

6. Basic properties of atoms ...

We will only discuss several interesting topics in this chapter:

- Basic properties of atoms—basically, they are really small, really, really, really small
- The Thomson Model—a really bad way to model the atom (with some nice E & M)
- The Rutherford nuclear atom—he got the nucleus (mostly) right, at least
- The Rutherford scattering distribution—Newton gets it right, most of the time
- The Rutherford cross section—still in use today
- Classical cross sections—yes, you can do this for a bowling ball
- Spherical mirror cross section—the disco era revisited, but with science
- Cross section for two unlike spheres—bowling meets baseball
- Center of mass transformation in classical mechanics, a revised mass, effectively
- Center of mass transformation in quantum mechanics—why is QM so strange?
- The Bohr model—one wrong answer, one Nobel prize
- Transitions in the Bohr model—it does work, I'm not Lyman to you
- Deficiencies of the Bohr Model—how did this ever work in the first place?

After this we get back to 3D QM, and it gets real again.

... Basic properties of atoms

Atoms are small!

e.g.

Iron (Fe): mass density, $\rho_{\text{Fe}} = 7.874 \text{ g/cm}^3$
molar mass, $M_{\text{Fe}} = 55.845 \text{ g/mol}$

$N_{\text{A}} = 6.0221413 \times 10^{23}$, Avogadro's number, the number of atoms/mole

$\therefore$ the volume occupied by an Fe atom is:

$$\frac{M_{\text{Fe}}}{N_{\text{A}}} \frac{1}{\rho_{\text{Fe}}} = \frac{55.845 \text{ [g/mol]}}{6.0221413 \times 10^{23} \text{ [1/mol]}} \frac{1}{7.874 \text{ [g/cm}^3\text{]}} = 1.178 \times 10^{-23} \text{ cm}^3$$

The radius is $(3V/4\pi)^{1/3} = 0.1411 \text{ nm} = 1.411 \text{ \AA}$ in Angström units.

Atoms are stable (usually)

$\therefore$ there must be a balance of:

an attractive force that keeps them together, and
a repulsive force that keeps them from collapsing.

The Thomson model of the atom ...

Atoms contain negative charges, called electrons

—J J Thomson, 1897, Nobel Prize 1906, for doing 3 experiments that determined:

1. Cathode rays (electron beams), are made up of small, inseparable packets of charge.
2. Those charges are negative.
3. Thomson measured the ratio e/m , and also determined that the electron's mass was about 1000 times less than H^+ .

Thomson proposed a “plum pudding” model, where the positive charge, he assumed, comprised the bulk the pudding (nothing about neutrons, protons or nuclei was known at the time), and electrons were situated in it like raisins in the pudding.

... The Thomson model of the atom ...



Those electrons, if perturbed, would oscillate.
We can calculate this by applying E&M theory.

Gauss's law states:

$$\oint_{S(V)} dS \vec{E} \cdot \hat{n} = \frac{1}{\epsilon_0} \int_V d\vec{x} \rho(\vec{x}) , \quad (6.1)$$

where the surface, S , encompasses a volume, V , $\hat{n}$ is a unit normal vector normal to the surface element, dS , and $\rho(\vec{x})$ is the charge density.

... The Thomson model of the atom ...

Let us consider a single electron in a Thomson model for a spherical atom with a positive charge of Ze .

In this case, applying (6.1) at radius r , and exploiting spherical symmetry $\Rightarrow$

$$E(r)S(r) = \frac{1}{\epsilon_0} \frac{Ze}{V_0} V(r) , \quad (6.2)$$

since $\rho(\vec{x}) = Ze/V_0$ where V_0 is the total volume of the spherical atom.

Using: $S(r) = 4\pi r^2$; $V(r) = \frac{4}{3}\pi r^3$; $V_0 = \frac{4}{3}\pi R^3$, (6.2) results in:

$$E(r) = \frac{Ze}{4\pi\epsilon_0} \frac{r}{R^3} . \quad (6.3)$$

$\therefore$ the force on the electron is attractive to the center of the sphere:

$$F(r) = -\frac{Ze^2}{4\pi\epsilon_0} \frac{r}{R^3} . \quad (6.4)$$

... The Thomson model of the atom

This is a spring, with a spring constant k .

$$F(r) = -kr \quad (6.5)$$

$$k = \frac{Ze^2}{4\pi\epsilon_0 R^3} \quad \text{with frequency} \quad (6.6)$$

$$\omega^2 = \frac{Ze^2}{4\pi\epsilon_0 R^3 m_e} \quad (6.7)$$

where m_e is the mass of the electron.

We know (and they knew then) that when an atom is excited, it radiates a lot of discrete frequencies.

The Thomson model predicts that a hydrogenic atom (Ze positive charges, one electron) has just one frequency.¹

However, even the hydrogen atom is known to radiate “discrete frequencies”, as discovered by Balmer in 1885.

¹That frequency, if converted to a photon energy, corresponds to about 27 eV for $Z = 1$ — in the right “ballpark”.

The Rutherford model of the atom ...

It turned out that the “plum pudding” model of the atom was wrong, as demonstrated by Rutherford. However, it took a long time to resolve this question satisfactorily.



Ernest Rutherford (1871–1937)

Rutherford was the “father” of nuclear physics.

... The Rutherford model of the atom ...

Rutherford decided to use α -particles, (now known as the nucleus of the ${}^4\text{He}$ atom) and bombard very thin foils of Au. (It is possible to hammer out gold to make foils. It is a popular roofing material!)

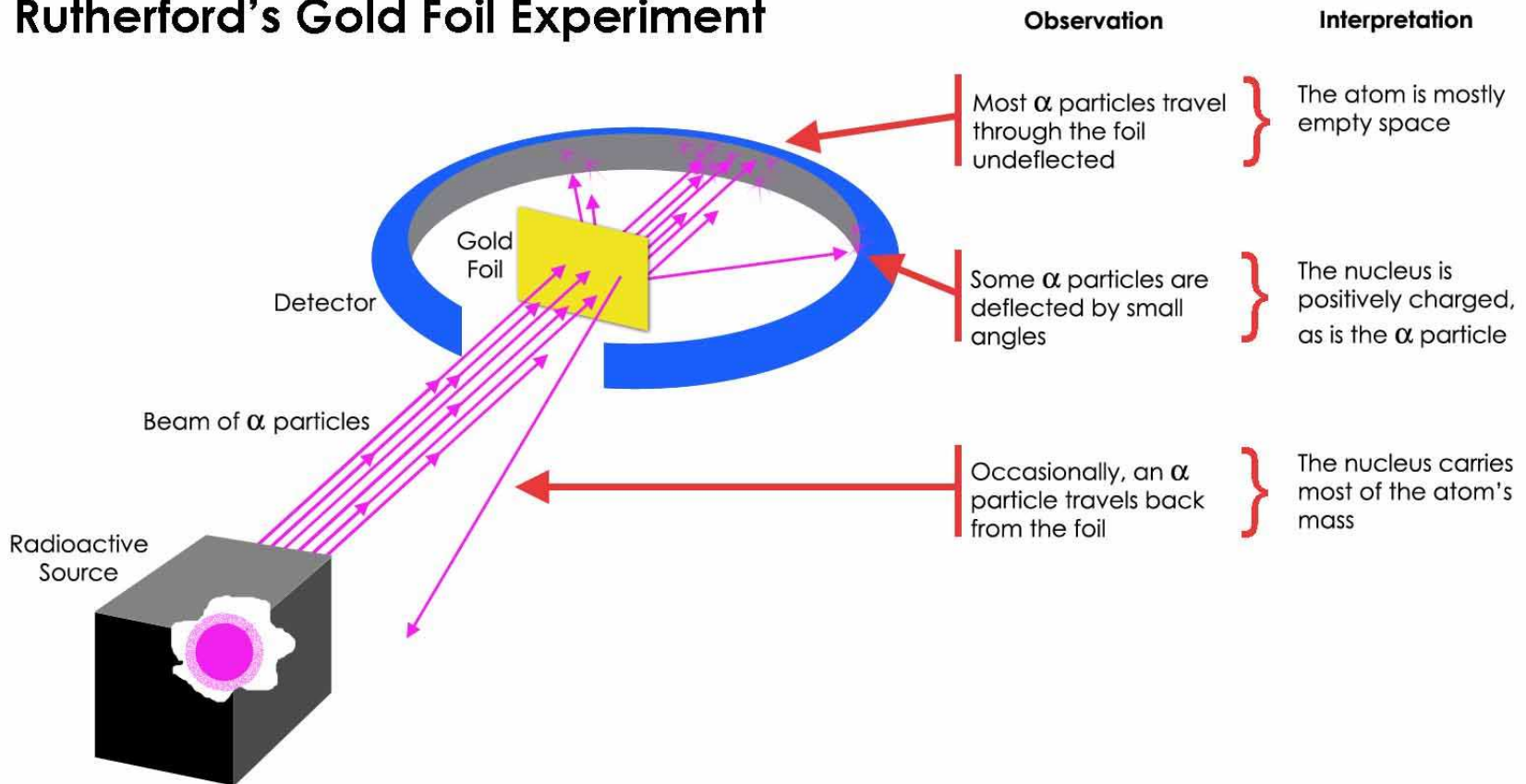


Quite a lot was known about the α -particle, at the time.
It has a positive charge, $2\times$ as large as the e^-
The α is about 8000 heavier than the e^- .

... The Rutherford model of the atom ...

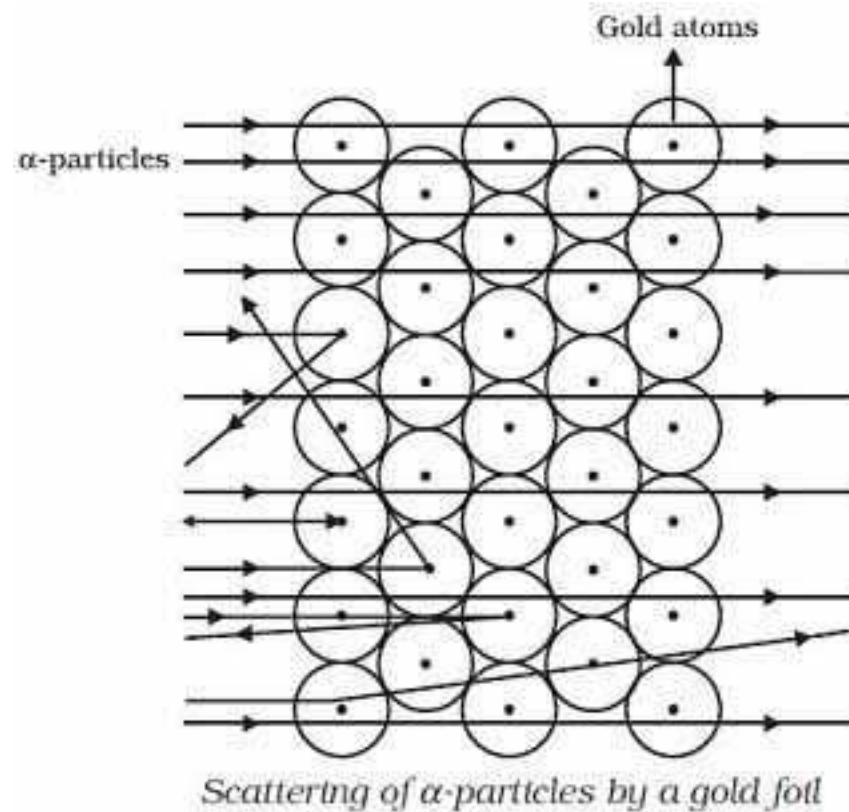
He did the following experiment (with Geiger and Marsden):

Rutherford's Gold Foil Experiment



... The Rutherford model of the atom ...

At the μ -scopic level...



... The Rutherford model of the atom ...

Rutherford found out that the positive charge is concentrated in the nucleus, much smaller than the atom, and also contains almost all of the mass of the atom. He was so surprised that occasionally an α would backscatter from the foil, that he issued this famous quote:

“It was quite the most incredible event that has ever happened to me in my life. Crikey! It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you.”²

Once he got over his surprise, he arrived at the inescapable conclusion that most of the atom is empty space, most of the mass is concentrated at the center, the matter that contains the positive charge, and called it the “nucleus”.

Nuclear Physics was born, giving rise to Nuclear Engineering.

In the absence of Quantum Mechanics, Rutherford envisaged the “Rutherford model of the atom”. A lot was known about electrostatics, and he went to work ... and later, Stockholm.

²Truth be told, I added the “Crikey” because Rutherford was a New Zealander, and New Zealand is close to Australia, and “Crikey” is a common exclamation of surprise there. The rest of the quote is historically accurate, however.

Rutherford scattering ...

Classical physics predicts the trajectory of the α as it encounters the nucleus.

