

1. Introduction

1.1. About this course ...

According to the College Bulletin:

NERS 311. Elements of Nuclear Engineering and Radiological Sciences I

Photons, electrons, neutrons, and protons. Particle and wave properties of radiation. Introduction to quantum mechanics. Properties and structure of atoms.

What is it really about? It's about Modern Physics.

See: http://en.wikipedia.org/wiki/Modern_physics

Modern Physics covers just about everything in physics from the 1900s onward, and explains those things that Classical Physics could not.

1.2 Modern Physics encompasses ...

Quantum Mechanics: The study of really small, light things, like neutrons, protons, nuclei, atoms, and slow, non-relativistic electrons ... the basic building blocks of Nuclear Engineering. *We spend most of our time in 311 doing this.*

Relativistic Quantum Mechanics: The study of very fast (approaching c), light things. *More cool stuff, but not covered in detail for NERS. We do make use of its results.*

General Relativity: The study of very fast (approaching c) objects, with external forces. Also, the study of very dense, massive objects, like black holes. *Cool stuff, but not covered for NERS.*

Modern Physics encompasses (continued)...

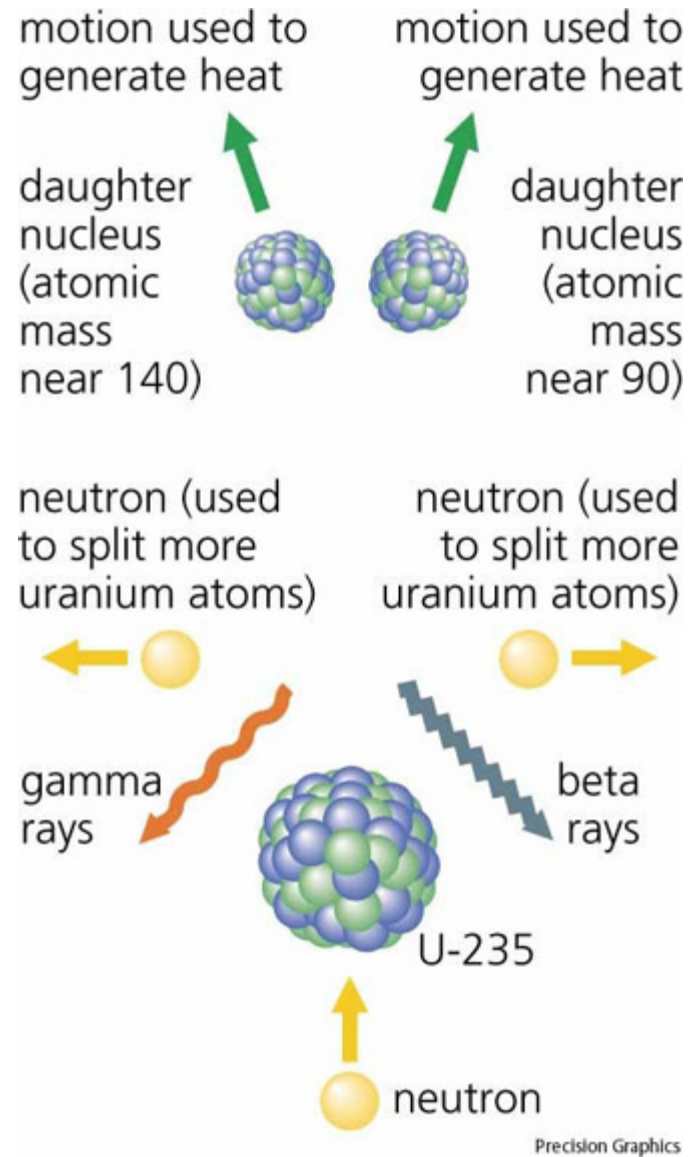
Special Relativity: The study of very fast (approaching c) objects, with no external forces. *We need to cover some of this in 311 to discuss photons, β -decay electrons, and their kinematics. Why?*

Because photons travel at the speed of light. In Nuclear Engineering applications, photons arise from

- nuclear disintegrations (fission)
- nuclear transitions (isotopes, excited states)
- electrons slowing down in materials (bremsstrahlung [Medical Physics, medical imaging using X-rays])
- Glow Blue! That's really Čherenkov radiation.

