Ad}{R_{\text{sph}}^2} \quad \text{from (1)}$$

It makes sense!

Slightly harder problem



disk is at some angle θ_0

It can be shown that

$$\lim_{d \rightarrow \infty} \Delta \mathcal{P}_{\text{det}} = \frac{\pi r_0^2}{d^2} \cos \theta_0$$

$$\lim_{d \rightarrow 0} \Delta \mathcal{P}_{\text{det}} = 2\pi$$

The case for general d is quite "interesting"

You have to go back thru (4.34) on p.1