Derived under the assumptions that:

- both the nucleus and the α are point particles,
- the nucleus does not recoil (tantamount to saying that $M_\alpha \ll M_N$), and
- the α sees the entire nuclear charge over its whole trajectory
(i.e. atomic electron shielding is ignored),

the trajectory is given by:

$$\frac{1}{r} = \frac{1}{b} \sin \theta - \frac{zZe^2}{8\pi\epsilon_0 b^2 K} (1 + \cos \theta) , \quad (6.8)$$

where

z = electric charge of the α

Z = electric charge of the nucleus

K = initial kinetic energy of the α

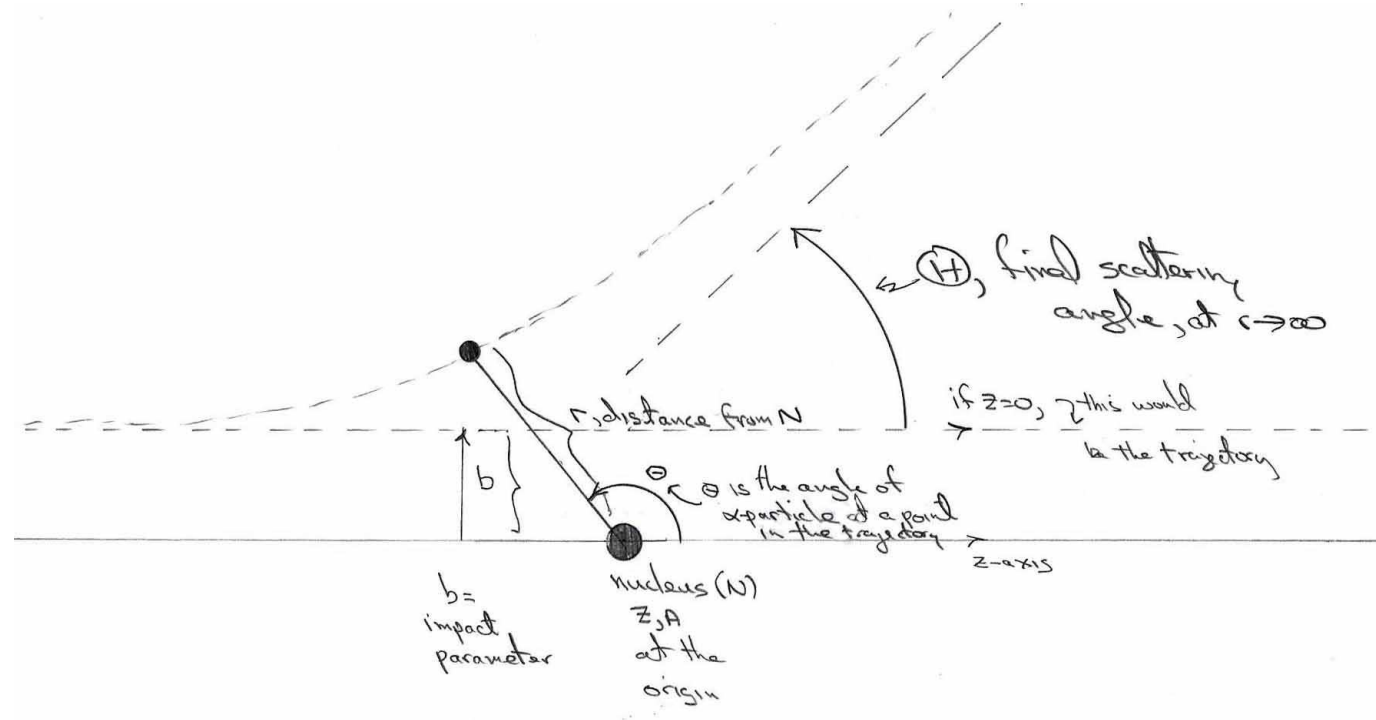
r = distance from the origin (location of the nucleus) to the α

θ = angle with respect to the z -axis

b = $\perp$ distance of the trajectory from the z -axis when r is very far away at $t = 0$

Θ = asymptotic value of θ as $t \rightarrow \infty$

... Rutherford scattering ...



Note that (6.8) has two solutions for $r \rightarrow \infty$, one when $\theta = \pi$, which corresponds to the starting point of the trajectory.

If we say $\theta = \Theta$ at the other end of the trajectory, then setting $r \rightarrow \infty$ and $\theta = \Theta$ in (6.8) gives the deflection of the α as:

$$\frac{\sin \Theta}{1 + \cos \Theta} = \frac{zZe^2}{8\pi\epsilon_0 bK}, \quad (6.9)$$

... Rutherford scattering ...

or,

$$b = \frac{zZ}{2K} \frac{e^2}{4\pi\epsilon_0} \frac{1 + \cos \Theta}{\sin \Theta}, \quad (6.10)$$

that makes explicit the relationship between the impact parameter, b , and the scattering angle, Θ .

Using the trigonometric identities,

$$\sin \Theta = \sin \left(\frac{\Theta}{2} + \frac{\Theta}{2} \right) = \sin \frac{\Theta}{2} \cos \frac{\Theta}{2} + \sin \frac{\Theta}{2} \cos \frac{\Theta}{2} = 2 \sin \frac{\Theta}{2} \cos \frac{\Theta}{2}$$

and

$$1 + \cos \Theta = 1 + \cos \left(\frac{\Theta}{2} + \frac{\Theta}{2} \right) = 1 + \cos^2 \frac{\Theta}{2} - \sin^2 \frac{\Theta}{2} = 2 \cos^2 \frac{\Theta}{2},$$

in (6.10) simplifies it to,

$$b = \frac{zZ}{2K} \frac{e^2}{4\pi\epsilon_0} \cot \frac{\Theta}{2}. \quad (6.11)$$

This is a celebrated equation!

Soon we shall discover that $b(\Theta)$ relationships play an important role in the determination of cross sections.

... Rutherford scattering ...

Distance of closest approach

Let us consider two cases, one where $b = 0$ (the α -particle's direction is in a direct line with the nucleus), and treat the case $b > 0$ as a minimization problem.

Case $b = 0$

When $b = 0$, eventually the α -particle will come to rest before turning around 180° and “retracing its steps”.

At the point where the α -particle stops, it has no kinetic energy—it's all in its potential energy. Balancing the initial kinetic energy, K , with the potential energy at that point gives:

$$K = \frac{zZe^2}{4\pi\epsilon_0 R_C} , \quad (6.12)$$

where R_C is the “Coulomb barrier”, the closest possible distance that these point charges, z and Z , could achieve, given the initial K .

... Rutherford scattering ...

Solving for R_C :

$$R_C = \frac{zZe^2}{4\pi\epsilon_0 K} . \quad (6.13)$$

Case $b > 0$

Let us use (6.13) to write (6.8) more compactly:

$$x = \sin \theta - \kappa(1 + \cos \theta) , \quad (6.14)$$

where $\kappa \equiv R_C/(2b)$ and $x = b/r$.

We will treat this case as a minimization problem, where we find the extremum associated with the trajectory, the distance of closest approach for a given b .

If we take the derivative of (6.14) and set it to zero, we can find the $\Theta = \Theta_0$ associated with the closest distance.

... Rutherford scattering ...

After some arithmetic:³

$$\tan \Theta_0 = -1/\kappa , \quad (6.15)$$

Calling R_0 the distance at $\Theta = \Theta_0$, it is also possible to show that:

$$R_0(b) = \sqrt{b^2 + (R_C/2)^2} + R_C/2 . \quad (6.16)$$

Note that $R_0(0) = R_C$, so it makes sense, and agrees with the much simpler derivation when we treated $b = 0$ as a special case.

Alternative form of the equation of motion

The compact form of (6.14) expresses the distance of the projectile from the nucleus as a function of θ . For plotting purposes, it is useful to express (6.14) in terms of the angle as a function of the distance.

Also shown in class will be the derivation of the relationship:

$$\cos \theta = \frac{-\kappa(x + \kappa) \pm \sqrt{1 - 2x\kappa - x^2}}{1 + \kappa^2} . \quad (6.17)$$

³There are some subtleties in this development. The details will be given in class. Be prepared to take notes!

... Rutherford scattering ...

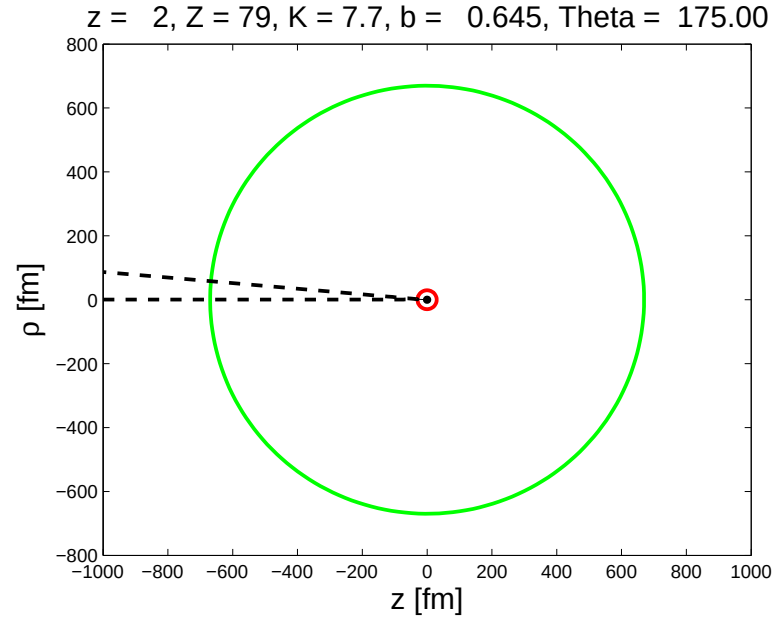
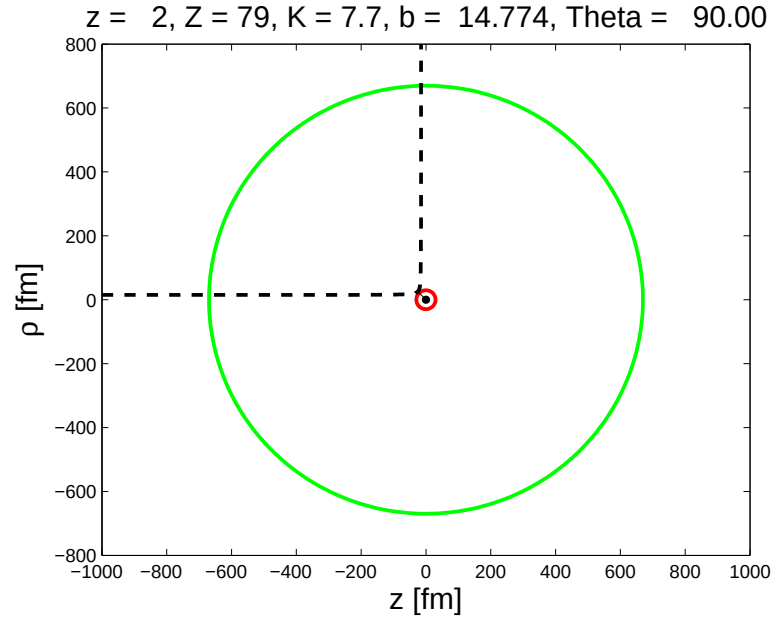
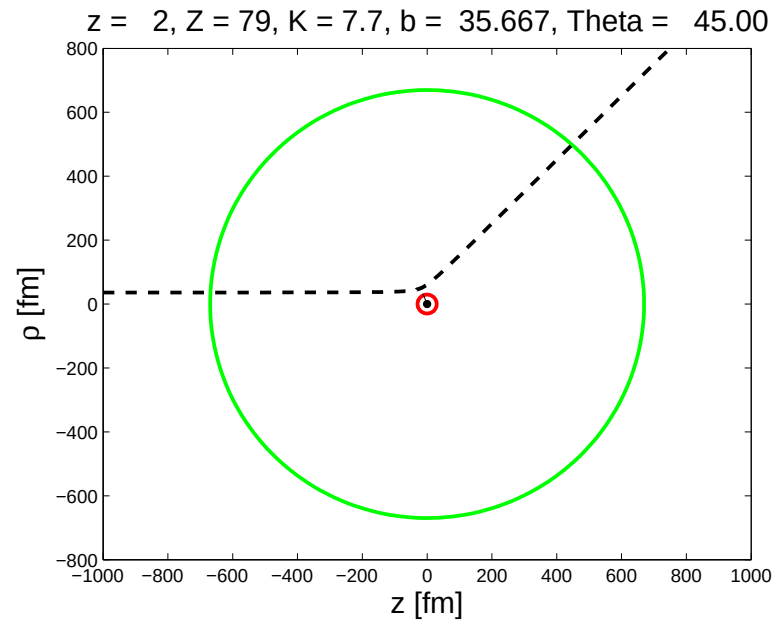
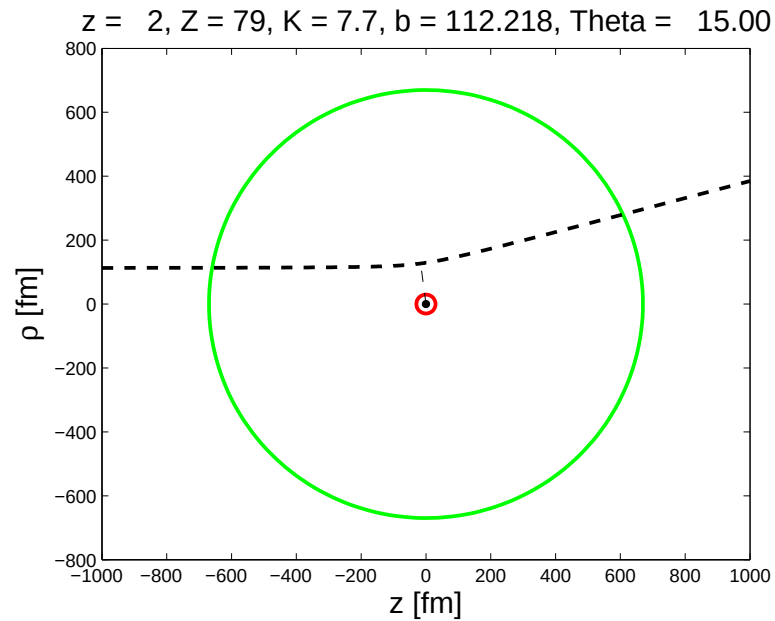
When the radical is zero, the extremum value of x can be obtained, the equivalent of (6.16), but expressed with the x variable.

