

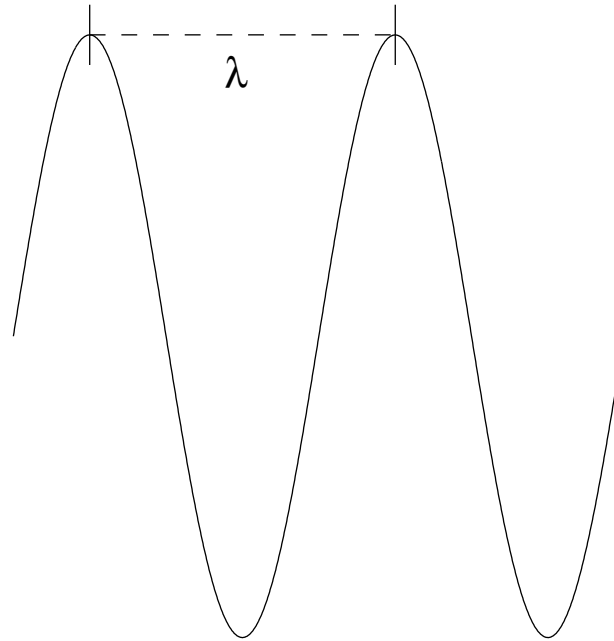
4. Wave-like properties of particles

4.1. Particles as waves

Statement	Acceptance
Light acts as waves	Common experience
Light is made up of photons	Seems reasonable
Particles are solid, indivisible entities	Common experience
Particles are waves	What ???

Some nomenclature

symbol	meaning
$\lambda, \tilde{\lambda}$	wavelength (see figure below) and “reduced wavelength”, $\tilde{\lambda} = \lambda/2\pi$
k	wave number, $k = 2\pi/\lambda = 1/\tilde{\lambda}$
$h, \hbar$	Planck’s constant and “reduced Planck’s constant”, $\hbar = h/2\pi$
ω	frequency, $\omega = kc$ for photons



Matter waves...



Enter Louis-Victor-Pierre-Raymond, 7th duc de Broglie (1892–1987) who won the Nobel Prize in 1929 for proposing (as a graduate student!) that all matter, in particular, particles with mass, have a wavelength inversely proportional to its momentum:

$$\lambda = \frac{h}{|\vec{p}|} \quad \text{or} \quad \lambda = \frac{\hbar}{|\vec{p}|} . \quad (4.1)$$

The latter form is often thought of as the “effect radius” of a particle.

...Matter waves...

de Broglie's idea unified the wave-particle concepts. Everything is both.

quantity	photons	non-relativistic mass	relativistic mass
Total energy	E $\hbar\omega$ $\hbar ck$	$mc^2 + \frac{1}{2}mv^2$ $mc^2 + p^2/2m$	$\sqrt{(pc)^2 + (mc^2)^2}$ $mc^2\gamma$
Kinetic energy	E $\hbar\omega$ $\hbar ck$	$\frac{1}{2}mv^2$ $p^2/2m$	$\sqrt{(pc)^2 + (mc^2)^2} - mc^2$ $mc^2(\gamma - 1)$
Momentum	E/c $\hbar\omega/c$ $\hbar k$ p	mv p	$mv\gamma$ $m\beta c\gamma$ p
Wavelength	h/p $\hbar/\lambda$	h/p $\hbar/\lambda$	h/p $\hbar/\lambda$

All things with mass or massless, are both waves and particles¹.

¹There was an attempt to call everything a wavicle, but the term fell into disuse.

...Matter waves...

Examples:	mass	v or K	λ	significance
speeding car	1000 kg	100 m/s	7×10^{-39} m	$< \ell_P!$ unphysical
bullet	10 g	500 m/s	7×10^{-34} m	$\approx \ell_P$
smoke particle	10^{-9} g	1 cm/s	7×10^{-20} m	$\approx r_{e^-}$
biological cell	10^{-12} g	0.011 cm/s	7×10^{-15} m	radius of a nucleus
electron	$\approx 10^{-28}$ g	1 eV	10^{-9} m	$\approx$ of atoms
		100 MeV	10^{-14} m	$\approx$ size of nuclei
		2 GeV	5×10^{-16} m	$\approx$ size of quarks

...Matter waves

The first 3 examples in the previous table produce unphysical distances. The Planck length, $\ell_P = 1.61619926 \times 10^{-35}$ m, is the assumed granularity of space.

A comparison of the next two would yield no practical information.

However, as soon as we can get a fundamental particle to wavelength of the order of other fundamental particles, we can measure some useful information about them.

The math to describe this is a little involved, and involves Fourier transforms, which we will describe shortly. The full description of how electrons may be used as probes of nuclei and quarks will come in NERS 312.

Some of these numbers are mind-boggling small.

Other mind-boggling numbers:

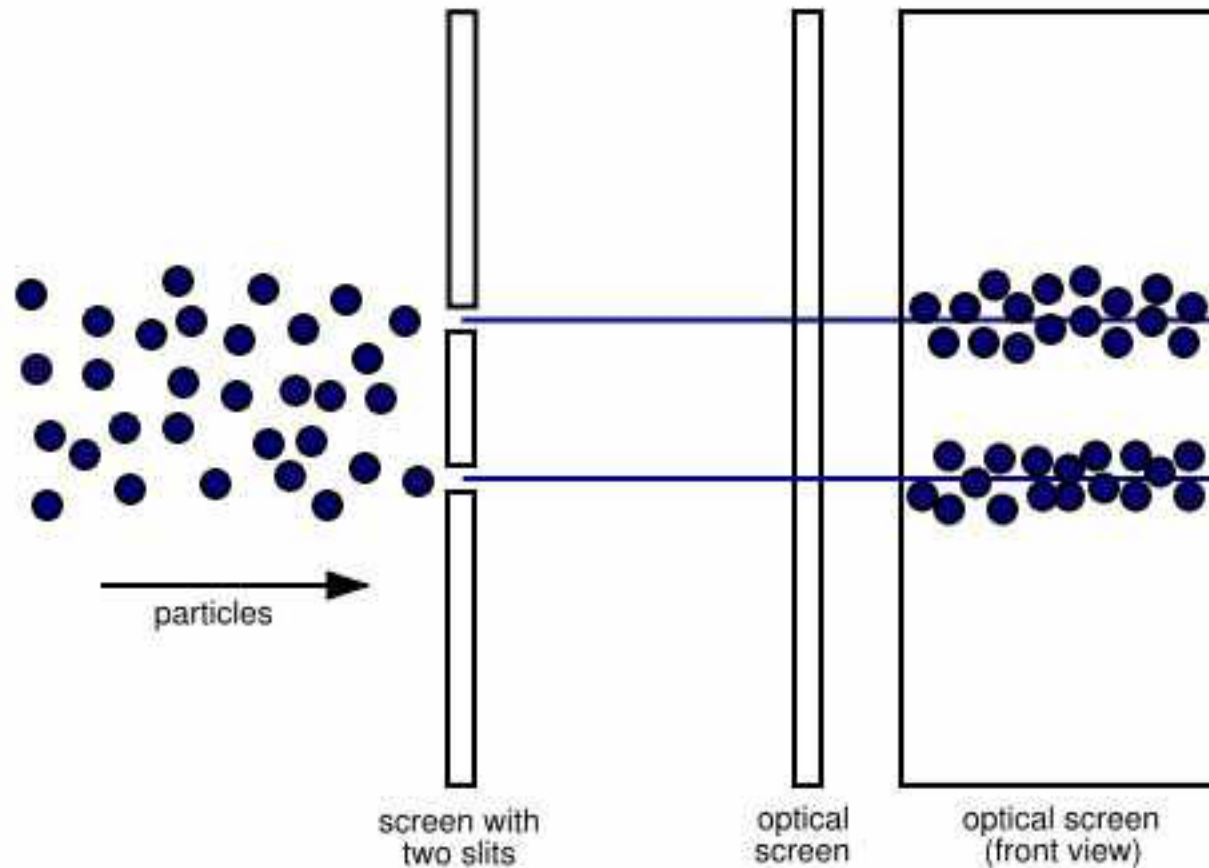
Age of the universe, in seconds: 3×10^{17}

Bill Gates' fortune in \$s: 80×10^9

Bill has made 3¢s per day, since the start of the universe!