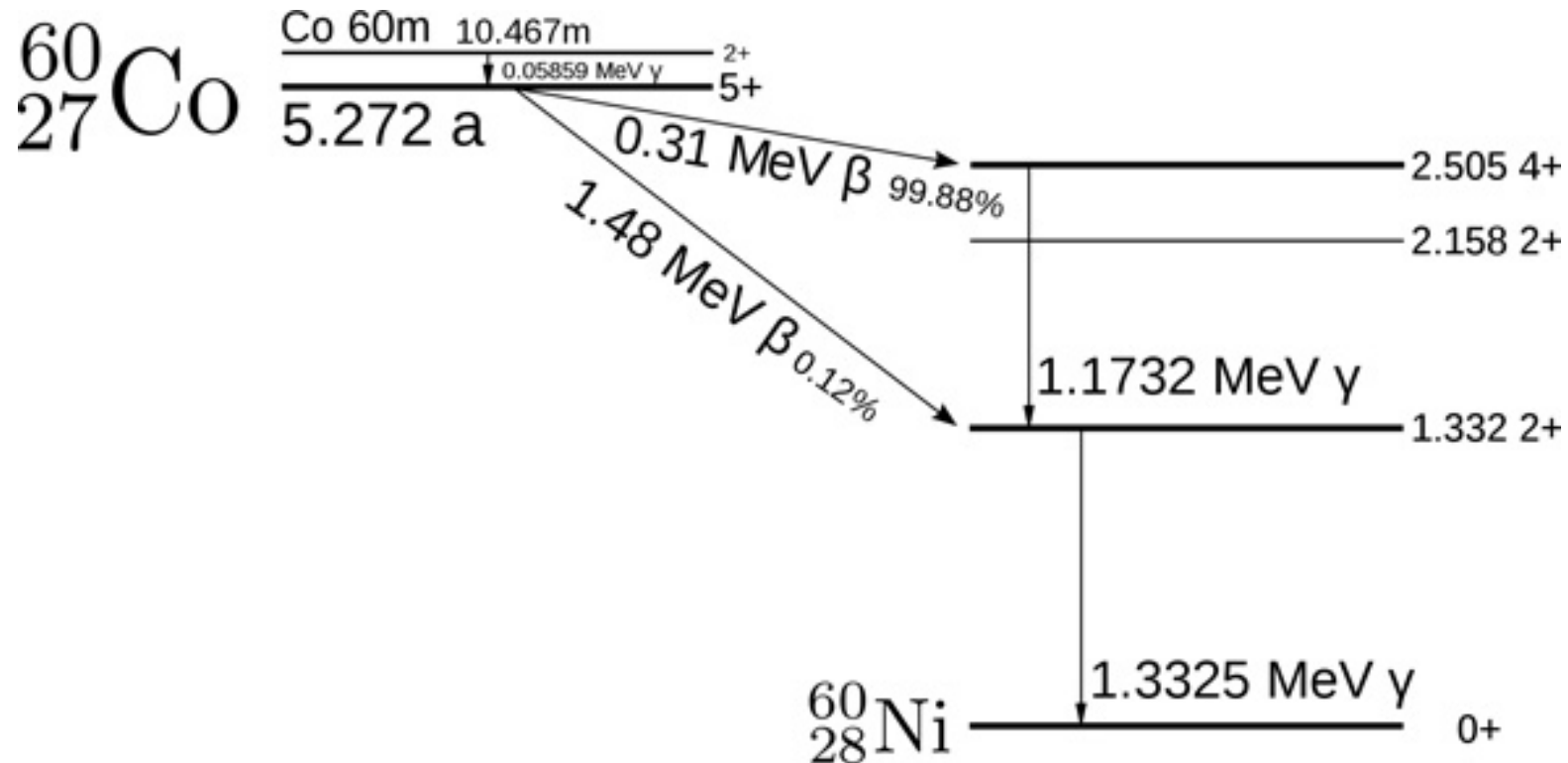
See http://en.wikipedia.org/wiki/Cherenkov_radiation

Fission



Isotopes

The ^{60}Co decay scheme ...



X-Rays for cancer treatment

A medical LINAC used for treatments of many diseases (particularly cancer)...



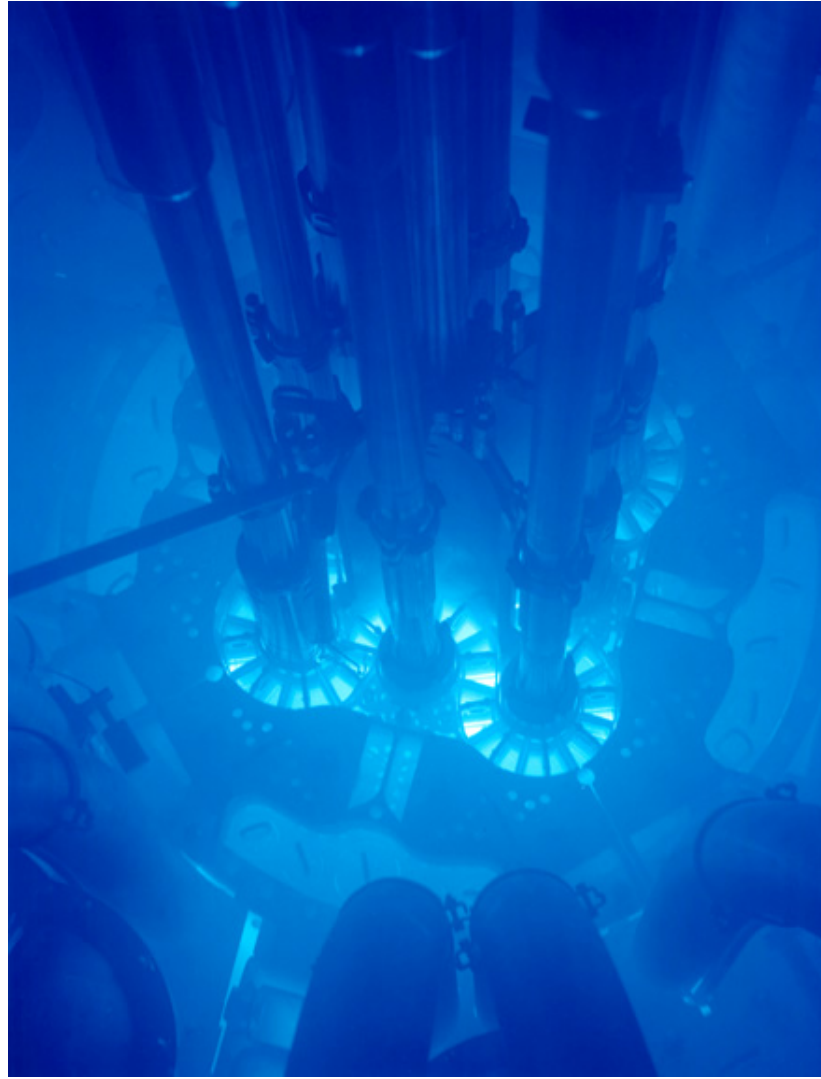
X-Rays for Diagnostic Imaging

A CT scanner...