Some plots are in order!

In the following plots, the black circle represents the nucleus, the larger red circle is the Coulomb barrier, and the largest green circle is the K -shell (innermost) electron radius.

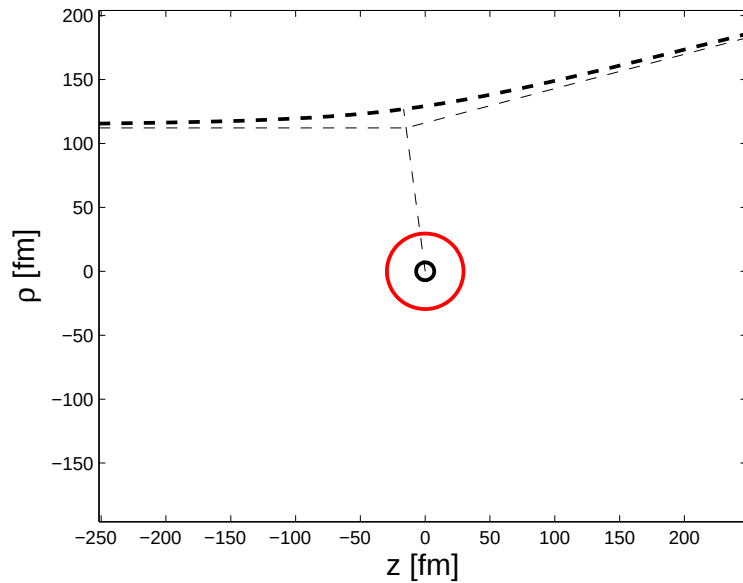
The second page of plots shows closeups as well as asymptotes and the closest approach line.

... Rutherford scattering ...

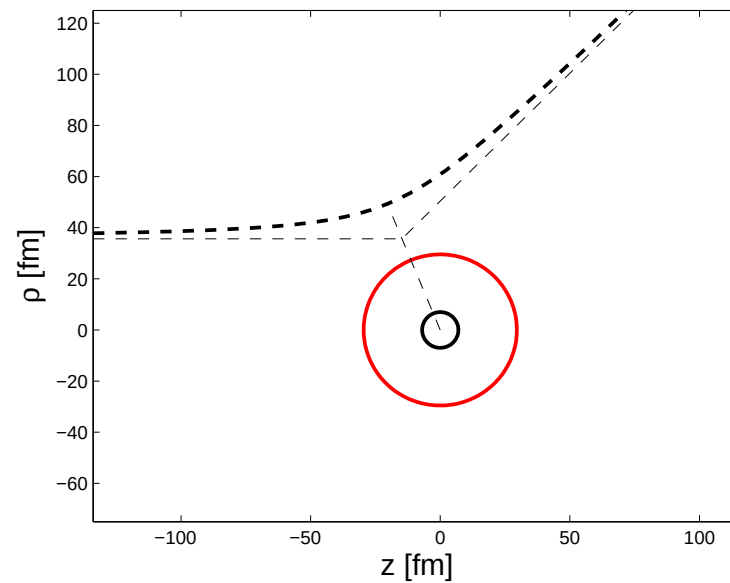


... Rutherford scattering ...

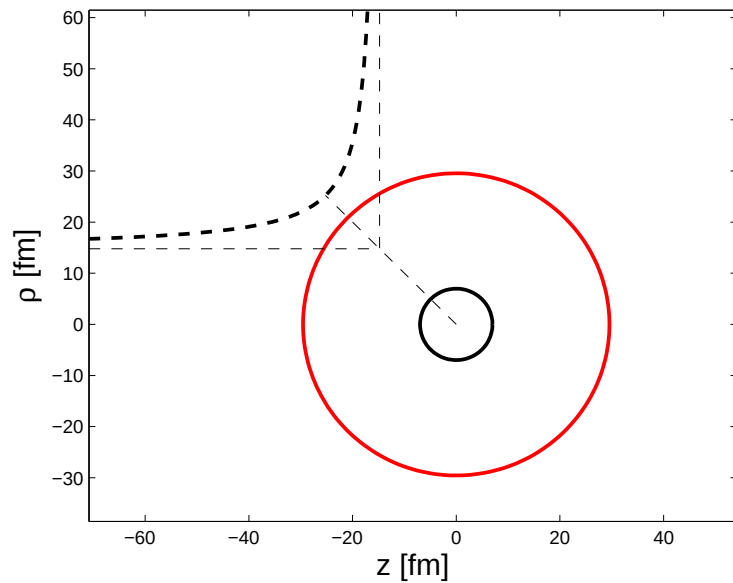
$z = 2, Z = 79, K = 7.7, b = 112.218, \text{Theta} = 15.00$



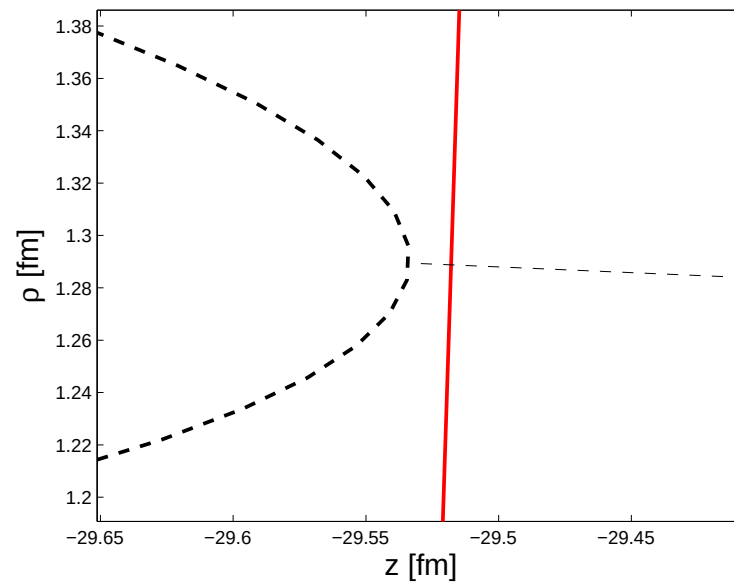
$z = 2, Z = 79, K = 7.7, b = 35.667, \text{Theta} = 45.00$



$z = 2, Z = 79, K = 7.7, b = 14.774, \text{Theta} = 90.00$



$z = 2, Z = 79, K = 7.7, b = 0.645, \text{Theta} = 175.00$



... Rutherford scattering ...

Before we start discussing cross sections, we should talk a little about the three approximations that were used to derive the Rutherford scattering formula.

These were discussed initially on Slide 12.

... Rutherford scattering ...

1. The nucleus and the α are point particles

As we shall see in NERS 312, the average radius of a nucleus is given as:

$$R(A) = R_0 A^{1/3} , \quad (6.18)$$

where $R_0 \approx 1.22$ fm, and A is the atomic mass, *i.e.* the number of nucleons ($n + p$) in the nucleus. This restricts the upper energy on the α s. At high energies, the α will enter the nucleus and formulas (6.8) and (6.11) no longer apply.

That upper energy is found from (6.12):

$$K_{\max}(z, A_p, Z, A_t) = \frac{zZe^2}{4\pi\epsilon_0[R(A_p) + R(A_t)]^{1/3}} , \quad (6.19)$$

where $R(A_p)$ is the radius of the projectile and $R(A_t)$ is the radius of the target. Evaluating for $A = 4$ for the α and $A = 197$ for Au, we get:

$$K_{\max}(2, 4, 79, 197) = 25.1 \text{ [MeV]} .$$

Since the maximum energy Rutherford had access to was about 7.7 MeV, his measurements were fine in this respect. However, accelerator-based Rutherford scattering may be problematic.

... Rutherford scattering ...

2. The nucleus does not recoil

It turns out that this did not really effect Rutherford either, because the mass of the Au atoms was almost 50 times greater than the α -particle.

Later in this Chapter we will talk about this in detail.

Even though the effect may be small, we can guess the answer. This effect will shift the angle a little to the forward direction.

... Rutherford scattering ...

3. *The α sees the entire nuclear charge Z*

This is an interesting discussion!

We shall see soon, that the the assumption that the projectile sees the entire nuclear positive charge over its whole trajectory has some remarkable consequences.

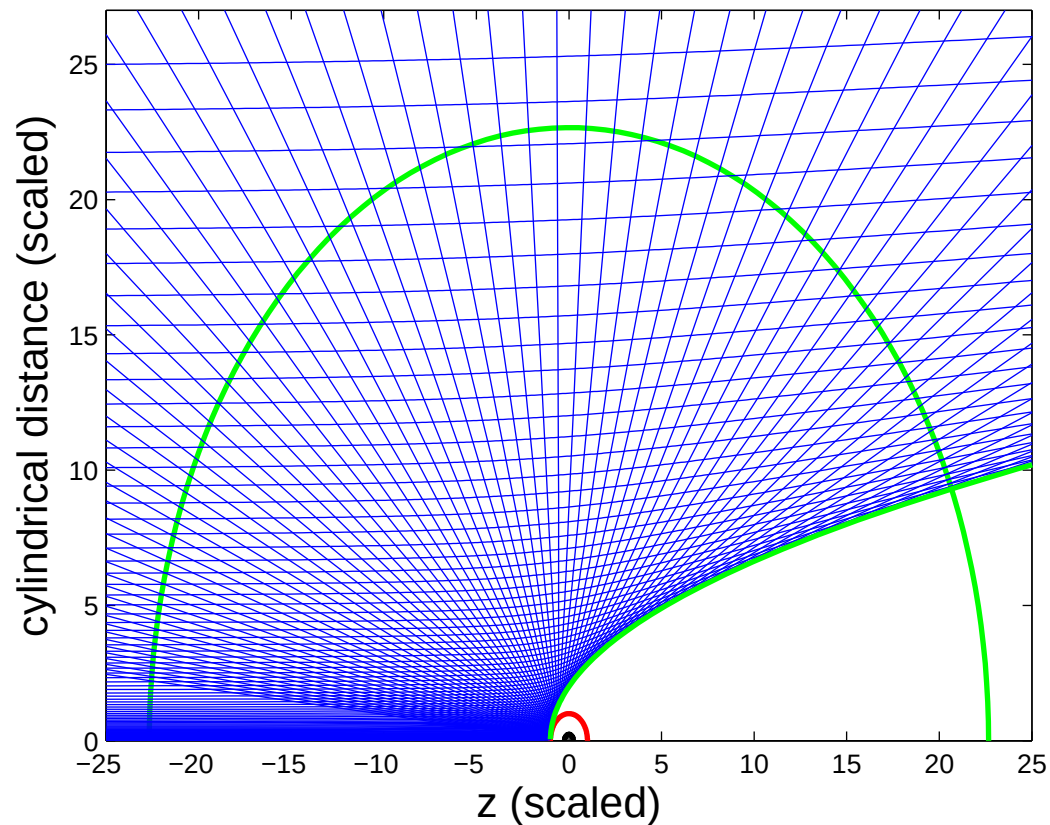
The total cross section of the scattering process (*i.e.* the effective size of the target-projectile system, is infinite!

This is illustrated by a figure on the next page.

... Rutherford scattering ...

This graph shows a variety of impact parameters, and the varied trajectories.

The unshielded nuclear charge causes an infinite shadow downstream of the source!



... Rutherford scattering ...

Actually, the nuclear charge is mostly screened by the atomic electrons.

On average the perceived charge of the nucleus is neutral.

This makes the total cross section finite, but it is still very large.

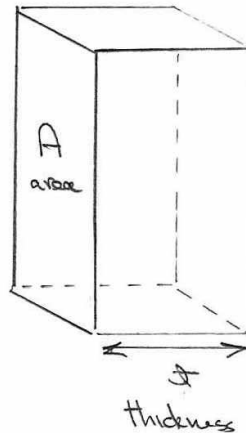
How little this affects the large angle scattering is seen from the graphs on Slide 19.

Most of the scattering occurs near the nucleus, well inside the atomic electron orbitals.

... Rutherford scattering ...

Q. *In what fraction of scatterings did Geiger and Marsden observe a scatter in the back direction?*

To answer this, first determine the density of scatterers ($n = \text{number of nuclei}/V$) in the target:



$$N = \frac{N_A \rho A t}{M} \quad [\text{total number of scatters}]$$

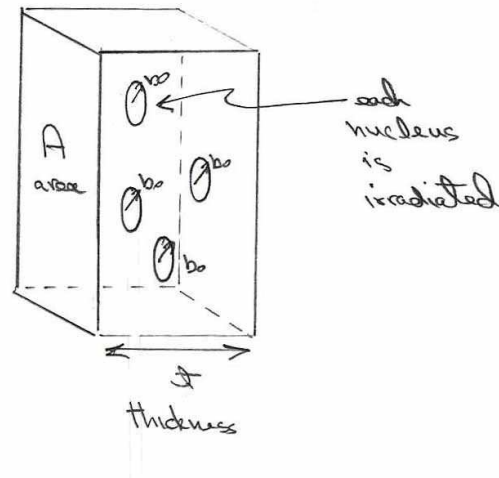
$$V = A t \quad [\text{volume of the target}]$$

$$n = \frac{N_A \rho}{M} \quad [\text{density of scatters}]$$

where ρ is the mass density. n is the “number” density.