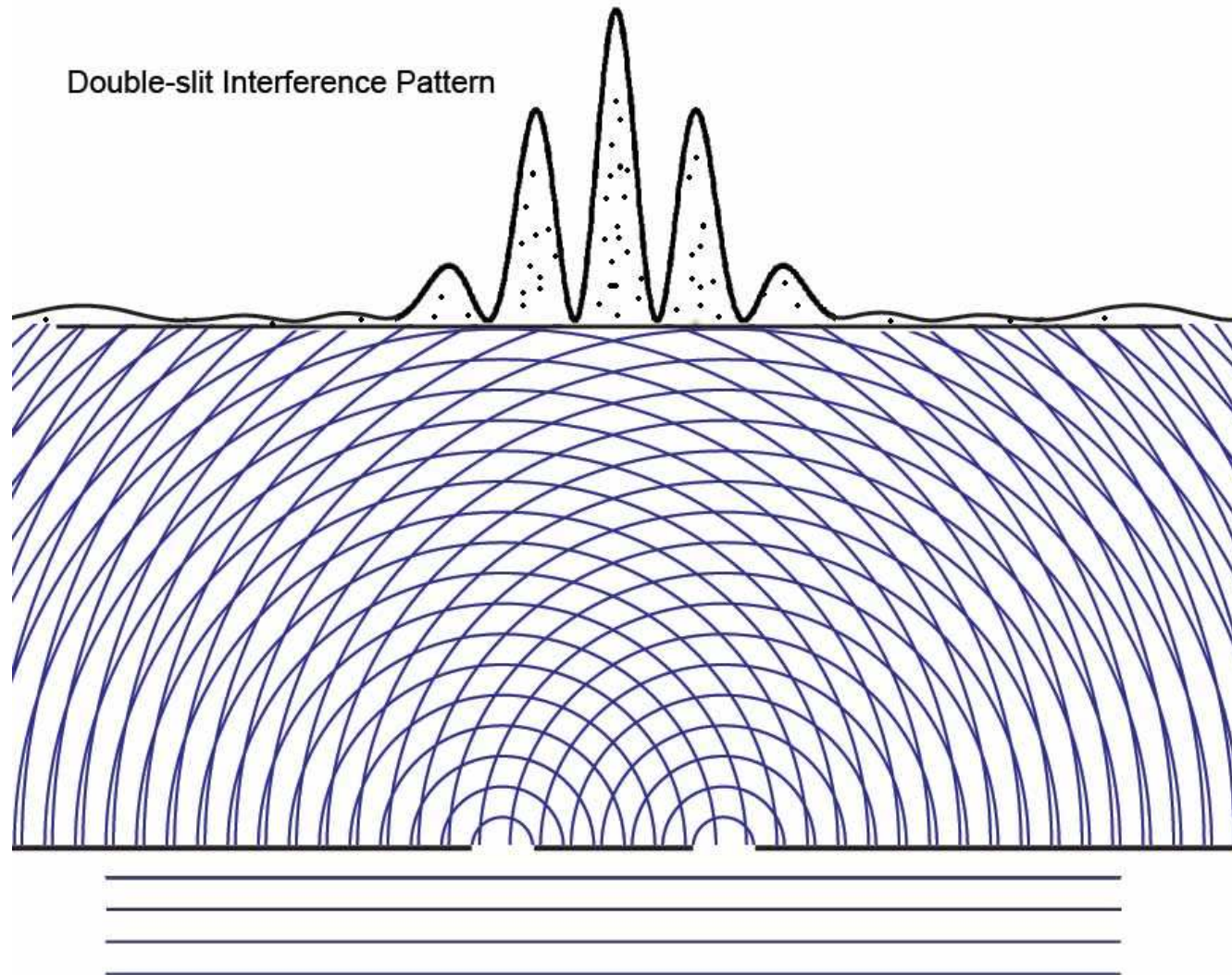
Double slit experiments ...

If one throws particles at a “wall” with two parallel slits, one would expect (if you did not accept the idea of “matter waves”) a result that looks like the figure below.



... Double slit experiments ...

However, if you arrange the slits properly (slits are about 1λ wide, distance between them not more than a few λ s), you see instead:



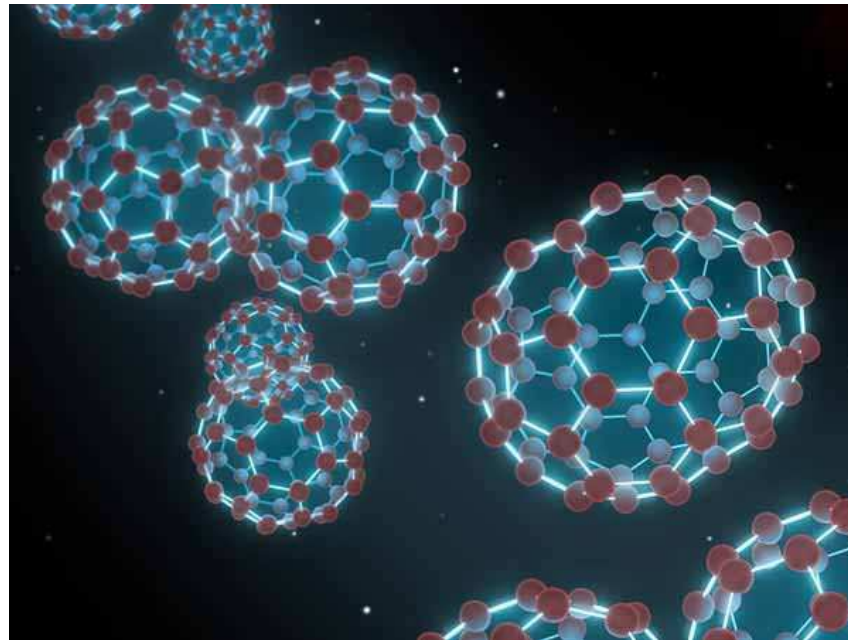
... Double slit experiments

The projectiles are spaced out so that only one particle is in motion at any time.

Note that the interference patterns is made up of single “strikes” against a detector, not some sort of smoothed-out wave.

If you block one of the slits, one obtains a result that looks more like the classical picture.

These experiments have been verified with photons, electrons, molecules (up to 810 atoms), and even buckyballs! (Buckminsterfullerene).



Waves, composites, superposition, and Fourier transforms ...

We have encountered an expression for a monochromatic (one energy) wave previously, in Chapter 3. Let us generalize the notation:

A monochromatic plane wave traveling along the $\hat{z}$ -axis is:

$$A(z, t) = A_0 e^{i[k_0 z - \omega(k_0)t]} , \quad (4.2)$$

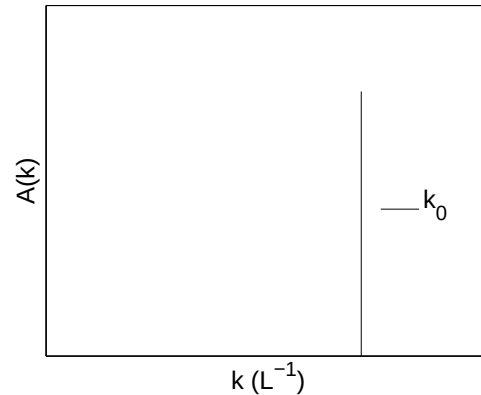
where $A(z, t)$ is the amplitude in space and time and A_0 represents the amplitude of the wave number k_0 component of this wave.

The direction of motion is given by the sign of k_0 . If k_0 is positive, it is traveling down the $+\hat{z}$ -axis. If k_0 is negative, it is traveling down the $-\hat{z}$ -axis.

ω is still the frequency, but here, for generality, we allow it to be a function of k . For example, $\omega = c|k|$ for photons. ω must always be positive, because $\hbar\omega$ is the energy of the wave. This applies for all types of particles.

...Waves, composites, superposition, and Fourier transforms ...

One way of representing these wavelength components of a wave is the “power spectrum” shown below, specific to (4.2).



Note that we used complex functions and numbers to represent the wave. Indeed, A , k and ω can all be complex.

Quantum mechanics uses the language of complex numbers to describe its solutions.

When it comes to light waves, this is very useful to describe the electric and magnetic field components together, in one expression.