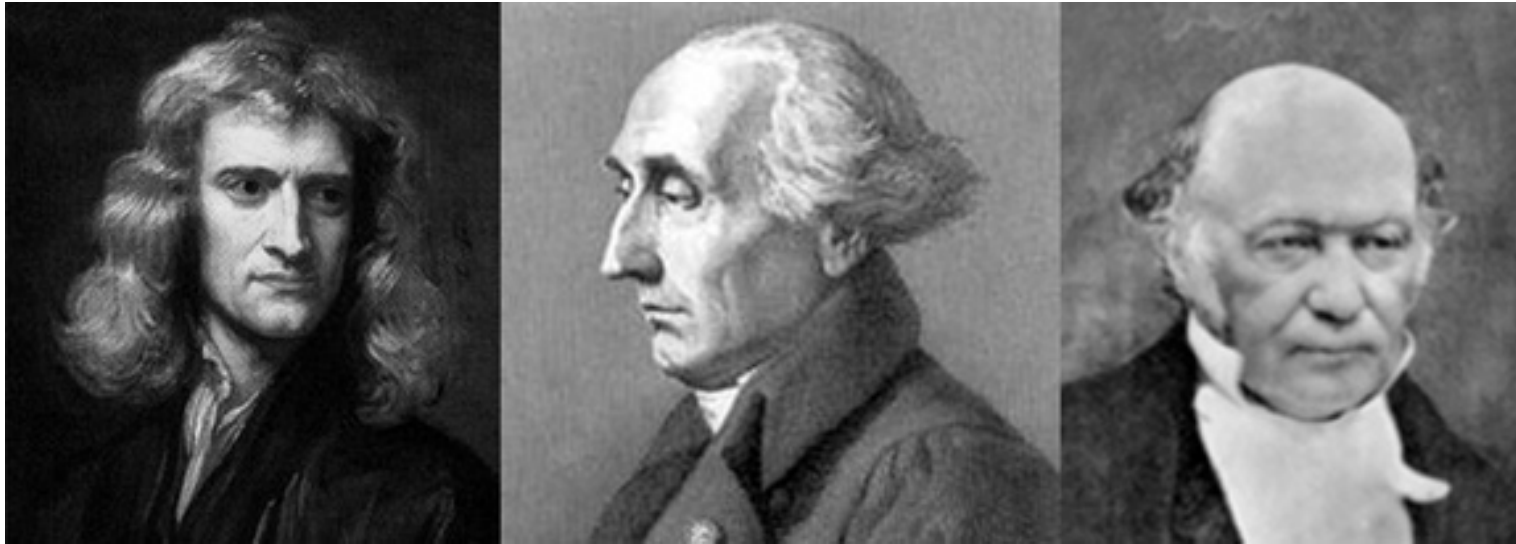


Blue Light

Pavel Alekseyevich Cherenkov (1904-1990) and his eponymous effect (ATR/INL)



1.3 Some History



(L→R): Isaac Newton (1642–1727), Joseph-Louis de Lagrange (1736–1813), and William Rowe Hamilton (1805–1865)

Around 1900 or so, all (in principle) was known!

—or so it seemed, to followers of the above austere-looking gentlemen

Classical Mechanics = “The Theory of Everything” *a.k.a.* “Holy Grail”

Some famous scientists said:

“If it has mass, and velocity, I can predict its evolution in time, forever.”

They later retracted their statements!

The motions of everything from dust particles to galaxies, were considered to be knowable, and predictable.

Maxwell, and his equations



(L→R): Michael Faraday (1791–1867), André-Marie Ampère (1775–1836)

James Clerk Maxwell (1831–1879) and Carl Friedrich Gauss (1777–1855)