... Rutherford scattering ...

Now, irradiate the entire face of the target with an α -particle-beam. Imagine a radius b_0 to around each scatterer. Choose b_0 to give a deflection of exactly 90° .



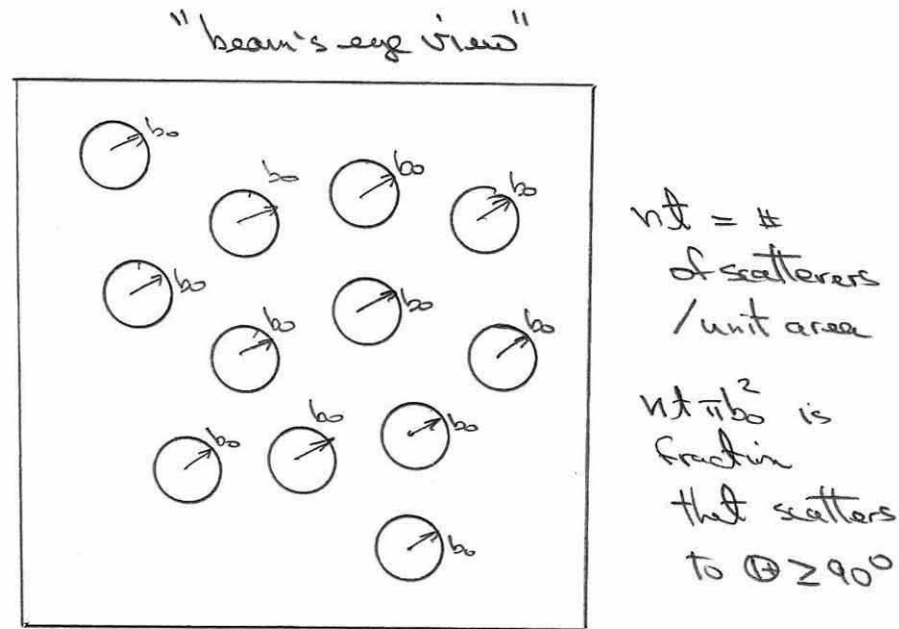
Hence, the entire illumination from the center of each nucleus to b_0 produces $\Theta \geq 90^\circ$ deflections.

Hence, the fraction of scatters observed in the back direction is:

$$f_{\Theta \geq 90^\circ} = f_{b \leq b_0} = nt\pi b_0^2, \quad (6.20)$$

since nt is the areal density (number of scatterers per unit area) of the scatterers, assuming there is no overlap among the scatterers. This is fine when the targets are thin enough. Recall that Rutherford also concluded that most of the target is empty space!

... Rutherford scattering ...



Recall that

$$b_0 = \frac{zZ}{2K} \frac{e^2}{4\pi\epsilon_0} \cot \frac{\Theta_0}{2} .$$

The above into (6.20) gives:

$$f_{\Theta \geq 90^\circ} = f_{b \leq b_0} = nt\pi \left(\frac{zZ}{2K} \frac{e^2}{4\pi\epsilon_0} \cot \frac{\Theta_0}{2} \right)^2 . \quad (6.21)$$

... Rutherford scattering ...

Evaluating for a target thickness of 400 atoms gives:

$$f_{\Theta \geq 90^\circ} = 2.6 \times 10^{-6}$$

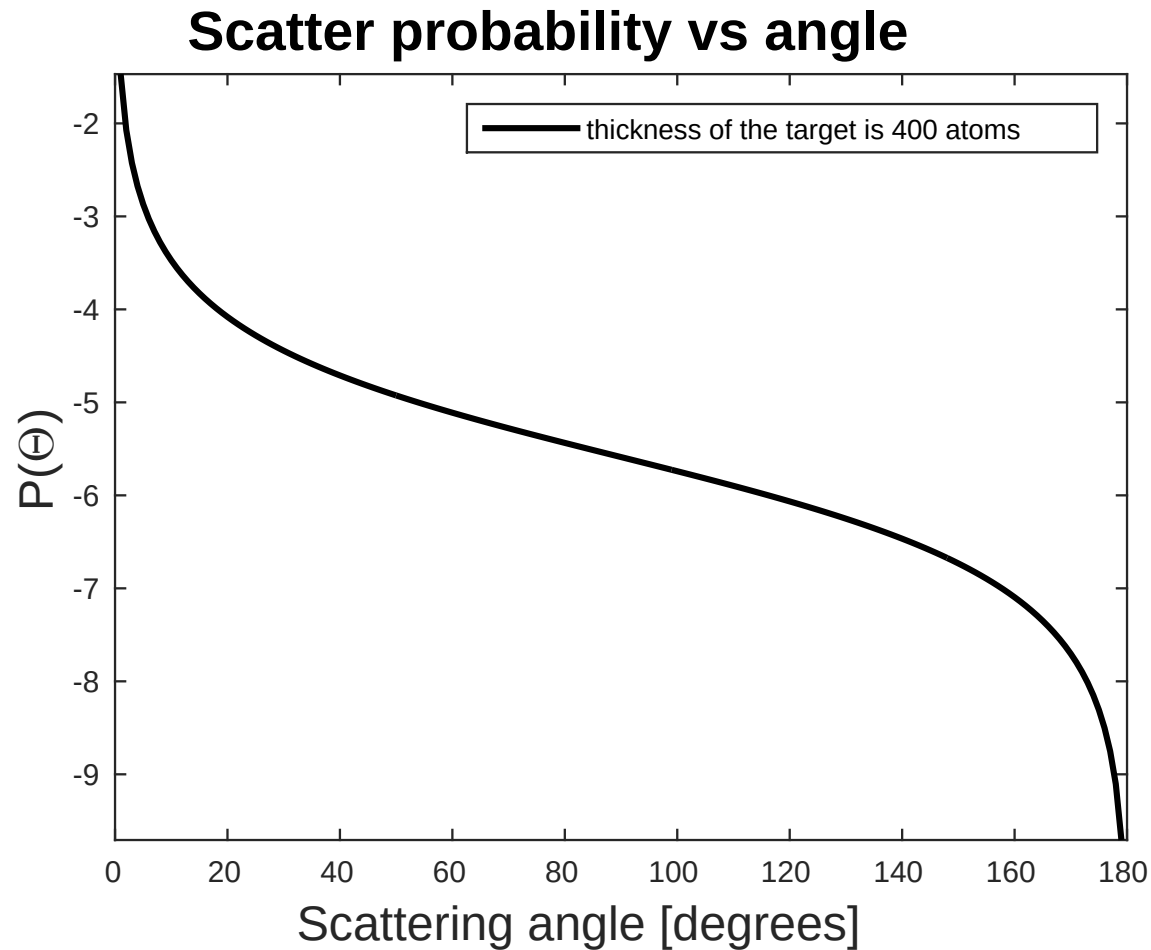
$$f_{\Theta \geq 45^\circ} = 1.5 \times 10^{-5}$$

$$f_{\Theta \geq 1^\circ} = 0.034$$

96.6% of the beam scatters less than 1° ! No wonder Rutherford was so surprised.

... Rutherford scattering ...

Here is what the entire range of scattering probabilities look like, down to 1° .

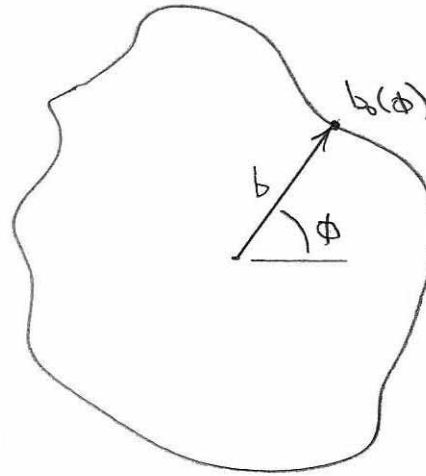


How do we characterize the “directedness” of the scattering?

Differential cross sections ...

Classical cross sections ...

A classical cross section is a well-defined concept — it is the effective area of a 3D object, seen from some point of view. In general, it will change depending on the orientation of the object.



This cross sectional area is called σ and has the units L^2 .

For the object shown above,

$$\sigma = \int_0^{2\pi} d\phi \int_0^{b_0(\phi)} b db . \quad (6.22)$$

For example, if the object was a sphere of radius R , $b_0(\phi) = R$, and $\sigma_{\text{sphere}} = \pi R^2$.

... Classical cross sections ...

The differential cross section is the integrand of (6.22):

$$d\sigma = |b \, db| \, d\phi . \quad (6.23)$$

The absolute value guards against a spurious negative sign that may arise from taking the derivative of b .

The differential cross section contains some additional information. It says something about the shape of the object.

We can measure this by scattering other objects “projectiles” against the “target” and measuring what angle they recoil at.

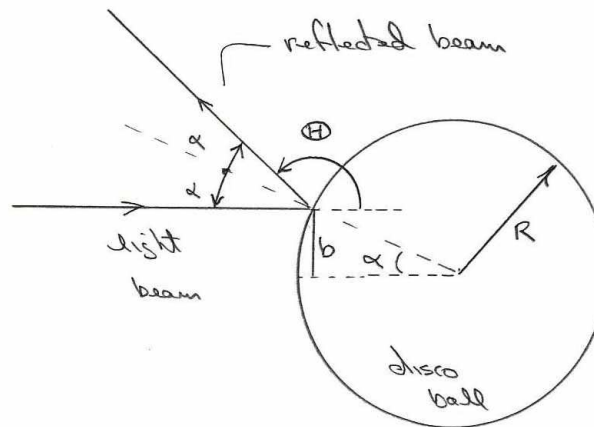
Let's imagine that the target is heavy enough so that it does not recoil from being struck, and that it perfectly reflects the projectiles. If the target is a flat shape, all the projectiles turns around 180° . If it irregularly shaped, then the projectiles scatter at various angles. The distribution of those angles says something about its shape.

... Classical cross sections ...

Let's work out a specific example, a target comprised of a perfectly spherical mirror irradiated uniformly by a parallel beam of photons, like a laser swept across a disco ball.



All we need is a relationship between b and the scattering angle, Θ . Consulting the figure below, $b = R \sin \alpha$.



... Classical cross sections ...

Also from the figure, $\Theta = \pi - 2\alpha$, Therefore the relationship between b and Θ is:

$$b = R \sin \left(\frac{\pi}{2} - \frac{\Theta}{2} \right) = R \cos \left(\frac{\Theta}{2} \right); \quad db = -R \sin \left(\frac{\Theta}{2} \right) \frac{d\Theta}{2} \quad (6.24)$$

Substituting (6.24) into (6.23) gives:

$$d\sigma = \frac{R^2}{2} \sin \left(\frac{\Theta}{2} \right) \cos \left(\frac{\Theta}{2} \right) d\phi = \frac{R^2}{4} \sin \Theta d\Theta d\phi, \quad (6.25)$$

and

$$\sigma = \frac{R^2}{4} \int_0^\pi \sin \Theta d\Theta \int_0^{2\pi} d\phi = \pi R^2, \quad (6.26)$$

as expected. Since

$$\frac{d\sigma}{d\Omega} = \frac{R^2}{4}, \quad (6.27)$$

there is no angular dependence to the differential cross section, which means the scattering is isotropic.

You will never look at a disco ball the same way again!

... Classical cross sections ...

Example: Unequal spherical mass cross section

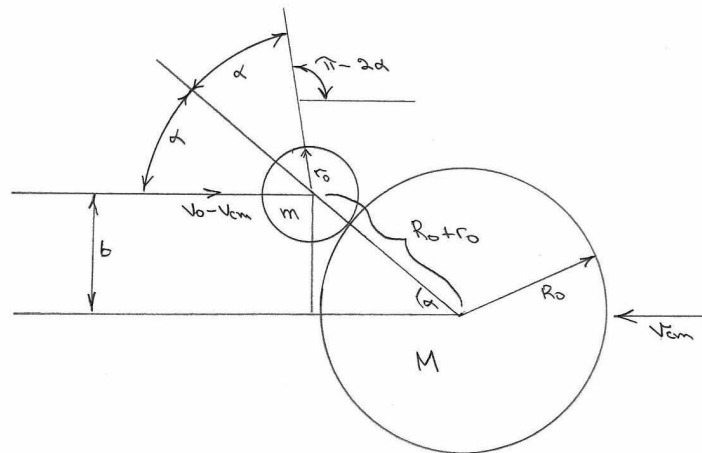
Consider a projectile with mass m , radius r_0 and velocity $v_0 \hat{z}$ striking a stationary target with mass M and radius R_0 .

It is best to work out this problem in the “zero momentum frame” frame by observing the interaction in a moving frame with velocity determined by:

$$m(v_0 - v_{\text{cm}})\hat{z} - Mv_{\text{cm}}\hat{z} = 0 , \quad (6.28)$$

which gives:

$$v_{\text{cm}} = \frac{m}{M + m}v_0 . \quad (6.29)$$



... Classical cross sections

From the figure, we see that:

$$b = (R_0 + r_0) \sin \left(\frac{\pi}{2} - \frac{\Theta}{2} \right) . \quad (6.30)$$

This is a similar result to the previous example, except that 1) we are in the center of momentum frame, and, 2) the size of the projectile now enters into the cross section.

Hence, the cross section is the effective size of the projectile-scatterer system.

We shall see, in NERS 312, that there is a similar relationship for waves, except that the projectile's "radius" is given by its "reduced de Broglie wavelength", $\tilde{\lambda} = \lambda/2\pi$.