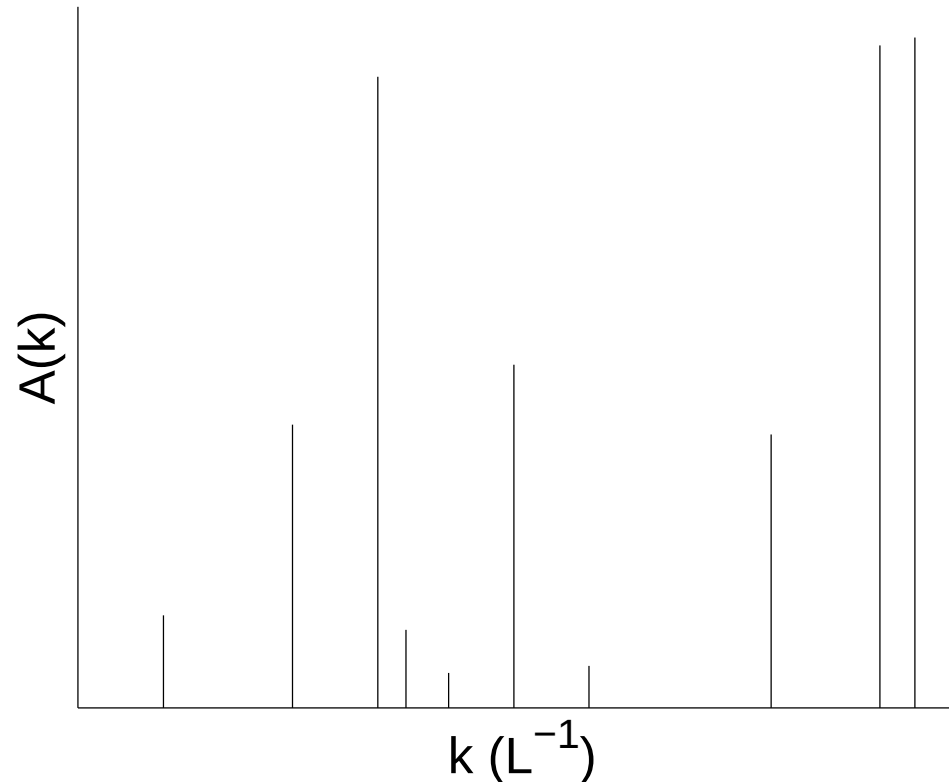
For matter waves, this mathematics is very compact and useful to describe constructive and destructive interference.

...Waves, composites, superposition, and Fourier transforms ...

Let's now consider a “composite” wave, or a “superposition”, made up of a number of monochromatic waves:

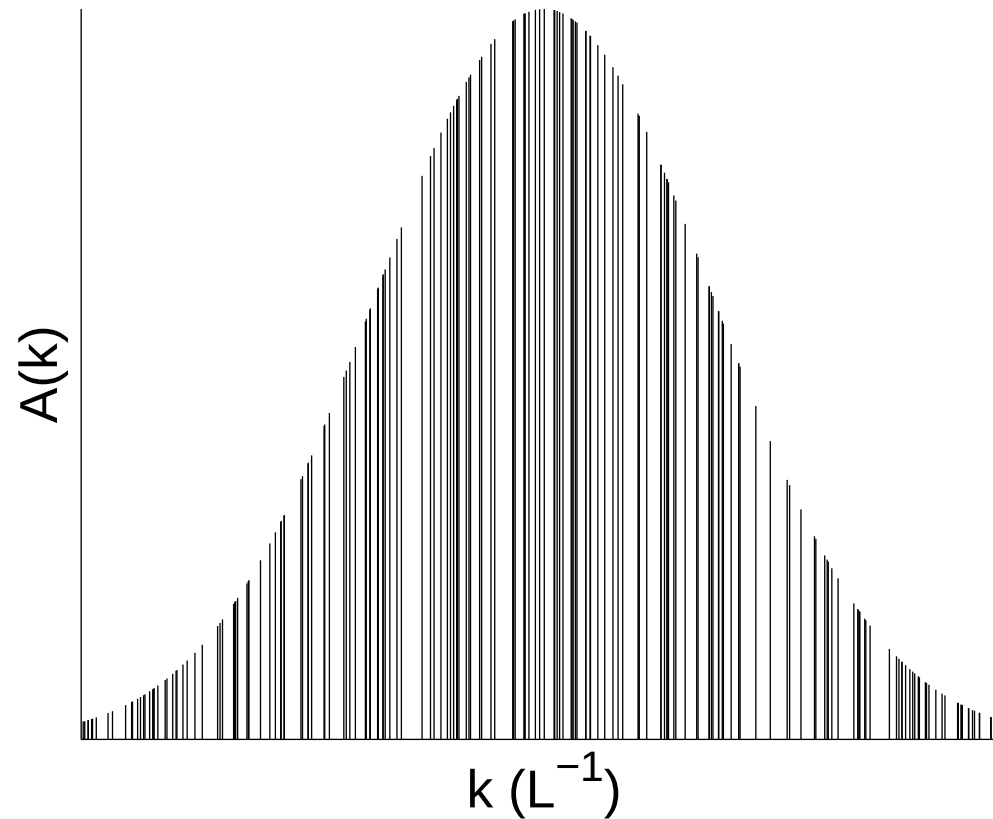
$$A(z, t) = \sum_i^N A_i e^{i[k_i z - \omega(k_i) t]} , \quad (4.3)$$

Here is an example where $N = 10$;



...Waves, composites, superposition, and Fourier transforms ...

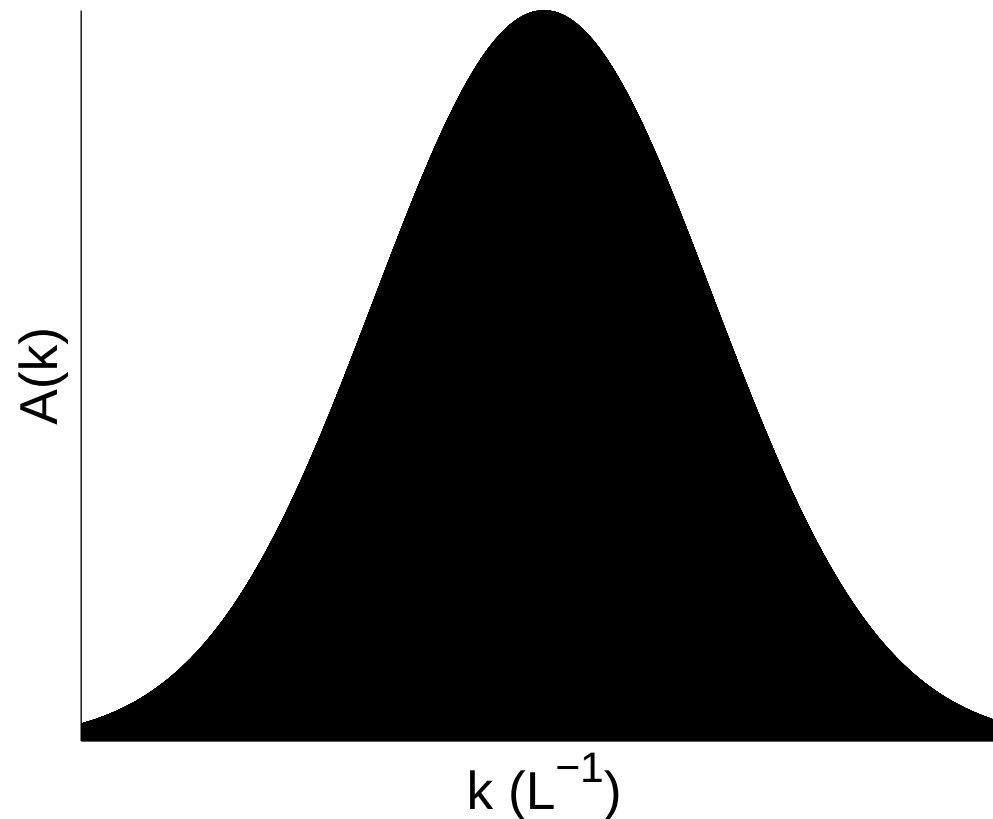
Here is another example where $N = 200$;



...Waves, composites, superposition, and Fourier transforms ...

As $N \rightarrow \infty$, the power spectrum looks like a continuum.

Here is another example for $N = 50000$.



...Waves, composites, superposition, and Fourier transforms ...

In 1D, the amplitude of a wave that has a continuum of frequencies, is given by a “Fourier transform”:

$$A(z, t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk A(k) e^{i[kz - \omega(k)t]} . \quad (4.4)$$

Note that the limits of integration suggest that this continuum composite is made up of components moving to the right, as well as to the left.