Gauss's Law of Electricity: $\vec{\nabla} \cdot \vec{D} = 4\pi\rho$

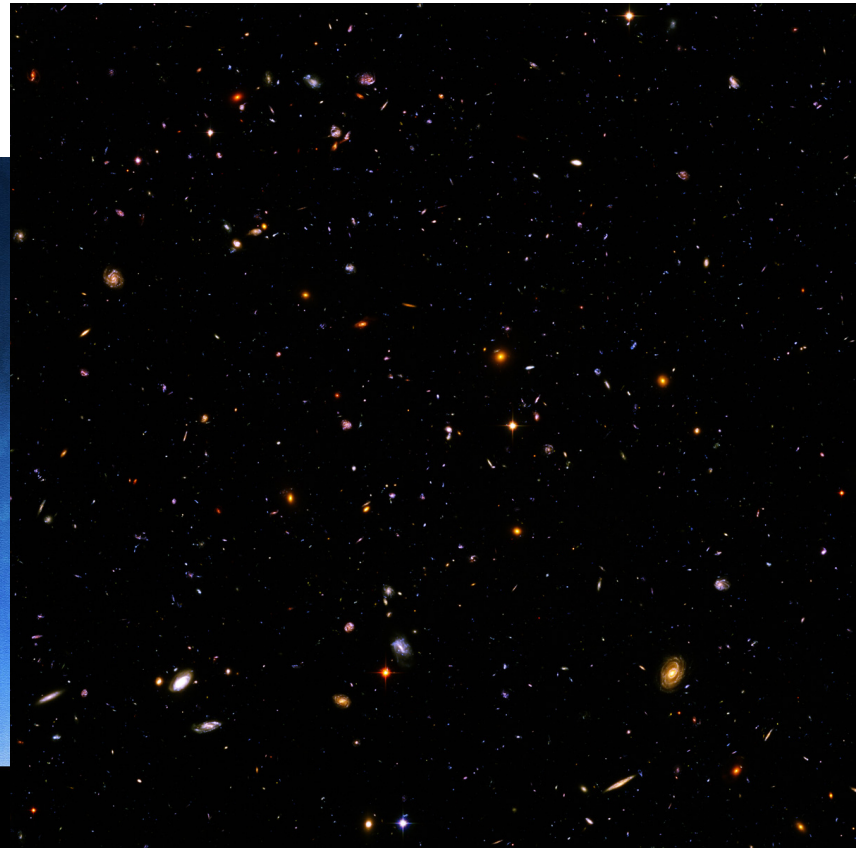
Gauss's Law of Magnetism: $\vec{\nabla} \cdot \vec{B} = 0$

Faraday's Law: $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$

Ampère-Maxwell Law: $\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$

Telescope Observations

Newer observations of distant galaxies that move with $v \approx c$, seemed to be



..... governed by different physical laws. **Unacceptable!**

1.4 Review of Classical Physics

1.4.1 Mechanics ...

Kinetic energy (a scalar) of a mass m moving with velocity $\vec{v}$ in some direction:

$$K = \frac{1}{2}mv^2 . \quad (1.1)$$

Momentum (a vector), is:

$$\vec{p} = m\vec{v} . \quad (1.2)$$

(1.1) $\rightarrow$ (1.2) $\Rightarrow$:

$$\vec{p} \cdot \vec{p} = p^2 = m\vec{v} \cdot m\vec{v} = m^2\vec{v} \cdot \vec{v} = m^2v^2 . \quad (1.3)$$

(1.1) & (1.3)/(2m) $\Rightarrow$:

$$\frac{p^2}{2m} = \frac{m^2v^2}{2m} = \frac{1}{2}mv^2 = K .$$

Hence:

$$K = \frac{p^2}{2m} . \quad (1.4)$$

1.4.1 ...Mechanics ...

The kinetic energy change across a potential difference of 1 volt is:

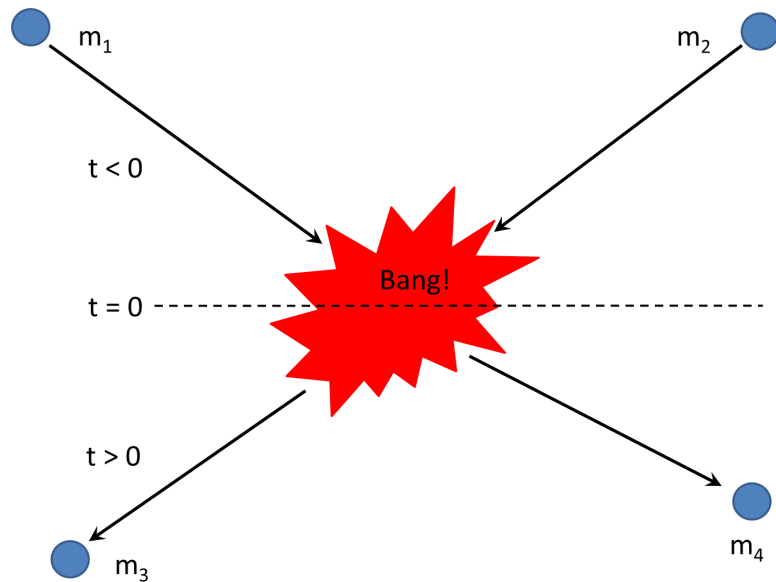
$$\Delta K = 1eV \approx 1.602 \times 10^{-19} \text{ J} \quad (1.5)$$

for a singly charged particle (one unit of charge e).

Review of 2-body Collision Kinematics

What is it?.....Newtonian mechanics

Bang two things together, and two things separate after the collision



Simplification/specification:

- Two bodies in, two bodies out
- No bodies are rotating, before or after the collision

Why do we study this? NE/RS applications!

neutron-neutron scattering

neutron-proton scattering (reactor cooling)

neutron-deuteron scattering (CANDU reactor)

proton/alpha/electron-nuclear

scattering (Rutherford scattering)

neutron-nuclear scattering (fuel bundles, containment)

neutron shielding

fission ...

How do we study this?