In the center-of-momentum frame, the scattering of these two unequal spheres is isotropic. The total cross section is independent of frame, and has the value,

$$\sigma = \pi(R_0 + r_0)^2 . \quad (6.31)$$

Differential Rutherford scattering ...

Recall, from (6.11) that

$$\begin{aligned} b &= \frac{zZe^2}{8\pi\epsilon_0 K} \cot \frac{\Theta}{2} \\ &= \frac{R_C}{2} \cot \frac{\Theta}{2} \quad [\text{using (6.13) for } R_C, \text{ the Coulomb barrier}] \end{aligned} \quad (6.32)$$

Given b in terms of the scattering the scattering angle, Θ , we may write the differential cross section for Rutherford scattering as:

$$\begin{aligned} d\sigma &= |b db| d\phi \\ &= \frac{R_C^2}{8} \frac{\cos(\Theta/2)}{\sin^3(\Theta/2)} d\Theta d\phi \quad [\text{after doing the } \Theta \text{ derivative}] \\ &= \frac{R_C^2}{16} \frac{\sin \Theta d\Theta d\phi}{\sin^4(\Theta/2)} \quad [\text{using } 2 \sin(\Theta/2) \cos(\Theta/2) = \sin \Theta] \end{aligned} \quad (6.33)$$

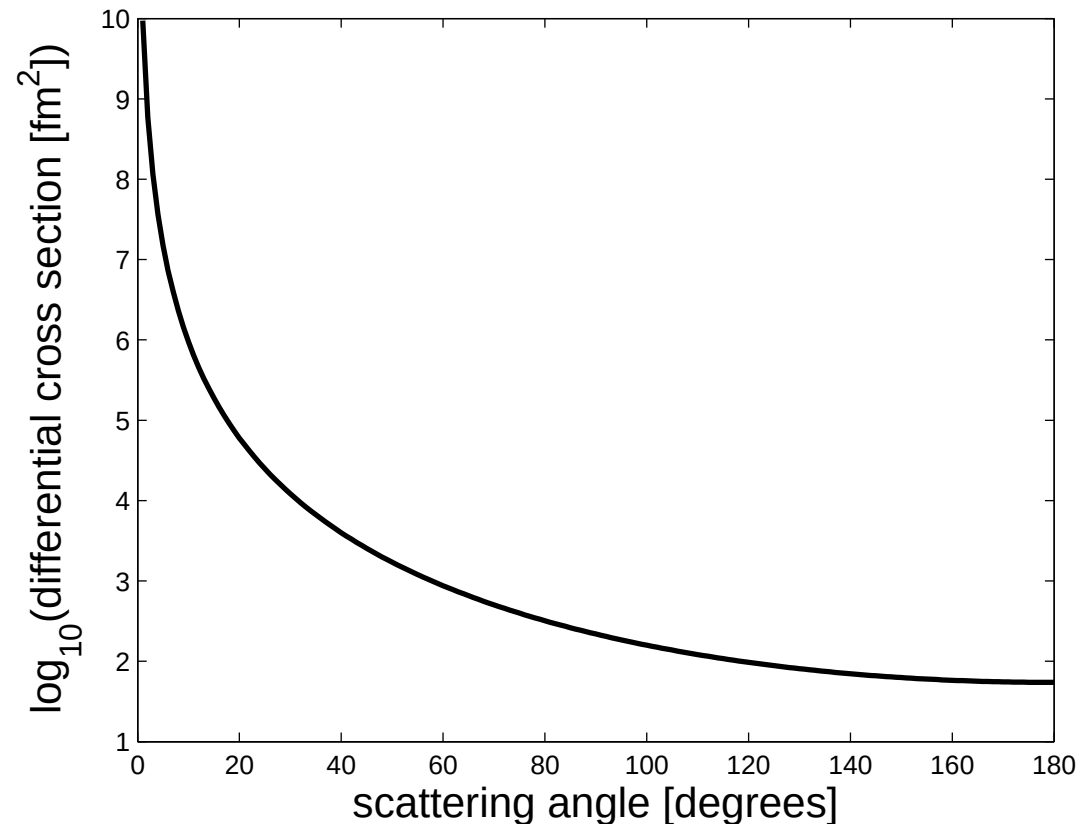
Thus,

$$\frac{d\sigma}{d\Omega} = \frac{R_C^2}{16} \frac{1}{\sin^4(\Theta/2)} . \quad (6.34)$$

The differential cross section is plotted on the next page.

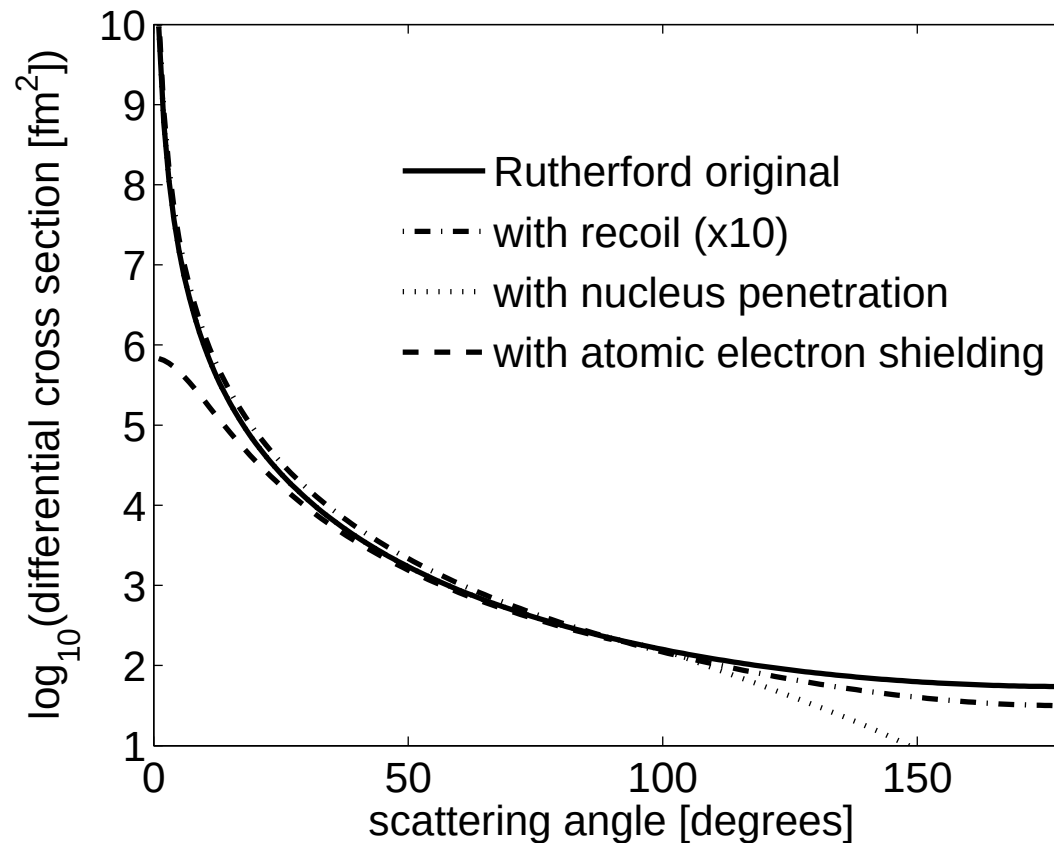
... Differential Rutherford scattering ...

This is (6.33) for 7.7 MeV α -particles on Au.



This shows the classical Rutherford cross section. Because of the $1/\sin^4(\Theta/2)$, it approaches ∞ at $\Theta \rightarrow 0$. Also, for the same reason, the total cross section, the integral over all angles, is infinite as well. Both of these behaviors arise from ignoring the shielding of the nuclear charge by the atomic electrons.

... Differential Rutherford scattering ...



The above figure shows the effect of undoing the approximations.

These effects are discussed on the next page.

... Differential Rutherford scattering ...

The nuclear recoil effect is only prominent when the projectile is heavy. For α -particles on Au at 7.7 MeV it would be hardly noticeable, so it was exaggerated by a factor of 10 to make it show. This effect pushes the angular distribution toward small angles.

The nuclear penetration correction was not a factor at this energy, but it was put in to show what effect it would have on the tail of the distribution, if it were. For this projectile-target combination, one would start to see these take effect at about 25 MeV. (See Slide 22).

Atomic electron shielding is always active and prominent at small angles. It makes the total and differential cross sections finite. One can see evidence of this in Rutherford scattering data.

Center of mass and central forces in Classical Physics ...

The correction for the nuclear recoil in Rutherford scattering will remain a mystery until we discuss center of mass analysis in classical mechanics.

We did this, to some extent, when we just talked about kinematics, earlier in the course. But we are now talking about dynamics, models of what produces those deflections, and so, the discussion gets a little more sophisticated...

Consider a 2-body classical system involving two particles, with masses m_1 and m_2 , at locations, $\vec{x}_1$ and $\vec{x}_2$, undergoing an interaction that is “central”. That is, we consider the most important problem in physics, the action implied by a force that only depends on the magnitude of the distance between two particles. Mathematically, this is expressed via a potential of the form, $V(\vec{x}) = V(|\vec{x}_1 - \vec{x}_2|)$.

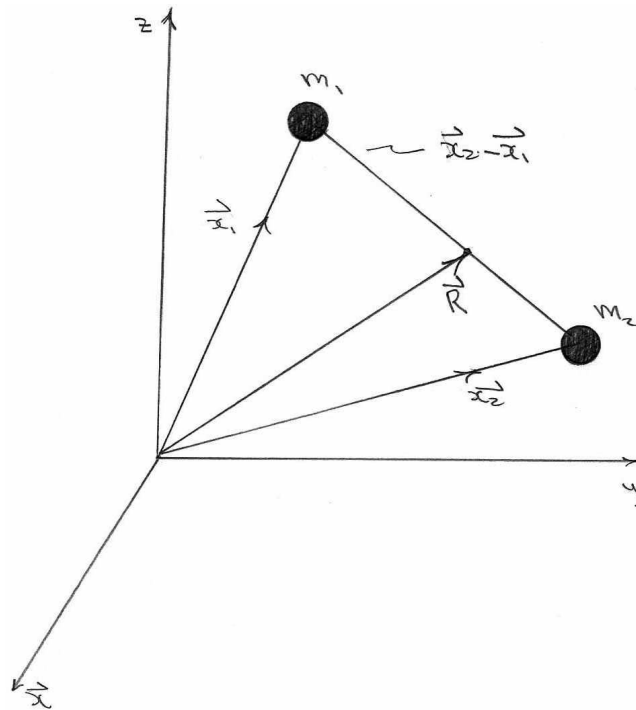
The total energy of the system of two particles is:

$$E_T = \frac{1}{2}m_1|\dot{\vec{x}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{x}}_2|^2 + V(|\vec{x}_1 - \vec{x}_2|) . \quad (6.35)$$

... Center of mass and central forces in Classical Physics ...

The position of the center-of-mass is given by:

$$\vec{R} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2} . \quad (6.36)$$



... Center of mass and central forces in Classical Physics ...

Define the relative position:

$$\vec{r} = \vec{x}_2 - \vec{x}_1 . \quad (6.37)$$

Rewrite $\vec{x}_1$ and $\vec{x}_2$ in terms of $\vec{R}$ and $\vec{r}$:

$$\vec{x}_1 = \vec{R} - \frac{m_2}{m_1 + m_2} \vec{r} \quad (6.38)$$

$$\vec{x}_2 = \vec{R} + \frac{m_1}{m_1 + m_2} \vec{r} \quad (6.39)$$

(6.38) and (6.39) $\rightarrow$ (6.35) $\Rightarrow$

$$E_T = \frac{m_1}{2} \left| \dot{\vec{R}} - \frac{m_2}{m_1 + m_2} \dot{\vec{r}} \right|^2 + \frac{m_2}{2} \left| \dot{\vec{R}} + \frac{m_1}{m_1 + m_2} \dot{\vec{r}} \right|^2 + V(r) . \quad (6.40)$$

... Center of mass and central forces in Classical Physics ...

The cross terms in the dot-product expansions of (6.40) vanish, and (6.40) becomes:

$$E_T = \frac{1}{2}M|\dot{\vec{R}}|^2 + \frac{1}{2}\mu|\dot{\vec{r}}|^2 + V(r) , \quad (6.41)$$

where

$$M = m_1 + m_2 \quad [\text{total mass}] \quad (6.42)$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad [\text{“reduced” mass}] \quad (6.43)$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad [\text{inverse “reduced” mass (easier to remember)}] \quad (6.44)$$

Interpretation:

- $V(r)$ only acts on r , so all the dynamics occurs in r .
- There is no potential acting on R , unless it comes from a source external and separate from the masses. Hence the energy of the “center of mass system” is all kinetic, and expressed by:

$$K_{\text{CM}} = \frac{1}{2}M|\dot{\vec{R}}|^2 \quad \text{with equation of motion: } \vec{R}(t) = \vec{R}(0) + \dot{\vec{R}}t .$$

... Center of mass and central forces in Classical Physics ...

- The energy of the motion between the particles is expressed by:

$$E_r = \frac{1}{2}\mu|\dot{\vec{r}}|^2 + V(r) . \quad (6.45)$$

The main conclusion of all of this is that, when the force is central (between two particles), we can treat their relative coordinate system separately (but using the reduced mass), and the center of mass system, the sum of both, is moving with constant velocity, since there is no force on it.

Cross sections in the laboratory frame ...