We also can recover (4.2) by specifying:

$$A(k) = (2\pi)^{1/2} A_0 \delta(k - k_0) , \quad (4.5)$$

and (4.3) by specifying:

$$A(k) = (2\pi)^{1/2} A_i \delta(k - k_i) \quad \text{for } i = 1, 2, 3, \dots N . \quad (4.6)$$

An aside on the Dirac delta function, $\delta(x - x_0)$...

... as well as the Kronecker delta and the Heaviside step function.

Dirac's δ -function is a mathematical tool for extracting the value of a function at a given point, *viz.*

$$\int_{x_1}^{x_2} dx \delta(x - x_0) f(x) = f(x_0), \text{ if } x_1 \leq x_0 \leq x_2 . \quad (4.7)$$

If x_0 is outside the range of integration, the integral is 0.

The Heaviside step function:

$$\begin{aligned} \theta(x - x_0) &= 1, \text{ if } x \geq x_0 \\ &= 0, \text{ if } x < x_0 \end{aligned} \quad (4.8)$$

It is easily shown that the indefinite integral of the Dirac δ is the step function.

$$\int_{-\infty}^x dx \delta(x - x_0) = \theta(x - x_0) \quad (4.9)$$

An aside on the Dirac delta function, $\delta(x - x_0)$...

Kronecker's δ is the integer form of the Dirac δ :

$$\begin{aligned}\delta_{mn} &= 1, \text{ if } m = n \\ &= 0, \text{ if } m \neq n\end{aligned}\tag{4.10}$$

There are many explicit mathematical forms of the Dirac δ -function. The one of greatest use for the manipulation of Fourier Transforms is:

$$\delta(k - k') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz e^{\pm i(k-k')z}\tag{4.11}$$

Some important properties of the Fourier Transform

From (4.4):

$$A(z, t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk A(k) e^{i[kz - \omega(k)t]} .$$

At $t = 0$:

$$A(z, 0) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk A(k) e^{ikz} . \quad (4.12)$$

The “inverse Fourier transform” is given by:

$$A(k) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dz A(z, 0) e^{-ikz} . \quad (4.13)$$

Note that if you combine (4.12) and (4.13):

$$\begin{aligned} A(k) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dz \left(\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk' A(k') e^{ik'z} \right) e^{-ikz} \\ &= \int_{-\infty}^{+\infty} dk' A(k') \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz e^{i(k'-k)z} \\ &= \int_{-\infty}^{+\infty} dk' A(k') \delta(k - k') \\ &= A(k) \end{aligned}$$

and the cycle is complete. That form of the δ -function **is** useful.

An example ...

As an example, let's make:

$$A(k) = A_0 \theta(k + k_0) \theta(k_0 - k) . \quad (4.14)$$

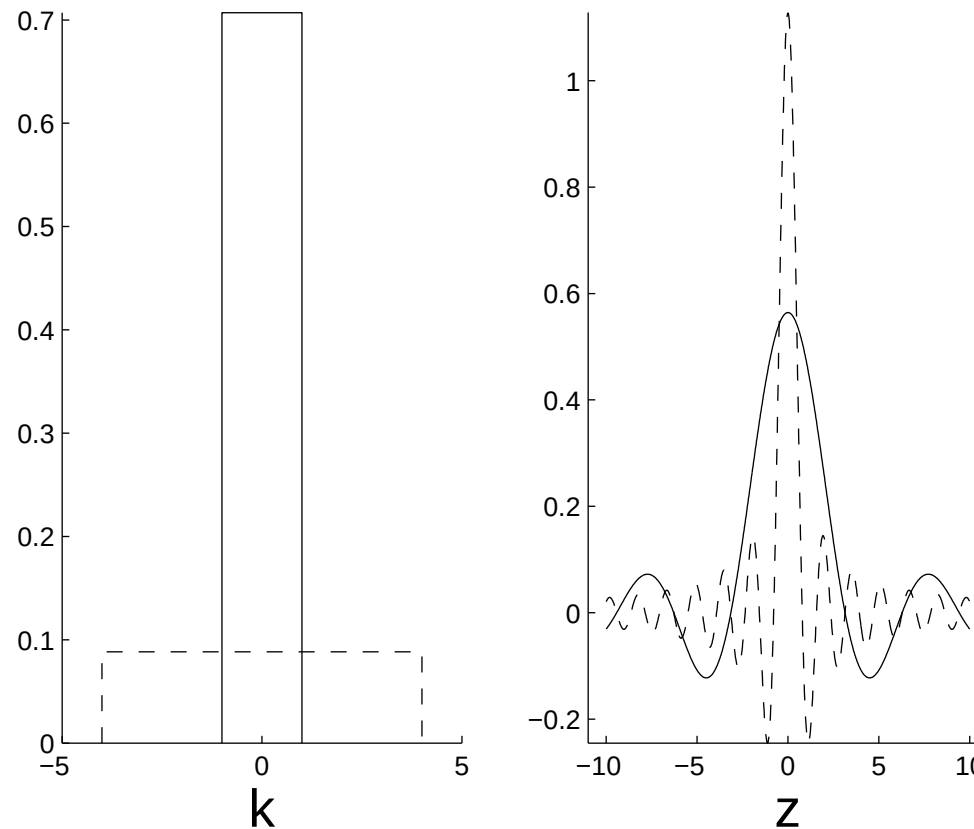
It can be shown²:

$$A(z, 0) = \left(\frac{2}{\pi} \right)^{1/2} A_0 \left(\frac{\sin k_0 z}{z} \right) . \quad (4.15)$$

On the next page is illustrated an example with different values of k_0 .

²The Mathematica code for this is: `Integrate[A0/Sqrt[2 Pi] Exp[-i k z], {k,-k0,k0}]`

... An example



Here are 2 examples of $A(k)$ and the corresponding $A(z, 0)$ for $k_0 = 1$ (solid line), and $k_0 = 4$ (dashed line). Note the inverse relationship between width in k -wavenumber space and width in z -dimension space.

Classical wave superposition ...

Recall (4.4):

$$A(z, t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk A(k) e^{i[kz - \omega(k)t]} ,$$

that contains the information of how a wave “packet” moves in physical space in time, $A(z, t)$, and its distribution (that is time-independent) in wavenumber space $A(k)$.

The “norm” of the wave is given by:

$$|A|^2 = \int_{-\infty}^{+\infty} dz A^*(z, t) A(z, t) , \tag{4.16}$$

where $A^*(z, t)$ is the “complex conjugate”³ of $A(z, t)$.

Let us expand the integral in (4.16) and see what we get ...

³Any complex function, f , may be split into two parts, $f() = f_R() + if_I()$, its complex and imaginary parts, where $f_R()$ and $f_I()$ are real functions. Then $f^*() = f_R() - if_I()$. An astute reader may object that (4.16) should contain a time dependence on the LHS (left hand side) because there is one on the RHS. She/he would be correct! This would describe non-static norms and arise when the $\omega(k)$'s have imaginary parts. We shall encounter that possibility only one time in NERS 312, and deal mostly, with time-independent norms.

Classical wave superposition ...

$$\begin{aligned}|A|^2 &= \int_{-\infty}^{+\infty} dz A^*(z, t) A(z, t) \\&= \int_{-\infty}^{+\infty} dz \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk' A^*(k') e^{-i[k'z - \omega^*(k')t]} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk A(k) e^{i[kz - \omega(k)t]} \\&= \int_{-\infty}^{+\infty} dk' \int_{-\infty}^{+\infty} dk A^*(k') A(k) e^{i[\omega^*(k') - \omega(k)]t} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz e^{i(k-k')z} \\&= \int_{-\infty}^{+\infty} dk' \int_{-\infty}^{+\infty} dk A^*(k') A(k) e^{i[\omega^*(k') - \omega(k)]t} \delta(k - k') \\&= \int_{-\infty}^{+\infty} dk |A(k)|^2 e^{2\Im[\omega(k)]t} \quad \text{where } \Im() \text{ means "Imaginary part"} \\&= \int_{-\infty}^{+\infty} dk |A(k)|^2 \quad \text{when } \Im[\omega(k)] = 0\end{aligned} \tag{4.17}$$

...Classical wave superposition

Therefore, for real ω

$$|A|^2 = \int_{-\infty}^{+\infty} dk |A(k)|^2$$

and we may compute the norm in either the wavenumber domain, or the space domain, whatever is more convenient.