Abbreviation	Conservation Law	Conserved quantity
CoE	Conservation of energy	kinetic energy, and “other” things
Co $\vec{M}$... CoM_x CoM_y CoM_z	Conservation of momentum ” in the $\hat{x}$-direction ” in the $\hat{y}$-direction ” in the $\hat{z}$-direction	$\vec{p}$ has 3 components... $\vec{p} \cdot \hat{x}$ or p_x $\vec{p} \cdot \hat{y}$ or p_y $\vec{p} \cdot \hat{z}$ or p_z
CoM _{mass} , CoM _a , CoM	Conservation of mass	no change in classical physics

Conservation of Energy implies (Shorthand: $\text{CoE} \Rightarrow$):

$$\frac{1}{2}m_1(v_{1,x}^2 + v_{1,y}^2 + v_{1,z}^2) + \frac{1}{2}m_2(v_{2,x}^2 + v_{2,y}^2 + v_{2,z}^2) + Q = \frac{1}{2}m_a(v_{a,x}^2 + v_{a,y}^2 + v_{a,z}^2) + \frac{1}{2}m_b(v_{b,x}^2 + v_{b,y}^2 + v_{b,z}^2) \quad (1.6)$$

Conservation of Momentum in the x -direction implies (Shorthand: $\text{CoM}_x \Rightarrow$):

$$m_1v_{1,x} + m_2v_{2,x} = m_av_{a,x} + m_bv_{b,x} \quad (1.7)$$

Conservation of Momentum in the other 2 spatial coordinates implies:

$\text{CoM}_y \Rightarrow$:

$$m_1v_{1,y} + m_2v_{2,y} = m_av_{a,y} + m_bv_{b,y} \quad (1.8)$$

$\text{CoM}_z \Rightarrow$:

$$m_1v_{1,z} + m_2v_{2,z} = m_av_{a,z} + m_bv_{b,z} \quad (1.9)$$

Conservation of Mass¹ implies (Shorthand: $\text{CoM}_a \Rightarrow$):

$$m_1 + m_2 = m_a + m_b \quad (1.10)$$

What is Q ?

¹Aside: Later on, when we discuss relativistic kinematics, we shall see that $Q = (m_1 + m_2 - m_a - m_b)c^2$, where c is the speed of light. The mass differences are usually so small that Conservation of Mass is employed without loss of accuracy in a non-relativistic analysis. However, classical analysis does not make use of this directly.

Q , the “reaction” Q -value ...

Describes the type of reaction:

$Q > 0$	“exothermic”, “exoergic”	$K_{\text{outgoing}} > K_{\text{incoming}}$
$Q < 0$	“endothermic”, “endoergic”	$K_{\text{outgoing}} < K_{\text{incoming}}$
$Q = 0$	“elastic”	$K_{\text{outgoing}} = K_{\text{incoming}}$

Examples:

$Q > 0$	“exothermic”, “exoergic”	fission
$Q < 0$	“endothermic”, “endoergic”	two magnets collide and stick
$Q = 0$	“elastic”	billiards (very close to zero)

Complexity!

Eqns: (1.6)–(1.10) have up to 17 unknowns: m_1 , $\vec{v}_1$, m_2 , $\vec{v}_2$, m_a , $\vec{v}_a$, m_b , $\vec{v}_b$, and Q

In more condensed vector notation,

$$\text{CoE} \Rightarrow \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + Q = \frac{m_a v_a^2}{2} + \frac{m_b v_b^2}{2} , \quad (1.11)$$

$$\text{Co}\vec{M} \Rightarrow m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_a \vec{v}_a + m_b \vec{v}_b . \quad (1.12)$$

If we know some of the variables, we can develop interesting relationships between them.

Solution Strategies

Here is an example demonstrating how to eliminate $\vec{v}_b$.

Isolate $\vec{v}_b$ and rearrange (1.12) and (1.11) using $(1.12)^2 - 2m_b \times (1.11)$ to give:

$$(m_1\vec{v}_1 + m_2\vec{v}_2 - m_a\vec{v}_a)^2 - m_b(m_1v_1^2 + m_2v_2^2 - m_av_a^2) - 2m_bQ = 0 , \quad (1.13)$$

v_b has vanished!

Fortunately, many interesting cases are much simpler.

2D elastic collision, equal masses ...

If you play billiards, or know how a Newton's cradle works, that helps! This is a good model for neutron-neutron scattering! It is 3D process. Why is it called 2D? (Discuss)

1. Set up the CoE and CoM equations assuming $\vec{v}_2 = 0$ and $Q = 0$:

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_a^2 + \frac{1}{2}mv_b^2 \quad (1.15)$$

$$m\vec{v}_1 = m\vec{v}_a + m\vec{v}_b \quad (1.16)$$

2. $|\vec{v}_a + \vec{v}_b|^2 = v_1^2$ is formed using $|(1.16)|^2/m^2$
 $v_a^2 + v_b^2 = v_1^2$ is formed using $(1.15)/(m/2)$

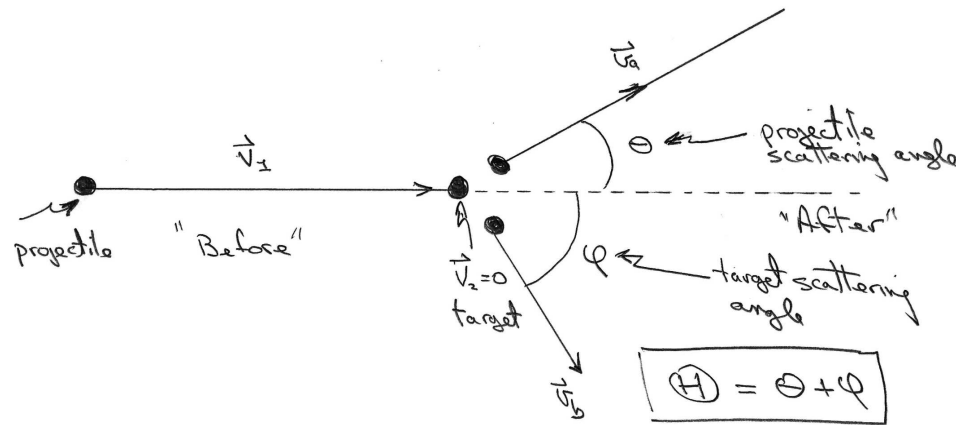
3. Subtract the above two equations to make v_1^2 vanish $\Rightarrow$