With this realization, we return to the problem of the collision between unequal masses. We first encountered this problem in Chapter 1, and worked out the kinematics in Slides 27–29, of the Chapter 1 notes. That was worked out in the laboratory frame, where m_2 was at rest.

Also in that Chapter 1, we discussed the “zero-momentum frame”, also known as the center-of-mass frame, and transformations to the laboratory frame, where the target particle is at rest. (See Slides 34–37 in Chapter 1.)

In this Chapter, we calculated the cross section for collision of the two unequal masses, in the center-of-mass frame, in Slides 35–36.

Now, we will close the loop by calculating the lab-frame cross section for unequal masses.

What we would like to do, is to find a relationship between the differential cross section in the laboratory and center-of-mass frame.

... Cross sections in the laboratory frame ...

It starts with a statement of the conservation of probability:

$$d\sigma_{\text{lab}} = d\sigma_{\text{cm}} \equiv d\sigma , \quad (6.46)$$

since particles can not disappear from the scattering ring in (6.23).

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{lab}} \sin \theta d\theta d\phi = \left(\frac{d\sigma}{d\Omega} \right)_{\text{cm}} \sin \Theta d\Theta d\phi , \quad (6.47)$$

where θ is the scattering angle in the laboratory frame.

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega} \right)_{\text{lab}} &= \left(\frac{d\sigma}{d\Omega} \right)_{\text{cm}} \frac{\sin \Theta d\Theta}{\sin \theta d\theta} \\ &= \left(\frac{d\sigma}{d\Omega} \right)_{\text{cm}} \frac{d(\cos \Theta)}{d(\cos \theta)} \end{aligned} \quad (6.48)$$

Now we need a relationship between Θ and θ .

... Cross sections in the laboratory frame ...

Consider a projectile moving with mass m_1 and velocity $v_0\hat{z}$ striking a stationary target with mass m_2 .

According to (1.37) [Chapter 1, Slide 34], the velocity in the center-of-mass frame (cm), $\vec{V}$, is:

$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} \quad [\text{general case}]$$

In this case, since the target is stationary,

$$\vec{V} = \frac{m_1 v_0}{m_1 + m_2} \hat{z} . \quad (6.49)$$

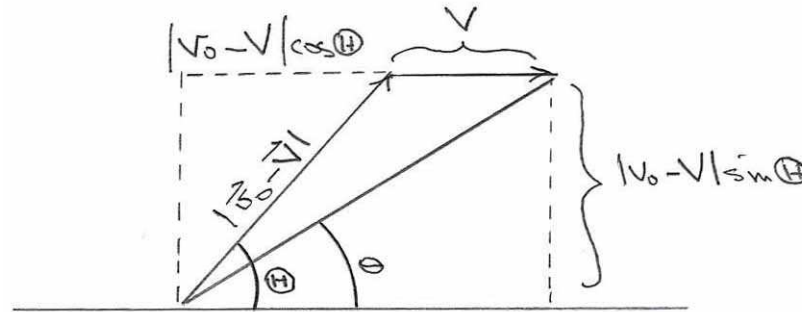
When the two masses collide in the cm frame, the scattering angle of m_1 is Θ .

The recoil angle of the target is $\pi - \Theta$, since they must be colinear to conserve momentum.

The velocity of m_1 before the collision is $|v_0 - V|\hat{z}$.

After the collision the velocity of m_1 is $|v_0 - V|\cos\Theta\hat{z} + |v_0 - V|\sin\Theta\hat{\rho}$, where $\hat{\rho}$ is a unit vector in a direction perpendicular to $\hat{z}$.

... Cross sections in the laboratory frame ...



Transforming back to the laboratory frame, by adding $\vec{V}$, gives us the relationship between Θ and θ :

$$\tan \theta = \frac{(v_0 - V) \sin \Theta}{V + (v_0 - V) \cos \Theta} = \frac{\left(v_0 - \frac{rv_0}{1+r}\right) \sin \Theta}{\frac{rv_0}{1+r} + \left(v_0 - \frac{rv_0}{1+r}\right) \cos \Theta} \longrightarrow \frac{\sin \Theta}{\cos \Theta + r}, \quad (6.50)$$

where $r = m_1/m_2$.

... Cross sections in the laboratory frame ...

As we saw in Chapter 1, Slides 34–37, when $r > 1$ the projectile is heavier than the target, and every θ has two velocities associated with it. When $r < 0$, the relationship between the particle velocities and their directions is unique.

We must manipulate the $\tan \theta = \sin \Theta / (\cos \Theta + r)$ relationship to use (6.48).

With

$$\sin \Theta = \sqrt{1 - \cos^2 \Theta} ,$$

(6.50) becomes:

$$\tan \theta = \frac{\sqrt{1 - \cos^2 \Theta}}{\cos \Theta + r} . \quad (6.51)$$

... Cross sections in the laboratory frame ...

Using (6.50) to express $\cos \Theta$ as a function of θ results in a quadratic equation for $\cos \Theta$:

$$\cos^2 \Theta + 2 \cos \Theta (r \sin^2 \theta) + [r^2 \sin^2 \theta - \cos^2 \theta] = 0 , \quad (6.52)$$

with solutions:

$$\begin{aligned} \cos \Theta &= -r \sin^2 \theta \pm \sqrt{r^2 \sin^4 \theta - r^2 \sin^2 \theta + \cos^2 \theta} \\ &= -r(1 - \cos^2 \theta) \pm \cos \theta \sqrt{1 - r^2 \sin^2 \theta} \end{aligned} \quad (6.53)$$

After some work, one finds⁴ $d \cos \Theta / d \cos \theta$, and

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{lab}} = \left(\frac{d\sigma}{d\Omega} \right)_{\text{cm}} \left(2r \cos \theta \pm \left[\frac{1 - r^2 \sin^2 \theta + r^2 \cos^2 \theta}{\sqrt{1 - r^2 \sin^2 \theta}} \right] \right) \quad (6.54)$$

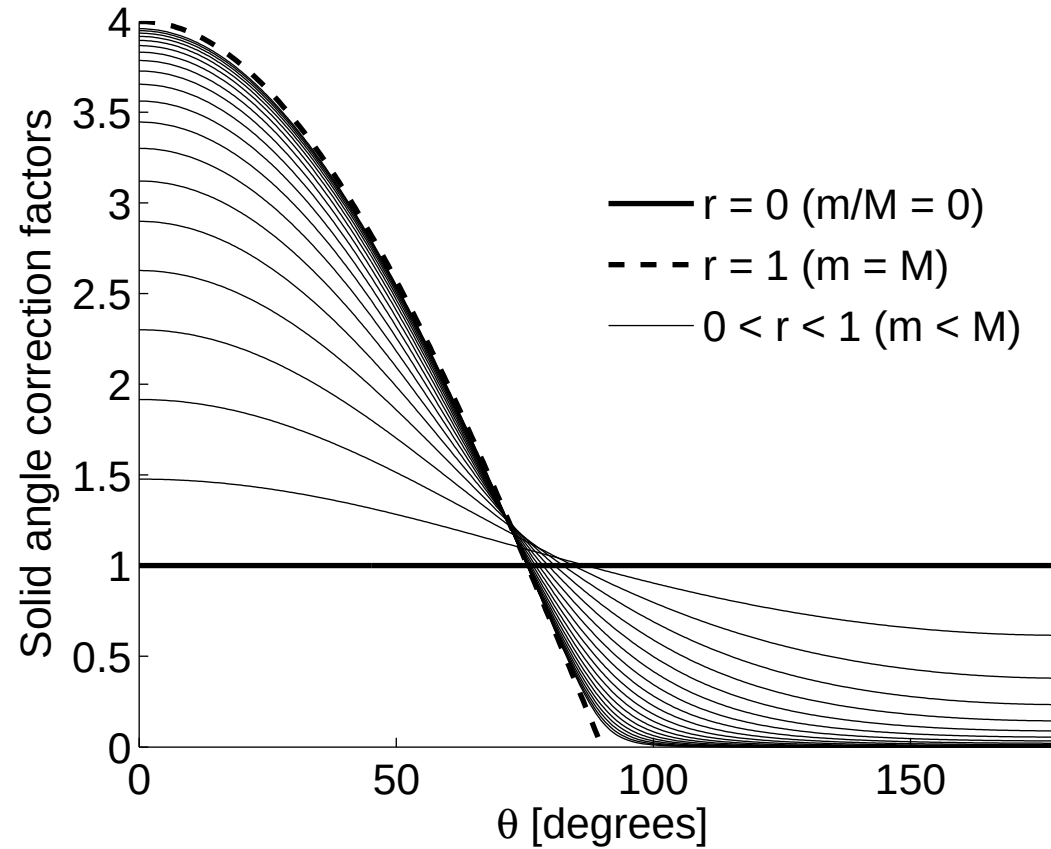
For $r \leq 1$, only the plus sign is used. For $r > 1$, both signs are in use, since that represents two different energies arriving at the same scattering angle, θ , in the laboratory frame.

Note that the cross sections are the same when $r = 0$, as expected.

This universal “solid-angle correction factor”, C_Ω , that multiplies the cm frame differential cross section, is plotted on the next page.

⁴You'll have to twist my arm to get me to show you this!

... Cross sections in the laboratory frame ...



... Cross sections in the laboratory frame ...

The case $r = 1$ is quite interesting as it is at the boundary of the double-valued, single-value solution. It also applies to fast neutrons slowing down in the moderator of a nuclear reactor.

When $r = 1$, (6.54) becomes:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = 4 \cos \theta \left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}} \quad (6.55)$$

Recall that in the cm frame, we found that

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}} = \frac{R_0^2}{4}.$$

However, to compute the total cross section, we only integrate θ from 0 to $\pi/2$ since there is no backscatter. The total cross section in either frame is πR_0^2 since the apparent size of an object is independent of the frame of the observer.

... Cross sections in the laboratory frame ...

The Rutherford cross section was found to be:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}} = \frac{1}{16} \left(\frac{zZe^2}{4\pi\epsilon_0 K_{\text{cm}}}\right)^2 \frac{1}{\sin^4(\Theta/2)}, \quad (6.56)$$

where we are now undoing the approximation that the target is infinitely heavy, and recognizing that we had performed a “center-of-mass” calculation.

In the laboratory frame, we use (6.54) directly, but with the positive sign, unless the projectile is heavier. Convert the $\sin^4(\Theta_{\text{cm}}/2)$ term to lab angles, we have to apply (6.50). The easiest way to do this is to use half-angle trigonometric relations, to find that:

$$\frac{1}{\sin^4(\Theta/2)} = \frac{4}{(1 - \cos \Theta)^2}, \quad (6.57)$$

and then use (6.53) directly.

We also have to determine the kinetic energy in the cm frame. Using (6.49), we would find that:

$$K_{\text{cm}} = K_0 \frac{1}{(1 + r)^2} \quad \text{or} \quad \frac{K_0}{K_{\text{cm}}} = (1 + r)^2. \quad (6.58)$$

... Cross sections in the laboratory frame ...

Summary: Lab frame conversion of the Rutherford cross section

Putting all the effects together, we can write:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \left[\frac{1}{16} \left(\frac{zZe^2}{4\pi\epsilon_0 K_0} \right)^2 \frac{1}{\sin^4(\theta/2)} C_K C_\theta \right] C_\Omega, \quad (6.59)$$

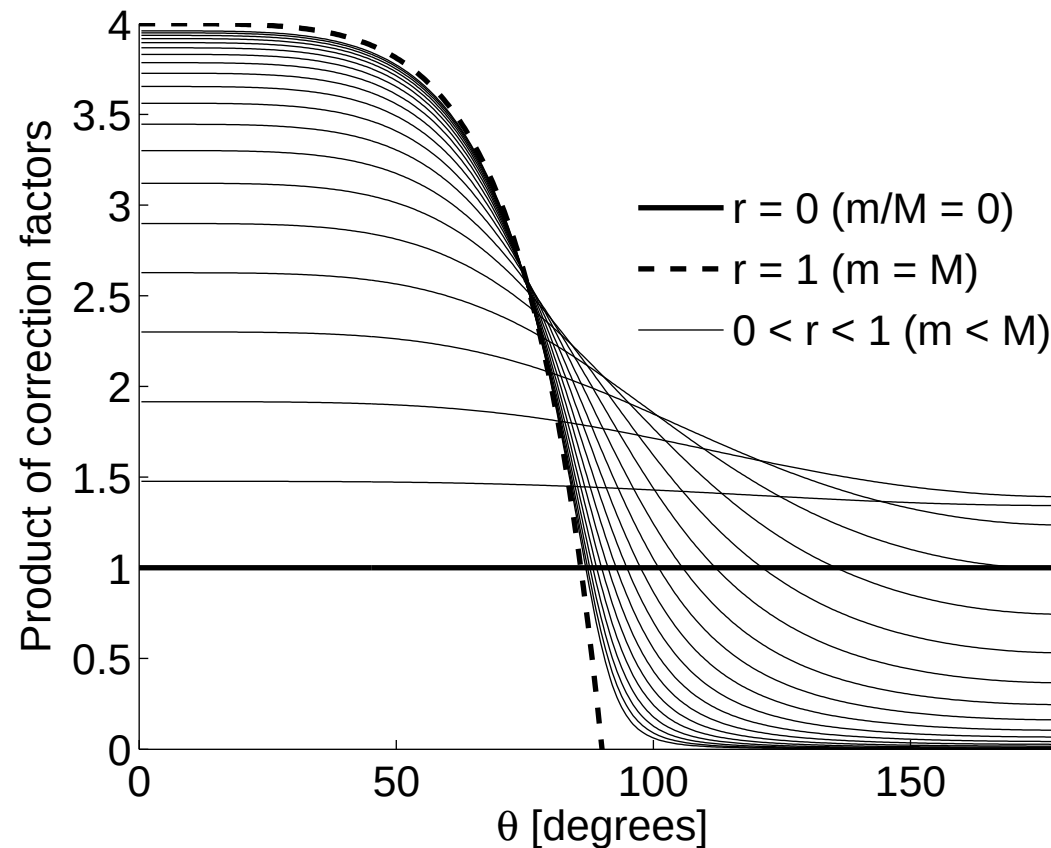
$$C_K = \left(\frac{K_0}{K_{\text{cm}}} \right)^2 = (1+r)^4 \quad [\text{from (6.58)}] \quad (6.60)$$

$$\begin{aligned} C_\theta &= \frac{\sin^4(\theta/2)}{\sin^4(\Theta/2)} = \frac{4 \sin^4(\theta/2)}{(1 - \cos \Theta)^2} \quad [\text{from (6.57)}] \\ &= \frac{4 \sin^4(\theta/2)}{\left(1 + r \sin^2 \theta \mp \cos \theta \sqrt{1 - r^2 \sin^2 \theta} \right)^2} \quad [\text{from (6.53)}] \end{aligned} \quad (6.61)$$

$$C_\Omega = \left(2r \cos \theta \pm \left[\frac{1 - r^2 \sin^2 \theta + r^2 \cos^2 \theta}{\sqrt{1 - r^2 \sin^2 \theta}} \right] \right) \quad [\text{from (6.54)}] \quad (6.62)$$

Note the the upper sign in $\mp$ or $\pm$ is to be used when $r \leq 1$. When $r > 1$ both signs must be used, and integrated over to conserve probability.