Alternatively, we are dealing with time-dependent norms.

Expectation values ...

Waves are spread out over all space.

Hence, the question, “Where is the wave?” is not a well-posed question, because it is everywhere!

Well-posed questions:

“What is the wave’s average position?”

Answer, the average position, $\langle z \rangle$, is:

$$\langle z \rangle = \int_{-\infty}^{+\infty} dz A^*(z, t) z A(z, t) \quad (4.18)$$

Note the position of the thing to be averaged, between the amplitudes, A .
For now, let’s just call this a convention (to be clarified later).

...Expectation values ...

“What is the average spread of the wave?”

Answer, the average spread, Δz , that is defined by:

$$\Delta z = \sqrt{\langle z^2 \rangle - \langle z \rangle^2} \quad (4.19)$$

where

$$\langle z^2 \rangle = \int_{-\infty}^{+\infty} dz A^*(z, t) z^2 A(z, t) , \quad (4.20)$$

and $\langle z \rangle$ is given by (4.18).

...Expectation values ...

“What is the average wavenumber of the wave?”

The average wavenumber, $\langle k \rangle$, can be obtained by:

$$\langle k \rangle = \int_{-\infty}^{+\infty} dz A^*(z, t) \left(-i \frac{\partial}{\partial z} \right) A(z, t) \quad (4.21)$$

where, after some “inspired intuition”, it is realized that the function that extracts the wavenumber in z -space is a differential operator, that operates to the right, on the wave amplitude $A(z, t)$.

The proof is obtained from substituting for $A^*(z, t)$ and $A(z, t)$ in the above, and grinding out a result, that seems, in retrospect, obvious.

This moderately long derivation is given on the next page.

...Expectation values ...

From the previous page (4.21):

$$\langle k \rangle = \int_{-\infty}^{+\infty} dz A^*(z, t) \left(-i \frac{\partial}{\partial z} \right) A(z, t)$$

Substituting for the definition of $A(z, t)$ from (4.4)

$$\begin{aligned} \langle k \rangle &= \int_{-\infty}^{+\infty} dz \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk' A^*(k') e^{-i[k'z - \omega^*(k')t]} \left(-i \frac{\partial}{\partial z} \right) \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk A(k) e^{i[kz - \omega(k)t]} \\ &= \int_{-\infty}^{+\infty} dz \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk' A^*(k') e^{-i[k'z - \omega^*(k')t]} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk (-i \partial / \partial z) A(k) e^{i[kz - \omega(k)t]} \\ &= \int_{-\infty}^{+\infty} dz \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk' A^*(k') e^{-i[k'z - \omega^*(k')t]} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk k A(k) e^{i[kz - \omega(k)t]} \\ &= \int_{-\infty}^{+\infty} dk' \int_{-\infty}^{+\infty} dk A^*(k') k A(k) e^{i[\omega^*(k') - \omega(k)]t} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz e^{i(k-k')z} \\ &= \int_{-\infty}^{+\infty} dk' \int_{-\infty}^{+\infty} dk A^*(k') k A(k) e^{i[\omega^*(k') - \omega(k)]t} \delta(k - k') \\ &= \int_{-\infty}^{+\infty} dk k |A(k)|^2 e^{2\Im[\omega(k)]t} \\ &= \int_{-\infty}^{+\infty} dk k |A(k)|^2 \quad \text{when } \Im[\omega(k)] = 0 \end{aligned} \tag{4.22}$$

...Expectation values ...

Therefore, for real ω

$$\langle k \rangle = \int_{-\infty}^{+\infty} dk k |A(k)|^2$$

and we may compute the norm in either the wavenumber domain, or the space domain, whatever is more convenient.

Similarly, the spread in wavenumbers, Δk , is given by:

$$\Delta k = \sqrt{\langle k^2 \rangle - \langle k \rangle^2} \quad (4.23)$$

where

$$\begin{aligned} \langle k^2 \rangle &= \int_{-\infty}^{+\infty} dz A^*(z, t) \left(-i \frac{\partial}{\partial z} \right)^2 A(z, t) \\ &= \int_{-\infty}^{+\infty} dz A^*(z, t) \left(-\frac{\partial^2}{\partial z^2} \right) A(z, t) \\ &= \int_{-\infty}^{+\infty} dk k^2 |A(k)|^2 \end{aligned} \quad (4.24)$$

and $\langle k \rangle$ is given by (4.22).

...Expectation values ...

We have reached certain conclusions when it comes to classical wave:

- Waves are everywhere. Therefore it does not make sense to ask where a wave or a particle (that is making its wave nature evident), "Where is it?". We can, however, say a lot about its average values, and the spread about those average values.
- Average values, e.g. z , z^2 , k , k^2 , ... are computed as follows. Sometimes these are also called "expectation values", or just "expectation".

$$\langle \mathcal{O} \rangle = \int_{-\infty}^{+\infty} dz A^*(z, t) \mathcal{O}(z) A(z, t) , \quad (4.25)$$

$$= \int_{-\infty}^{+\infty} dk A^*(k) \mathcal{O}(k) A(k) , \quad (4.26)$$

$$\langle \mathcal{O}^2 \rangle = \int_{-\infty}^{+\infty} dz A^*(z, t) \mathcal{O}(z) \mathcal{O}(z) A(z, t) , \quad (4.27)$$

$$= \int_{-\infty}^{+\infty} dk A^*(k) \mathcal{O}(k) \mathcal{O}(k) A(k) , \quad (4.28)$$

...Expectation values ...

- The spread, or “variance” associated with an operator is given by:

$$\Delta\mathcal{O} = \sqrt{\langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2} \quad (4.29)$$

- For generality, we allow the quantity to be determined to be allowed to have an “operator” representation, indicated by the $\mathcal{O}$ -symbol, not to be confused with the same symbol used in Taylor expansions. For example, we have already seen operator forms of k , in terms of a derivative.
- **All** measurables have representations in k -space **and** z -space.

...Expectation values ...

- The operators of classical waves and their expectation values

Measurable	z -space $\mathcal{O}$	z -space calculation	k -space $\mathcal{O}$	k -space calculation
$\langle z \rangle$	z	$\langle z \rangle$	$i \frac{\partial}{\partial k}$	$\langle i \frac{\partial}{\partial k} \rangle$
$\langle z^2 \rangle$	z^2	$\langle z^2 \rangle$	$-\frac{\partial^2}{\partial k^2}$	$\langle -\frac{\partial^2}{\partial k^2} \rangle$
Δz		$\sqrt{\langle z^2 \rangle - \langle z \rangle^2}$		$\sqrt{\langle -\frac{\partial^2}{\partial k^2} \rangle - \langle i \frac{\partial}{\partial k} \rangle^2}$
$\langle k \rangle$	$-i \frac{\partial}{\partial z}$	$\langle -i \frac{\partial}{\partial z} \rangle$	k	$\langle k \rangle$
$\langle k^2 \rangle$	$-\frac{\partial^2}{\partial z^2}$	$\langle -\frac{\partial^2}{\partial z^2} \rangle$	k^2	$\langle k^2 \rangle$
Δk		$\sqrt{\langle -\frac{\partial^2}{\partial z^2} \rangle - \langle -i \frac{\partial}{\partial z} \rangle^2}$		$\sqrt{\langle k^2 \rangle - \langle k \rangle^2}$

Uncertainty relationships for classical waves

We have seen from the previous example, starting with (4.14), that wave packets that are spread out in wavenumber, are more tightly spread out in space, and vice-versa.

The extreme example of this is a plane wave (spread out over all space), with a monochromatic frequency “spectrum”.

Conversely, if you said that $A(z, 0) = \delta(z)$, you would have, from (4.13), that $A(k)$, viz. **all** frequencies, going in the $-z$ -direction and in the $+z$ -direction, are equally represented!

Waves are curious things indeed.

This behavior is encapsulated by the uncertainty relationship for classical waves, that states⁴:

$$\Delta k \Delta z \geq \frac{1}{2} \quad (4.30)$$

The conclusion of this is a more precise statement of the inverse relationship between the spread in wavenumber space and z -space, for a wavepacket.