$$\begin{aligned} v_1^2 - v_1^2 &= |\vec{v}_a + \vec{v}_b|^2 - (v_a^2 + v_b^2) \\ \cancel{(v_1^2 - v_1^2)}^0 &= |\vec{v}_a|^2 + 2\vec{v}_a \cdot \vec{v}_b + |\vec{v}_b|^2 - (v_a^2 + v_b^2) \\ \cancel{(v_1^2 - v_1^2)}^0 &= (\cancel{|\vec{v}_a|^2 - v_a^2})^0 + 2\vec{v}_a \cdot \vec{v}_b + (\cancel{|\vec{v}_b|^2 - v_b^2})^0 \end{aligned}$$

$\Rightarrow$

$$\vec{v}_a \cdot \vec{v}_b = 0 \text{ !!!!!!!!!!!!! } \odot \quad (1.17)$$

...2D elastic collision, equal masses ...



Interpretation?

Rewrite (1.17), $\vec{v}_a \cdot \vec{v}_b = 0$ as²:

$$|\vec{v}_a| |\vec{v}_b| \cos \Theta = v_a v_b \cos \Theta = 0$$

² There are some important conventions here:

Θ : angle between the projectile and the target particles, after the collision. Sometimes you see θ_{ab} for this.

θ : angle between the projectile's initial and final directions.

φ : angle between the target's final direction and the projectile's final direction. Sometimes ϕ is used for the φ symbol

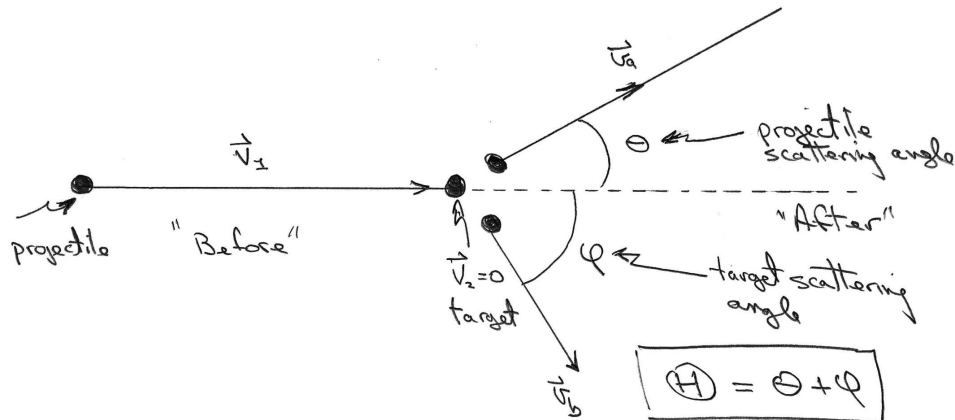
Note that : $\Theta + \varphi = \Theta$.

...2D elastic collision, equal masses ...

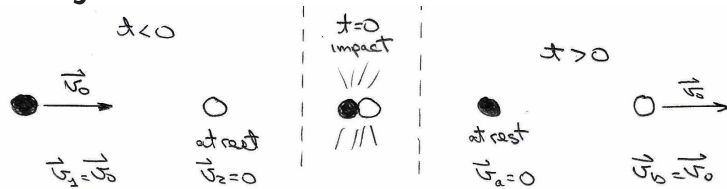
Conclusions, for starting condition: $v_1 = v_0$, $v_2 = 0$ (target at rest).

We enumerate all the possibilities described by the solution $v_a v_b \cos \Theta = 0$.

- If $v_a \neq 0$ and $v_b \neq 0$ then $\cos \Theta = 0 \Rightarrow \Theta = \pi/2$ or 90° .
 θ falls in the range, $0 < \theta < \pi/2$, and φ is anti-correlated with it by $\varphi = \pi/2 - \theta$.



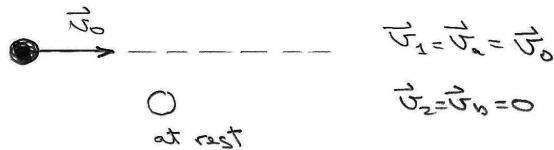
- After the collision, the projectile is at rest, viz. $v_a = 0$. Θ is undefined. Projectile scores a direct hit! You can show that $\vec{v}_a = \vec{v}_0$ in this case.



It goes straight ahead!

...2D elastic collision, equal masses ...

- After the collision, the target is at rest, *viz.* $v_b = 0$. Θ is undefined. Projectile missed the target! You can show that $\vec{v}_a = \vec{v}_0$ in this case.