... Cross sections in the laboratory frame



Note the similarity of this correction with graph on Slide 52, as most of shape comes from C_Ω . The other two correction factors, in the case, $C_K C_\theta$ that arise from the change in kinetic energy and the $1/\sin^4(\theta/2)$, provide the rest of the shape. Note how the distribution shifts the large angles forward, as expected.

Center of mass coordinates in Quantum Mechanics ...

The equation of motion in Quantum Mechanics for a two particle system is:

$$\left[-\frac{\hbar^2}{2m_1} \nabla_{\vec{x}_1}^2 - \frac{\hbar^2}{2m_2} \nabla_{\vec{x}_2}^2 + V(|\vec{x}_1 - \vec{x}_2|, t) \right] \psi(\vec{x}_1, \vec{x}_2, t) = i\hbar \frac{\partial}{\partial t} \psi(\vec{x}_1, \vec{x}_2, t) , \quad (6.63)$$

where $\nabla_{\vec{x}_1}^2$ operates on $\vec{x}_1$ and $\nabla_{\vec{x}_2}^2$ operates on $\vec{x}_2$.

If we assume that the potential contains no explicit time dependence, then,

$$\psi(\vec{x}_1, \vec{x}_2, t) = U(\vec{x}_1, \vec{x}_2) e^{-i(E_T/\hbar)t} , \quad (6.64)$$

and (6.63) becomes:

$$\left[-\frac{\hbar^2}{2m_1} \nabla_{\vec{x}_1}^2 - \frac{\hbar^2}{2m_2} \nabla_{\vec{x}_2}^2 + V(|\vec{x}_1 - \vec{x}_2|) \right] U(\vec{x}_1, \vec{x}_2) = E_T U(\vec{x}_1, \vec{x}_2) , \quad (6.65)$$

where $U(\vec{x}_1, \vec{x}_2)$ contains all the spatial information of the wavefunctions, in analogy with one-particle systems.

We now perform the same change of coordinate system that we did in the classical system, namely:

$$\vec{R} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2}, \quad \vec{r} = \vec{x}_2 - \vec{x}_1, \quad \text{and} \quad M = m_1 + m_2$$

... Center of mass coordinates in Quantum Mechanics ...

With the change of variables $\vec{\nabla}_{\vec{x}_1}$ and $\vec{\nabla}_{\vec{x}_2}$ become⁵

$$\vec{\nabla}_{\vec{x}_1} = \frac{\partial \vec{R}}{\partial \vec{x}_1} \frac{\partial}{\partial \vec{R}} + \frac{\partial \vec{r}}{\partial \vec{x}_1} \frac{\partial}{\partial \vec{r}} = \frac{m_1}{M} \vec{\nabla}_{\vec{R}} - \vec{\nabla}_{\vec{r}} \quad (6.66)$$

$$\vec{\nabla}_{\vec{x}_2} = \frac{\partial \vec{R}}{\partial \vec{x}_2} \frac{\partial}{\partial \vec{R}} + \frac{\partial \vec{r}}{\partial \vec{x}_2} \frac{\partial}{\partial \vec{r}} = \frac{m_2}{M} \vec{\nabla}_{\vec{R}} + \vec{\nabla}_{\vec{r}} \quad (6.67)$$

(6.66) and (6.67) $\rightarrow$ (6.65) $\Rightarrow$

$$\left[-\frac{\hbar^2}{2m_1} \left| \frac{m_1}{M} \vec{\nabla}_{\vec{R}} - \vec{\nabla}_{\vec{r}} \right|^2 - \frac{\hbar^2}{2m_2} \left| \frac{m_2}{M} \vec{\nabla}_{\vec{R}} + \vec{\nabla}_{\vec{r}} \right|^2 + V(r) \right] U(\vec{R}, \vec{r}) = E_T U(\vec{R}, \vec{r}) . \quad (6.68)$$

As in the classical case, the cross terms vanish when the $|\cdot|^2$ are expanded, and the above becomes:

$$\left[-\frac{\hbar^2}{2M} \nabla_{\vec{R}}^2 - \frac{\hbar^2}{2\mu} \nabla_{\vec{r}}^2 + V(r) \right] U(\vec{R}, \vec{r}) = E_T U(\vec{R}, \vec{r}) , \quad (6.69)$$

noting the appearance of the reduced mass, μ .

⁵Please note that this is very condensed notation that may be unfamiliar to you. If you do the change of variable component by component, you will see what it means. e.g. the x -component of (6.66) is $\frac{\partial}{\partial x_1} = \frac{\partial R_x}{\partial x_1} \frac{\partial}{\partial R_x} + \frac{\partial r_x}{\partial x_1} \frac{\partial}{\partial r_x} = \frac{m_1}{M} \frac{\partial}{\partial R_x} - \frac{\partial}{\partial r_x}$, and so on.

... Center of mass coordinates in Quantum Mechanics ...

Using separation of variables we can write $U(\vec{R}, \vec{r}) = U_{\text{CM}}(\vec{R})u(\vec{r})$, and conclude that

$$K_{\text{CM}} U(\vec{R}) = -\frac{\hbar^2}{2M} \nabla_{\vec{R}}^2 U(\vec{R}) , \quad (6.70)$$

$$E_{\text{r}} u(\vec{r}) = \left[-\frac{\hbar^2}{2\mu} \nabla_{\vec{r}}^2 + V(r) \right] u(\vec{r}) , \quad (6.71)$$

$$E_{\text{T}} = K_{\text{CM}} + E_{\text{r}} \quad (6.72)$$

where a free-wave equation describes the motion of the center of mass with kinetic energy K_{cm} and wavefunction $U(\vec{R})$, and a wavefunction in r that describes the effect of the internal force on the system. It has a constant energy E_{r} . The sum of these two energies, is the total energy of the system, E_{T} .

The wavefunction of the CM system is:

$$\Psi(\vec{R}, t) = N \exp \left\{ i \left[\vec{k}_{\text{CM}} \cdot \vec{R} - \frac{K_{\text{CM}} t}{\hbar} \right] \right\} \quad \text{where} \quad \vec{k}_{\text{CM}} = \sqrt{\frac{2MK_{\text{CM}}}{\hbar^2}} \quad (6.73)$$

Summary: Center of mass coordinates

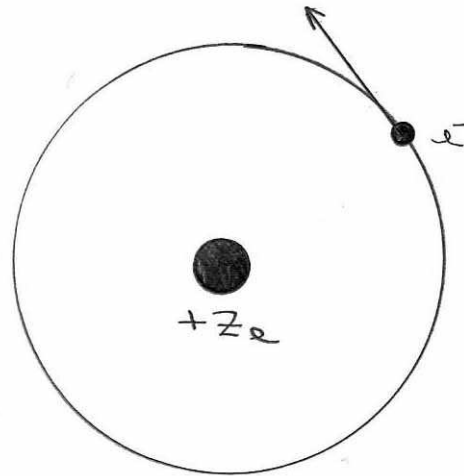
	Classical	Quantum
equation of motion of center of mass ($m_1 + m_2$)	$K_{\text{CM}} = \frac{1}{2}M \dot{\vec{R}} ^2$	$K_{\text{CM}} U(\vec{R}) = -\frac{\hbar^2}{2M}\nabla_{\vec{R}}^2 U(\vec{R})$
solution	$\vec{R}(t) = \vec{R}(0) + \dot{\vec{R}}t$	$\Psi(\vec{R}, t) = N \exp \left[i \left(\vec{k}_{\text{CM}} \cdot \vec{R} - \frac{K_{\text{CM}}t}{\hbar} \right) \right]$
interpretation	center of mass moves as one particle with constant energy and momentum	center of mass moves as one particle wave with constant energy and momentum
equation of motion of the distance between m_1 & m_2	$E_{\text{r}} = \frac{1}{2}\mu \dot{\vec{r}} ^2 + V(r)$	$E_{\text{r}} u(\vec{r}) = \left[-\frac{\hbar^2}{2\mu}\nabla_{\vec{r}}^2 + V(r) \right] u(\vec{r})$
interpretation	trajectory of m_1 & m_2 seen by an observer moving with the CM frame is mathematically equivalent to a single particle with mass μ	wavefunction of m_1 & m_2 seen by an observer moving with the CM frame is mathematically equivalent to a single particle with mass μ

The Rutherford model of the atom ...

- As we learned in Chapter 4, the Thomson model of the atom had some difficulties associated with it.
- Rutherford discovered that the positive charge of the atom was concentrated in a “point-like” object at the center (the nucleus) that also contained most of the mass (99.975%, approximately)

The Rutherford model

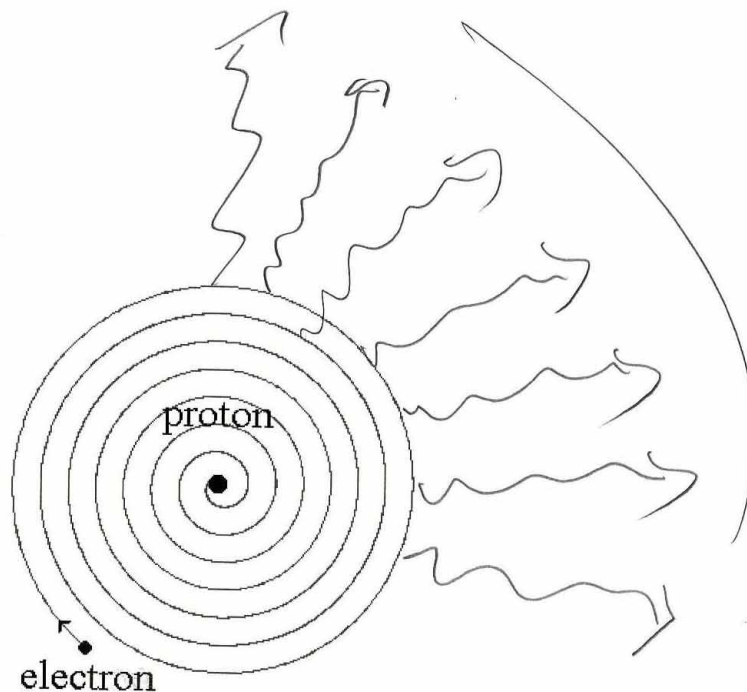
The atom must be like a planetary system where the nucleus was the sun, and the electrons acted like planets in orbit.



... The Rutherford model of the atom

Nice idea! But wrong!

If this were true, and Maxwell's equations were to be believe, this would imply that the H-atom would live only 10^{-12} s, because charges under acceleration, that is, centrifugal force, would quickly radiate away all their energy.⁶



radiates light
continuously
according to
Maxwell's
eq's
lives only 10^{-12} s!

⁶Maxwell **was** right. Any charged particle undergoing deflection gives off synchrotron radiation (bremsstrahlung), a major shielding problem, and energy drain, for people in the business of accelerating particles.