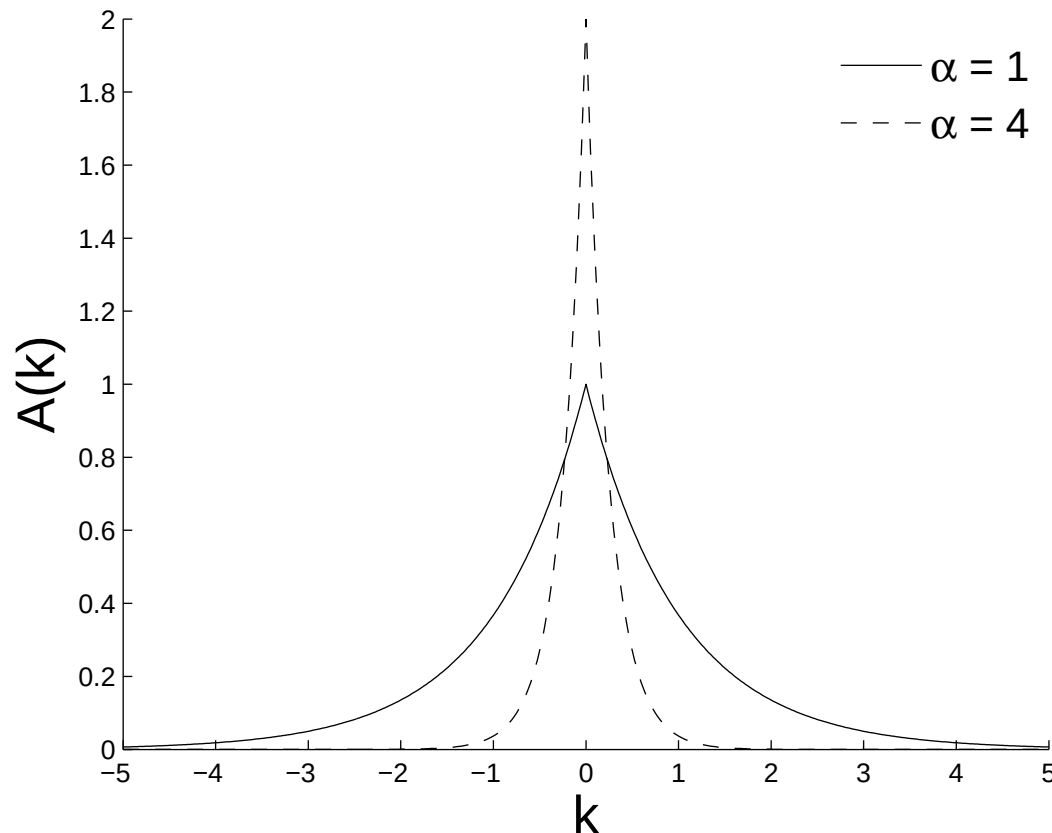
⁴This is a long proof that can be hard to find, but it can be found in Messiah's book on Quantum Mechanics.

Example of a wavepacket...

Let's just pick an example out of the air.

$$A(k) = \sqrt{\alpha} e^{-\alpha|k|}, \quad (4.31)$$

where α is a real constant, with possible value $0 < \alpha < \infty$.



... Example of a wavepacket

First let's check the norm:

$$\begin{aligned} |A|^2 &= \int_{-\infty}^{+\infty} dk |A(k)|^2 \quad [\text{general expression from (4.17)}] \\ &= \int_{-\infty}^{+\infty} dk (\sqrt{\alpha} e^{-\alpha|k|})^* \sqrt{\alpha} e^{-\alpha|k|} \quad [\text{substitution from (4.31)}] \\ &= \alpha \int_{-\infty}^{+\infty} dk e^{-2\alpha|k|} \quad [\text{some reorganization, noted that example is all real}] \\ &= 2\alpha \int_0^{\infty} dk e^{-2\alpha k} \quad [\text{noted that integral is even}] \\ &= \int_0^{\infty} du e^{-u} \quad [\text{used change of variable } u = 2\alpha k] \\ &= -e^u \Big|_0^{\infty} = 0 - (-1) \quad [\text{doing the integral and evaluating}] \\ &= 1 \quad [\text{It's normalized!}] \end{aligned}$$

... Example of a wavepacket ...

Now let's get $A(z, 0)$:

$$\begin{aligned} A(z, 0) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk A(k) e^{ikz} \quad [\text{general expression from (4.12)}] \\ &= \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dk e^{-\alpha|k|} e^{ikz} \quad [\text{substitution from (4.31)}] \\ &= \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dk e^{-\alpha|k|} (\cos kz + i \sin kz) \quad [\text{using Euler's formula}] \\ &= \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dk e^{-\alpha|k|} \cos kz \quad [\text{using antisymmetry}] \\ &= \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_0^{\infty} dk e^{-\alpha k} \cos kz \quad [\text{using symmetry}] \\ &= \left(\frac{2\alpha}{\pi}\right)^{1/2} \frac{\alpha}{\alpha^2 + z^2} \quad [\text{using Mathematica!}] \end{aligned} \tag{4.32}$$

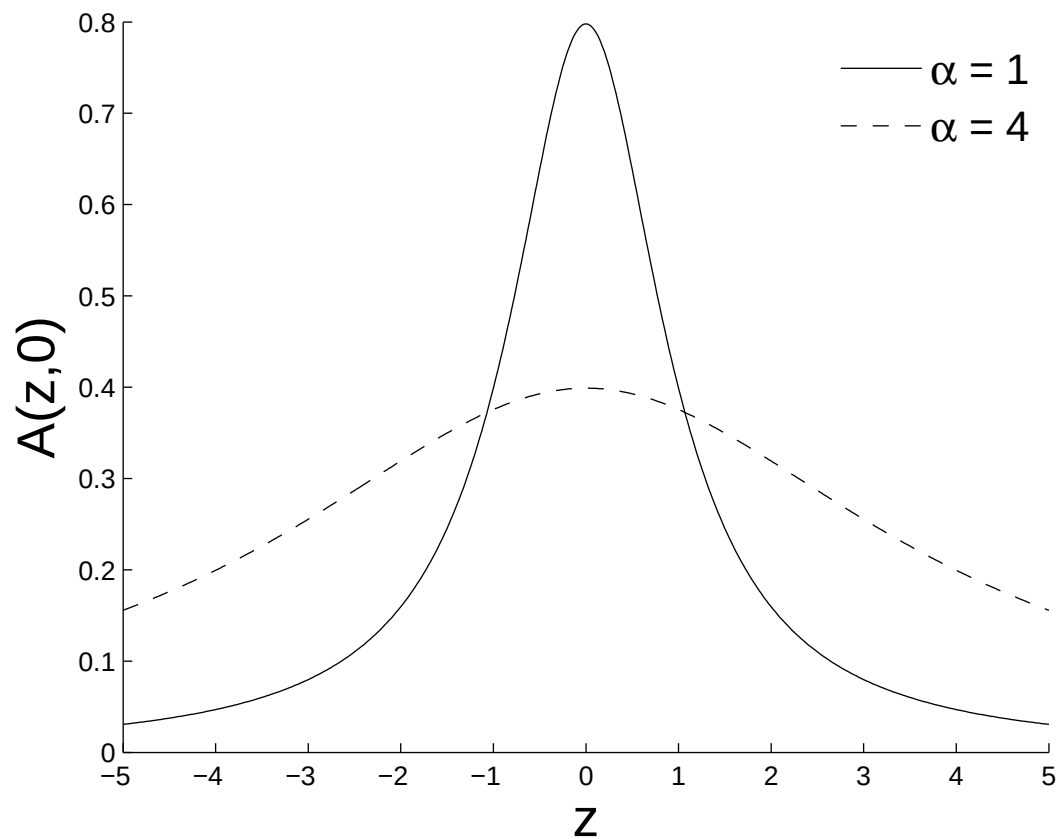
The Mathematica code used was: `Integrate[Exp[-a k] Cos[k z], {k,0,Infinity}]`

You can verify for yourselves that this is normalized properly (when ω is real), *i.e.*:

$$\int_{-\infty}^{+\infty} dz A^*(z, t) A(z, t) = 1 .$$

... Example of a wavepacket ...

Here is a plot of the $A(z, 0)$ for the the k -space examples shown previously. Note, again, the inverse relationship of widths in k -space and z -space.



... Example of a wavepacket ...