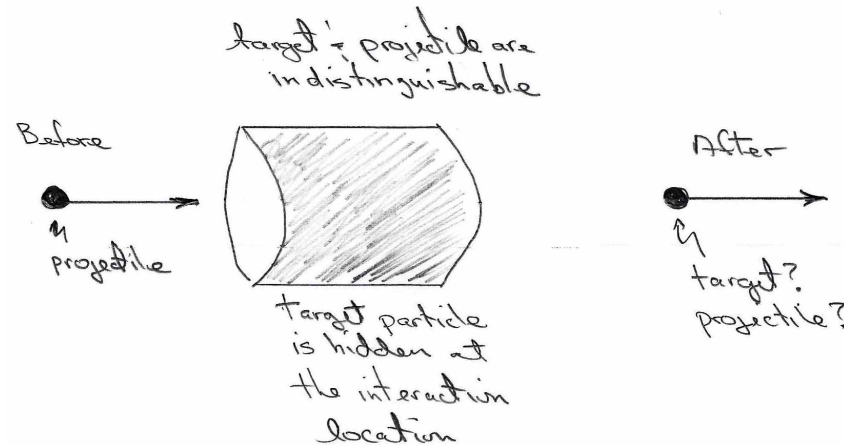
- The projectile and the target are at rest. $v_1 = v_2 = v_a = v_b = 0$. Θ is undefined. Nothing is happening!



All these scenarios are described by the one equation $v_a v_b \cos \Theta = 0$.

...2D elastic collision, equal masses ...

The math is even more general than that!
Imagine that the target object is hidden.



This is always the case when dealing with nuclear radiations.

There are a few things you can say:

1. the projectile came out, or
2. the target particle came out, or
3. the target and projectile exchanged matter, and something looking identical came out!

You can **never** be sure!³

³Oddly enough, if you suggested the third scenario to most people, they would justifiably give you a wide berth! However, in Quantum Mechanics, the answer is more analogous to the third possibility. Particles are also waves. Their wave functions interact with each other constructively and destructively, and participate in the propagation of a composite wavefunction. There is much more about this story that we will talk about later in the course... It gets curiously, and curiously!

...2D elastic collision, equal masses

Kinematic analysis is a powerful tool, and a gross over-simplification!

It does not tell anything about *how* the particles interact.

That requires a discussion of “dynamics” and is much more complicated.

See Wiki: [http://en.wikipedia.org/wiki/Dynamics_\(mechanics\)](http://en.wikipedia.org/wiki/Dynamics_(mechanics))

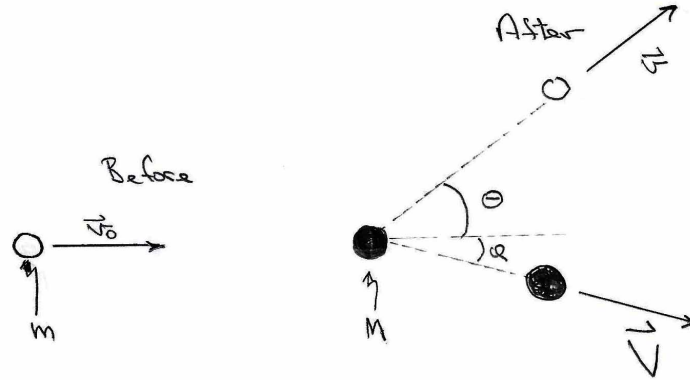
However, the conservation laws still apply, and they allow us to say many things about collisions (sometimes ... surprising things!)

Moreover, these kinematical relations hold for subatomic particles, billiard balls, asteroids, planets and galaxies.

And who knows, maybe even for universes.

Example: Elastic collision, unequal masses ...

This is like the billiard ball collision, but the masses of the target and projectile are different.



With $Q = 0$, CoE and CoM $\Rightarrow$:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}MV^2 \quad ; \quad m\vec{v}_0 = m\vec{v} + M\vec{V} \quad (1.18)$$

We are looking for θ , therefore, we must eliminate V . Reorganize (1.18)

$$v_0^2 - v^2 = nV^2 \quad (1.19)$$

$$\vec{v}_0 - \vec{v} = n\vec{V} \quad (1.20)$$

where $n \equiv M/m$, the ratio of the target's mass to the projectile's mass.

$$n \times (1.19) - |(1.20)|^2 \Rightarrow n(v_0^2 - v^2) - (v_0^2 - 2\vec{v}_0 \cdot \vec{v} + v^2) = 0 \quad (1.21)$$

V is eliminated!

... Elastic collision, unequal masses ...

Reorganizing (1.21) gives a quadratic equation in v :

$$(n + 1)v^2 - 2vv_0 \cos \theta - (n - 1)v_0^2 = 0 \quad (1.22)$$

that is easily solved⁴ to give the relationship between v , and θ , that comes in two forms:

$$\frac{v}{v_0} = \frac{\cos \theta \pm \sqrt{n^2 - 1 + \cos^2 \theta}}{n + 1} \quad (1.23)$$

$$\frac{\cos \theta \pm \sqrt{n^2 - \sin^2 \theta}}{n + 1} \quad (1.24)$$

Case 1: $M > m \Rightarrow n > 1$, target is heavier

In this case, only the $+$ sign in (1.24) applies because v must be ≥ 0 (v is the magnitude of a vector). Hence, the projectile can scatter forward or backward, in fact, any angle! That is $0 \leq \theta \leq \pi$, and therefore, $0 \leq v \leq v_0$.