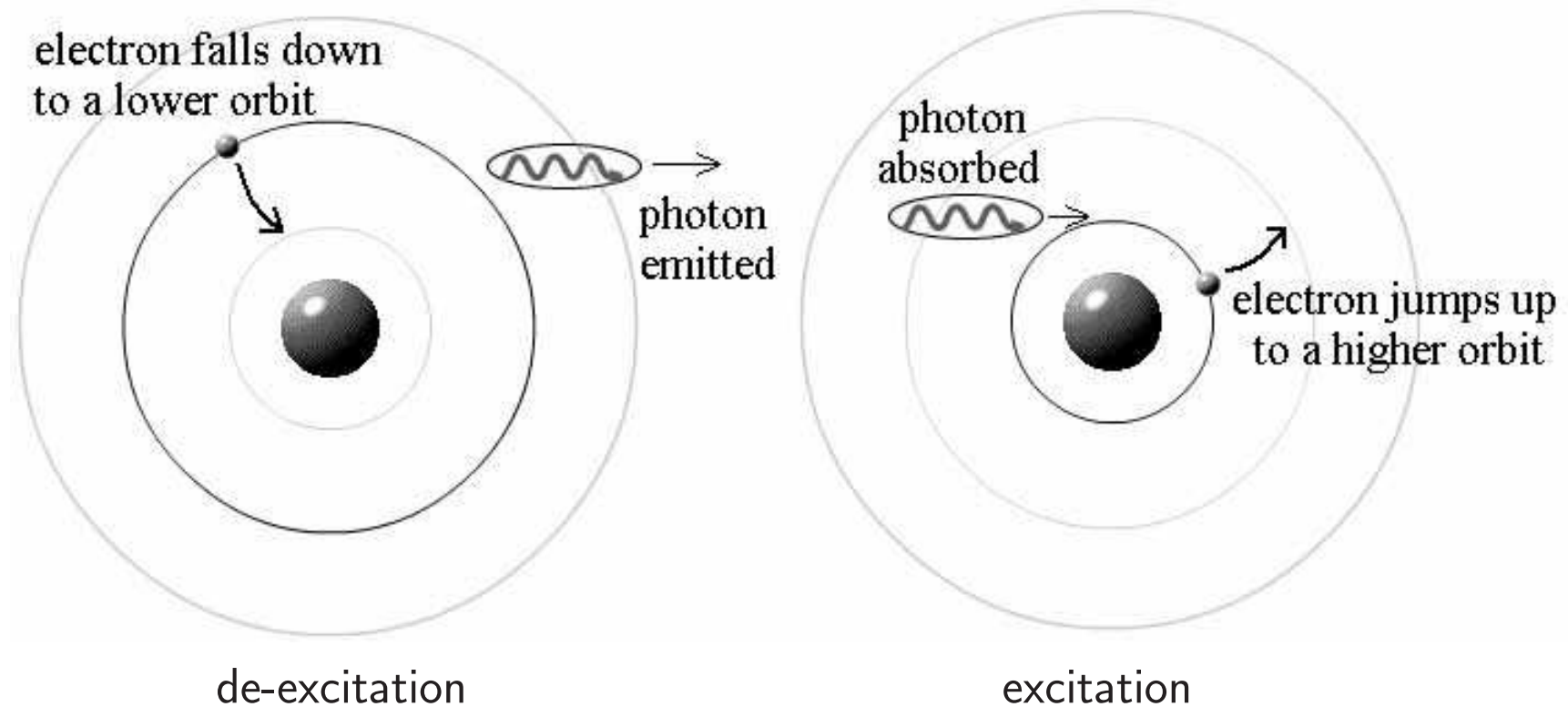
The Rutherford-Bohr model of the atom ...

In 1913, Niels Henrik David Bohr (1885—1962, Nobel Prize 1922) has a great idea!

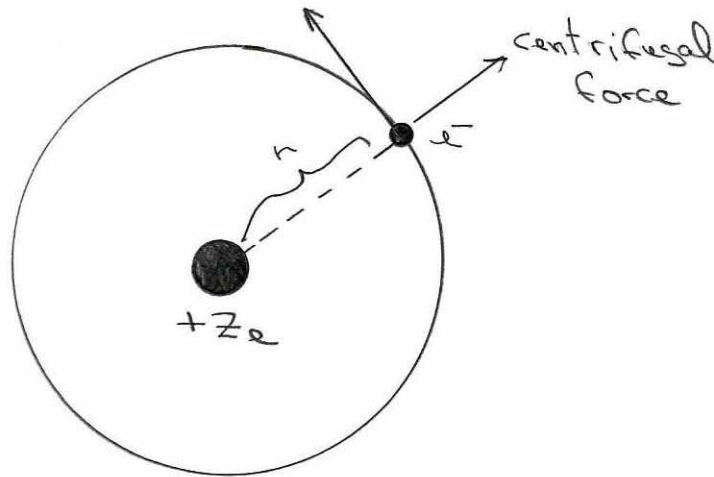
“Planck showed us that photons are really ‘quanta’.

Hence their orbits must have fixed energy and photons arise by means of a discrete jump in orbits downward in energy.”

Also, an electron may jump to a higher level by absorbing a photon with the right energy.



... The Rutherford-Bohr model of the atom ...



Then Bohr did his infamous calculation.

The centrifugal force (making the e^- want to fly off) is:

$$F_{\text{out}} = \frac{mv^2}{r} . \quad (6.74)$$

The centripetal force (making it stay in orbit) is the electrostatic attraction:

$$F_{\text{in}} = -\frac{Ze^2}{4\pi\epsilon_0 r^2} . \quad (6.75)$$

... The Rutherford-Bohr model of the atom ...

(6.75) and (6.74) $\Rightarrow$

$$\frac{mv^2}{r} = \frac{Ze^2}{4\pi\epsilon_0 r^2} . \quad (6.76)$$

$(r/2) \times (6.76) \Rightarrow$

$$\frac{mv^2}{2} = K = \frac{Ze^2}{8\pi\epsilon_0 r} . \quad (6.77)$$

This is a closed system, therefore the total energy must be a constant. This $\Rightarrow$

$$\begin{aligned} E &= K + V \\ &= \frac{Ze^2}{8\pi\epsilon_0 r} - \frac{Ze^2}{4\pi\epsilon_0 r} \\ E &= -\frac{Ze^2}{8\pi\epsilon_0 r} \end{aligned} \quad (6.78)$$

... The Rutherford-Bohr model of the atom ...

And now, the Nobel Prize-winning moment:

Bohr quantized the electron's angular momentum:

$$mv_n r_n = n\hbar . \quad (6.79)$$

$$\therefore K_n = \frac{mv_n^2}{2} = \frac{m}{2} \left(\frac{n\hbar}{mr_n} \right)^2 = \frac{Ze^2}{8\pi\epsilon_0} \frac{1}{r_n} \quad (6.80)$$

Solving for $1/r_n$ in (6.80):

$$\frac{1}{r_n} = Z \left(\frac{me^2}{4\pi\epsilon_0\hbar^2} \right) \frac{1}{n^2} , \quad (6.81)$$

and

$$r_n = \left(\frac{1}{Z} \right) \left(\frac{4\pi\epsilon_0\hbar^2}{me^2} \right) n^2 = \left(\frac{a_0}{Z} \right) n^2 , \quad (6.82)$$

which defines a_0 , known as the “Bohr radius”, 0.0529 nm, about $\frac{1}{2}\text{\AA}$.

... The Rutherford-Bohr model of the atom ...

From (6.78) and (6.82)

$$E_n = -\frac{Ze^2}{8\pi\epsilon_0} \left(\frac{mZe^2}{4\pi\epsilon_0\hbar^2} \right) \frac{1}{n^2} = -Z^2 R_y \frac{1}{n^2} . \quad (6.83)$$

The R_y is the Rydberg energy constant, with value 13.605 692 53(30) eV.

Another way of writing the energy is through use of the “fine-structure constant”:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137.035\,999\,074(44)} . \quad (6.84)$$

Thus,

$$E_n = -Z^2 \frac{mc^2\alpha^2}{2} \frac{1}{n^2} . \quad (6.85)$$

Atomic physicists tend to favor (6.83) while high-energy physicists tend to choose (6.85).

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The γ -transition energies in the Rutherford-Bohr models are given by:

$$E_{\gamma} = Z^2 R_y \left[\frac{1}{n_f^2} - \frac{1}{n_i^2} \right] , \quad (6.86)$$

where n_i is the quantum number of the initial (excited) states, and n_f is the quantum number of the final state of the transition, the ground state ($n_f = 1$) or excited state such that $n_f < n_i$.

(6.86) was discovered by Swedish Physicist Johannes (Janne) Robert Rydberg in 1888 of Lund University, using a phenomenological approach.⁷

⁷From the “Concise Dictionary of Physics”: *Phenomenological Theory. A theory that expresses mathematically the results of observed phenomena without paying detailed attention to their fundamental significance.* Basically, Rydberg was guessing! There is a pub night held in Rydberg’s honor every Wednesday at the Department of Physics at Lund University. You have to be a member of a club, to go to the pub. I am going to crash this party one day.

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Name conventions:

n_f	Series	energy range	discoverer	year
1	Lyman	(10.2—13.6 eV) uv	Theodore Lyman (1874—1954)	1906
2	Balmer	(1.89—3.40 eV) visible	Johann Jakob Balmer (1825—1898)	1885
3	Paschen	(0.661—1.51 eV) infrared	Louis Karl Heinrich Freidrich Paschen (1865—1947)	1908
4	Brackett	(0.301—0.850 eV) infrared	Frederick Sumner Brackett (1896—1988)	1922
5	Pfund	(0.166—0.540 eV) infrared	August Herman Pfund (1879—1949)	1924
6	Humphreys	(0.100—0.378 eV) infrared	Curtis Judson Humphreys (1898—1986)	1953
:	Not named	(< 0.278 eV)	:	:

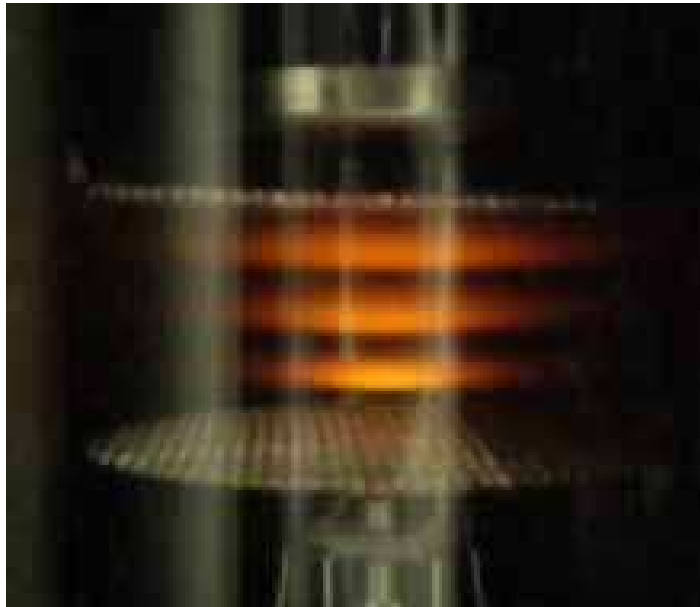
All of these transitions have been measured, and show excellent agreement with Bohr's model.

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Other successes of the Rutherford-Bohr model:

- James Franck (1882—1964) and Gustav Ludwig Hertz (1887—1975) presented a series of measurements in 1914 that demonstrated excitation levels of various atoms, in agreement with the Rutherford-Bohr model. The “Franck-Hertz” experiments eventually earned them the Nobel Prize in 1926.

Franck-Hertz experiment showing visible transitions in Neon:



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- At large quantum numbers, it can be shown that the energy spectra of photons emitted from atoms agreed with Classical E&M theory. This agreement with the Correspondence Principle strongly supported the Rutherford-Bohr model.

But ... there were problems ...

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Deficiencies of the Rutherford-Bohr model:

- The finite mass of the nucleus was neglected

Recall:

$$E_n = -Z^2 \frac{m_e c^2 \alpha^2}{2} \frac{1}{n^2} ,$$

where m_e is the electron mass. Since the electron is bound by a central potential, the reduced mass, $\mu = m_e m_N / (m_e + m_N)$ should be used. (m_N is the mass of the nucleus.) For the H-atom, the worst possible case, this reduced the E_n 's by a factor $1/1.00055$, a small amount, easily accounted for.

- The measured wavelengths were measured in air, not vacuum.
This decreases the γ wavelengths (increases their energy) by a factor of 1.00029, partially undoing the reduced-mass effect. Again, an easy fix.

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Deficiencies of the Rutherford-Bohr model: (con't)

- The transition lines are doublets, not singlets

The transition lines are actually two very closely spaced lines, with slightly different energies, for which there is no prediction from the Rutherford-Bohr model. This is now called the “fine structure” of the energy levels, and is caused by the spin of the electron having a force exerted on it, by the magnetic field generated by its orbit. This was measured by the celebrated Stern-Gerlach experiment in 1922, by Otto Stern (1888—1969) and Walther Gerlach (1889—1979). The reason was discovered in 1925 by Samuel Abraham Goudsmit (1902—1978)⁸ and George Eugene Uhlenbeck (1900—1988)⁹ who proposed that the cause was due to an intrinsic spin of the e^- .

- The Rutherford-Bohr model predicted that the ground state has angular momentum
The Rutherford-Bohr model predicted that the ground state has one unit angular momentum ($1 \hbar$). It was demonstrated that there was no angular momentum associated with the ground state.

⁸Goudsmit was a Professor of Physics at the University of Michigan from 1927—1946).

⁹Uhlenbeck was an Instructor in the Physics Department of the University of Michigan from 1927—1935.

... The Rutherford-Bohr model of the atom ...

Deficiencies of the Rutherford-Bohr model: (con't)

- Although it is after the fact, the Rutherford-Bohr model was in violation of Heisenberg's Uncertainty Principle $\Delta p_r \Delta r \geq \hbar/2$, that was discovered by Werner Karl Heisenberg (1901—1976). The classical calculation of the Rutherford-Bohr model could not possibly be true, despite its successes.

⇒ We had to do better!

That happened in short order with the development of the Schrödinger equation by Erwin Rudolf Josef Alexander Schrödinger (1887—1961) in 1926.

The great early success of the Schrödinger equation was its use to find the energy levels, and wavefunctions of the H-atom ... and this is the topic of Chapter 7.