What about the moments $\langle k \rangle$, and $\langle k^2 \rangle$?

$$\begin{aligned}\langle k \rangle &= 0 \quad [\text{by antisymmetry of the integral}] \\ \langle k^2 \rangle &= \int_{-\infty}^{+\infty} dk A^*(k) k^2 A(k) \quad [\text{general case}] \\ &= \int_{-\infty}^{+\infty} dk k^2 |A(k)|^2 \quad [\text{since } k \text{ is not a differential}] \\ &= \alpha \int_{-\infty}^{+\infty} dk k^2 e^{-2\alpha|k|} \quad [\text{this example}] \\ &= 2\alpha \int_0^{+\infty} dk k^2 e^{-2\alpha k} \quad [\text{by symmetry}] \\ &= \frac{1}{4\alpha^2} \int_0^{+\infty} du u^2 e^{-u} \quad [\text{change of variable, } u = 2\alpha k] \\ \langle k^2 \rangle &= \frac{1}{2\alpha^2} \quad [\text{from Mathematica}]\end{aligned}\tag{4.33}\tag{4.34}$$

Mathematics aside 1: $\int_0^{+\infty} du u^n e^{-u} = n!$, the factorial. n is an integer ≥ 0 .

Mathematics aside 2: $\int_0^{+\infty} du u^{t-1} e^{-u} = \Gamma(t)$, the gamma function. $\Gamma(n) = (n-1)!$.

Mathematica aside: The Mathematica code used was: `Integrate[u^2 Exp[-u], {u,0,Infinity}]`

... Example of a wavepacket ...

What about the moments $\langle z \rangle$, and $\langle z^2 \rangle$?

$$\begin{aligned}\langle z \rangle &= 0 \quad [\text{at } t = 0, \text{ by antisymmetry of the integral}] & (4.35) \\ \langle z^2 \rangle &= \int_{-\infty}^{+\infty} dz A^*(z, 0) z^2 A(z, 0) \quad [\text{general case at } t = 0] \\ &= \int_{-\infty}^{+\infty} dz z^2 |A(z, 0)|^2 \quad [\text{since } z \text{ is not a differential}] \\ &= \left(\frac{2\alpha}{\pi} \right) \int_{-\infty}^{+\infty} dz z^2 \frac{\alpha^2}{(\alpha^2 + z^2)^2} \quad [\text{this example, note: } \omega \text{ is assumed to be real}] \\ &= \left(\frac{2\alpha^2}{\pi} \right) \int_{-\infty}^{+\infty} du \frac{u^2}{(1 + u^2)^2} \quad [\text{change of variable, } u = z/\alpha] \\ \langle z^2 \rangle &= \alpha^2 \quad [\text{from Mathematica}] & (4.36)\end{aligned}$$

Mathematica aside: The Mathematica code used was: `Integrate[u^2/(1 + u^2)^2, {u,-Infinity,Infinity}]`

... Example of a wavepacket ...

Summary of results so far:

$$A(k) = \sqrt{\alpha} e^{-\alpha|k|} \quad [\text{this example, the rest was derived}]$$

$$A(z, 0) = \left(\frac{2\alpha}{\pi} \right)^{1/2} \frac{\alpha}{\alpha^2 + z^2}$$

$$\langle k \rangle = 0$$

$$\langle k^2 \rangle = \frac{1}{2\alpha^2}$$

$$\langle z \rangle = 0 \quad [\text{at } t = 0]$$

$$\langle z^2 \rangle = \alpha^2 \quad [\text{at } t = 0]$$

$$\Delta k = \frac{(1/2)^{1/2}}{\alpha} \quad [\text{Recall: } \sqrt{\langle k^2 \rangle - \langle k \rangle^2}]$$

$$\Delta z = \alpha \quad [\text{at } t = 0]$$

$$\Delta k \Delta z = (1/2)^{1/2} \geq 1/2 \quad [\text{at } t = 0] \quad \checkmark$$

Note: If you do not obtain $\Delta k \Delta z \geq 1/2$, you have made a mistake somewhere!

... Example of a wavepacket ...

What about the the time dependence?

To answer this, we have to go back to the definition. From (4.4)

$$A(z, t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk A(k) e^{i[kz - \omega(k)t]} .$$

Then, we must specify the relationship for $\omega(k)$.

To complete this example, let's choose a light wave, for which $\omega = c|k|$.

With this choice:

$$\begin{aligned} A(z, t) &= \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dk e^{-\alpha|k|} e^{i(kz - c|k|t)} \\ &= \left(\frac{\alpha}{2\pi}\right)^{1/2} \left(\int_{-\infty}^0 dk e^{-\alpha|k|} e^{i(kz - c|k|t)} + \int_0^{\infty} dk e^{-\alpha|k|} e^{i(kz - c|k|t)} \right) \\ &\quad \dots \text{split the integral into positive and negative } k \\ &= \left(\frac{\alpha}{2\pi}\right)^{1/2} \left(\int_{-\infty}^0 dk e^{\alpha k} e^{ikz} e^{ickt} + \int_0^{\infty} dk e^{-\alpha k} e^{ikz} e^{-ickt} \right) \\ &\quad \dots \text{changed } |k| \text{ to } k, \text{ changing signs where needed} \\ &= \left(\frac{\alpha}{2\pi}\right)^{1/2} \left(\int_0^{\infty} dk e^{-\alpha k} e^{-ikz} e^{-ickt} + \int_0^{\infty} dk e^{-\alpha k} e^{ikz} e^{-ickt} \right) \\ &\quad \dots \text{changed } k \text{ to } -k, \text{ in the first integral} \end{aligned}$$

... Example of a wavepacket ...

continuing ...

$$\begin{aligned} &= \left(\frac{2\alpha}{\pi} \right)^{1/2} \int_0^\infty dk e^{-\alpha k} \cos(kz) e^{-ickt} \\ &\quad \dots \text{combined the integrals using } e^{iu} + e^{-iu} = 2 \cos u \\ &= \left(\frac{2\alpha}{\pi} \right)^{1/2} \frac{\alpha + ict}{(\alpha + ict)^2 + z^2} \\ &\quad \dots \text{Mathematica to the rescue (see below)} \\ &= \left(\frac{2\alpha}{\pi} \right)^{1/2} \left(\frac{1}{\alpha - i(z - ct)} + \frac{1}{\alpha + i(z + ct)} \right) \end{aligned} \tag{4.37}$$

The last step is a simplification of the step above.

And, it has a clear interpretation⁵.

The term with $z - ct$ represents a wave traveling along +ve z .

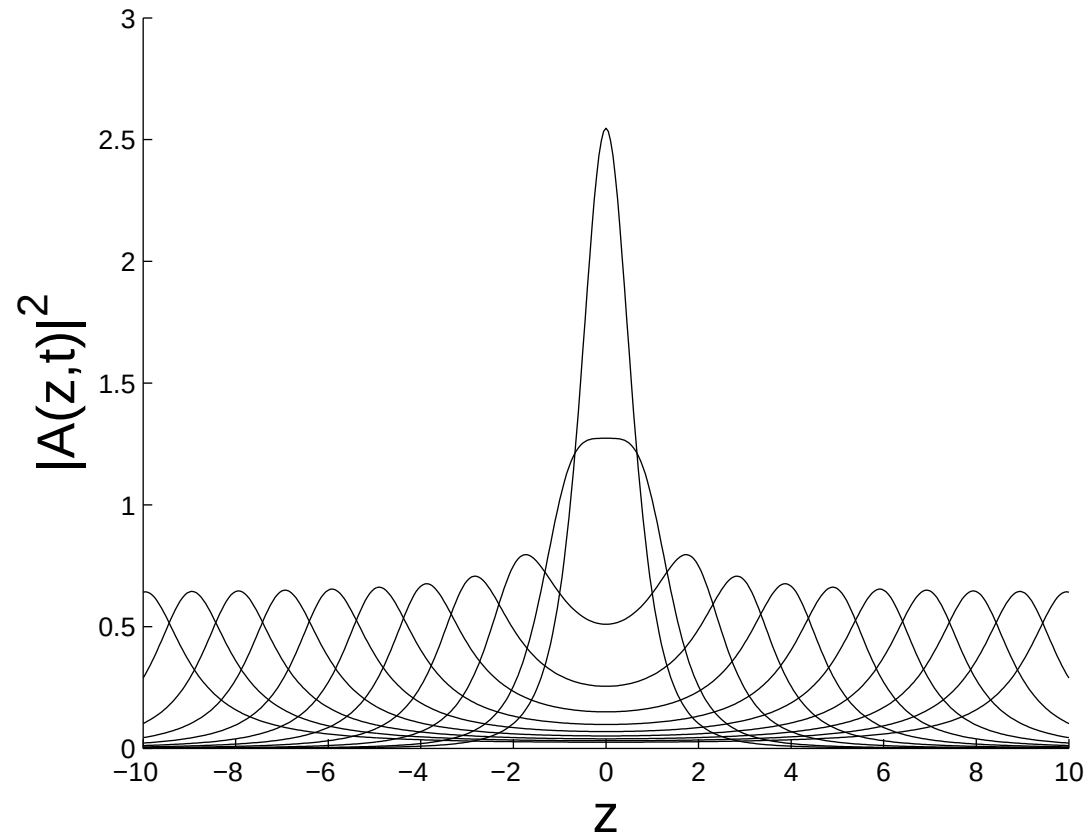
The term with $z + ct$ represents a wave traveling along -ve z .

$|A(z, t)|^2$ is plotted on the next page

The Mathematica code used above was: `Integrate[Exp[-a k] Cos[k z] Exp[ i c k t], {k,0,Infinity}]`

⁵If $f(z, t)$ reduces to $f(z \mp vt)$, then an observer with speed $\pm v$ sees a static $f(z)$ in his moving frame.

... Example of a wavepacket ...



The central peak is for $t = 0$, and then the sequence $t = 1, 2, 3, \dots, 10$ are plotted. The local maxima occur at $z = \pm t$ for each time instance.

... Example of a wavepacket ...

Here is the Matlab code that plotted this figure.

It is interesting to watch the figure being written to the screen.

You can see the wave splitting and moving to the left and right.

```
close all % Closes any open figures
clear all % Clears the Workspace

alpha= 1; % This is alpha, use this for the graph

z0 = 10; % Limits of the function to be plotted

N = 401; % Choose an odd number so that z = 0 is a point in the linspace below((
z = linspace(-z0, z0, 401);

c = 1; % We'll be in "speed of light" units

figure(1)
hold on

for t = 0:1.0:10
    Azt = 1./(alpha - i*(z - c*t)) + 1./(alpha + i*(z + c*t)); % This is A(z,t), un-normalized
    A2 = Azt.*conj(Azt); % A(z,t) times its complex conjugate
    A2 = (2*alpha/pi)*A2; % Constant normalization factor out front
    plot(z,A2,'k-')
    pause(0.1) % It's great to watch this unfold slowly
end

xlabel('z','fontsize',20)
ylabel('|A(z,t)|^2','fontsize',20)
```

Group velocity ...

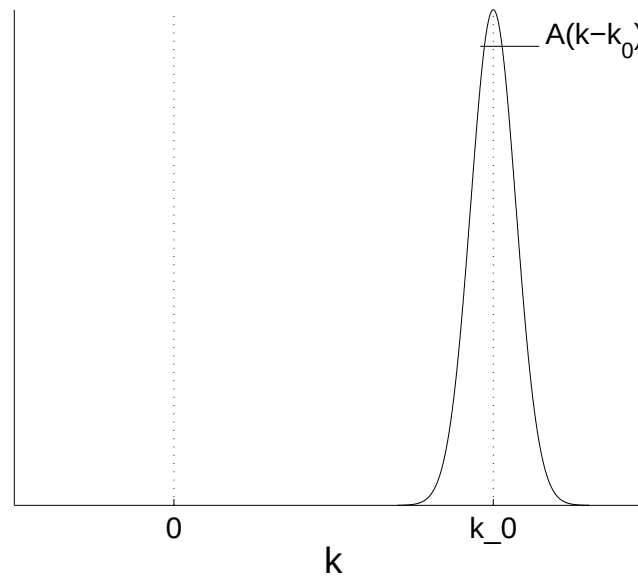
Recall (4.4):

$$A(z, t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk A(k) e^{i[kz - \omega(k)t]} .$$

We have learned that this amplitude represents the kinematics of waves.

Now, we want to draw a strong connection between the kinematics of waves (all of Chapter 4), to the kinematics of particles.

Imagine that our wavepacket's k -distribution, $A(k)$, is tightly constrained around a principle value, k_0 , as indicated in the figure below.



... Group velocity ...

Consider that $A(k)$ is so tightly constrained in the vicinity of k_0 , that one can Taylor-expand $\omega(k)$ and ignore all but the first 2 terms. That is:

$$\omega(k) \approx \omega_0 + v_g(k - k_0) , \quad (4.38)$$

where the principle frequency is defined as

$$\omega_0 \equiv \omega(k_0) , \quad (4.39)$$

kinematics and

$$v_g \equiv \left. \frac{d\omega}{dk} \right|_{k=k_0} \quad [\text{Note: has the dimensions of velocity (L/T)}] \quad (4.40)$$

Higher-order terms, $\mathcal{O}[(k - k_0)^2]$, are ignored. Then, rewrite (4.4) as:

$$\begin{aligned} A(z, t) &\approx \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk A(k) e^{i[kz - \omega_0 t - v_g(k - k_0)t]} \\ &\approx \frac{e^{it(v_g k_0 - \omega_0)}}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dk A(k) e^{ik(z - v_g t)} \\ &\approx e^{it(v_g k_0 - \omega_0)} A(z - v_g t) , \end{aligned} \quad (4.41)$$

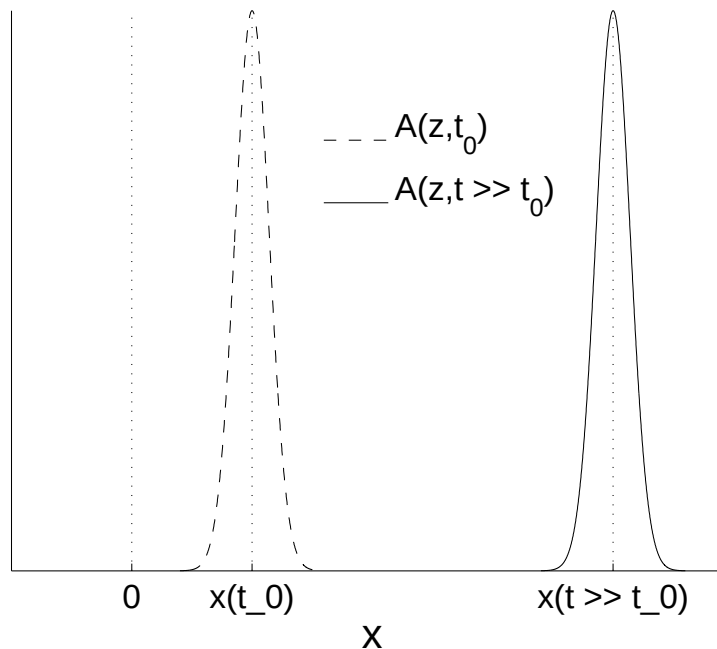
after taking the inverse Fourier transform.

... Group velocity ...

What does this all mean?

The factor $e^{it(v_g k_0 - \omega_0)}$ is a time-dependent phase. It factors out of every moment that extracts information from the wave, e.g. $\langle z \rangle$, etc.

The result of the calculation is that $A(z, t) = A(z - v_g t)$. This means that the amplitude in z -space appears to be a coherent shape that travels at a single velocity, v_g , the “group” velocity.



... Group velocity

Does it make sense? Does it jibe with the particle picture?

photons

$$\omega = kc; \quad v_g = \frac{d(kc)}{dk} = c \quad \checkmark \quad (4.42)$$

non-relativistic particles with mass

$$v_g = \frac{d\omega}{dk} = \frac{\hbar d\omega}{\hbar dk} = \frac{dK}{dp} = \frac{dp^2/2m}{dp} = \frac{p}{m} = v \quad \checkmark \quad (4.43)$$

relativistic particles with mass

$$v_g = \frac{d\omega}{dk} = \frac{\hbar d\omega}{\hbar dk} = \frac{dE}{dp} = \frac{d[p^2c^2 + m^2c^4]^{\frac{1}{2}}}{dp} = \frac{(1/2)2c^2p}{[p^2c^2 + m^2c^4]^{\frac{1}{2}}} = \frac{c^2p}{E} = \frac{c^2mv\gamma}{mc^2\gamma} = v \quad \checkmark \quad (4.44)$$

Yes it does!