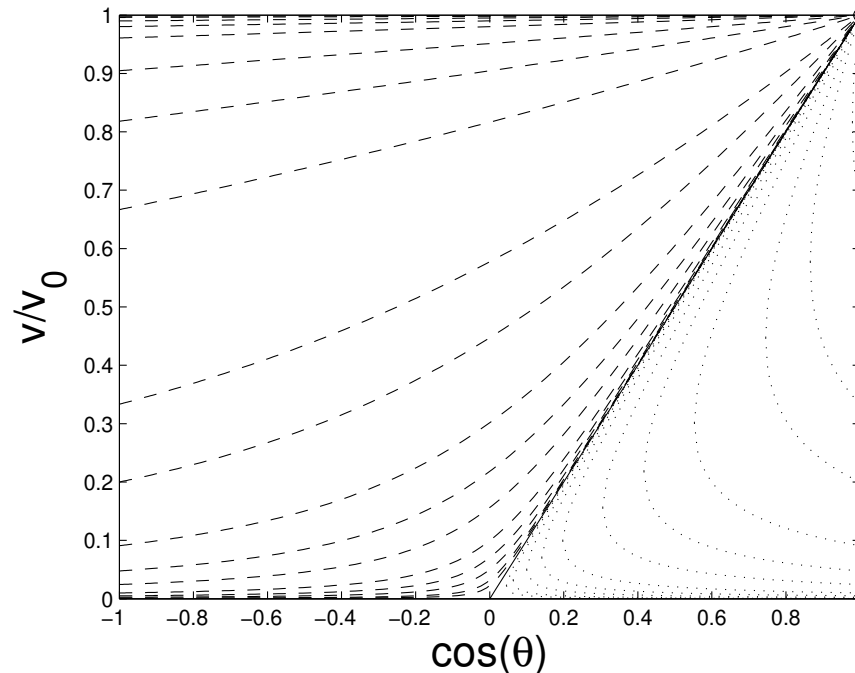
⁴We shall encounter, quite often in this course, quadratic equations of the form $ax^2 + 2bx + c = 0$, that have the solutions: $x = [-b \pm \sqrt{b^2 - ac}]/a$.

... Elastic collision, unequal masses

Case 2: $M < m \Rightarrow n < 1$, target is lighter

In this case both signs apply because the radical $\sqrt{\quad}$ is less in magnitude than $\cos \theta$. There are two consequences:

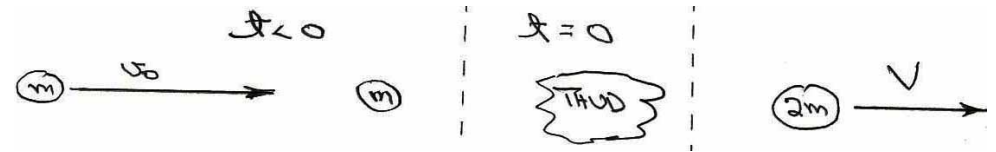
- there is a maximum angle of scatter viz. $\theta_{\max} = \arcsin n$.
- for each θ there are two values of v , associated with the two signs in the solution of the quadratic equation



dashed lines: $M > m$ target heavier, dotted lines: $M < m$ target lighter, solid line: $M = m$, equal masses

Inelastic collisions ...

Imagine the scenario described in the picture below where an incoming particle strikes a particle of equal mass, and sticks to it:



CoE and Co $\vec{M} \Rightarrow$:

$$\frac{1}{2}mv_0^2 = 2 \times \frac{1}{2}mV^2 \quad (1.25)$$

$$mv_0 = 2 \times m\vec{V} \quad (1.26)$$

The $2 \times m$ above accounts for the two particles being stuck together after the collision – equivalent to a single particle of mass $2m$.

Simplifying the above gives:

$$v_0^2 = 2V^2 \quad (1.27)$$

$$\vec{v}_0 = 2\vec{V} \text{ that implies}$$

$$v_0^2 = 4V^2 \quad (1.28)$$

which, of course, can not be true. We have missed something! Include Q !

Inelastic collisions with Q

If we include Q , CoE and Co $\vec{M} \Rightarrow$:

$$\frac{1}{2}mv_0^2 + Q = 2 \times \frac{1}{2}mV^2 \quad (1.29)$$

$$m\vec{v}_0 = 2 \times m\vec{V} \quad (1.30)$$

The $2 \times m$ above accounts for the two particles being stuck together after the collision – equivalent to a single particle of mass $2m$.

Simplifying the above gives:

$$v_0^2 + 2Q/m = 2V^2 \quad (1.31)$$

$$\vec{v}_0 = 2\vec{V} \text{ that implies}$$

$$v_0^2 = 4V^2 \quad (1.32)$$

We can solve the above $2 \times (1.31) = (1.32) \Rightarrow Q = -\frac{1}{4}mv_0^2 = -\frac{1}{2}K_0$

Conclusions:

- $Q < 0$, it's an endothermic reaction
- half the initial kinetic energy goes into the kinetic energy of the outgoing mass
- the remaining half goes into heat, deformation *etc.* of the particles
- in the interest of self-preservation, don't veer for deer

Inelastic collisions, equal masses $Q \geq -K_0/2$...

We know, from the previous problem, that when equal masses are involved, the absolute minimum Q is $-K_0/2$. Now we examine all the other possibilities.

Special case $|\vec{v}_1| = |\vec{v}_2| \equiv |\vec{v}|$

In this case, the outgoing particles (there is no exchange of mass) have equal speed, and hence, the same kinetic energy.

CoE and CoM $\Rightarrow$:

$$\frac{1}{2}mv_0^2 + Q = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \quad (1.33)$$

$$m\vec{v}_0 = m\vec{v}_1 + m\vec{v}_2 \quad (1.34)$$

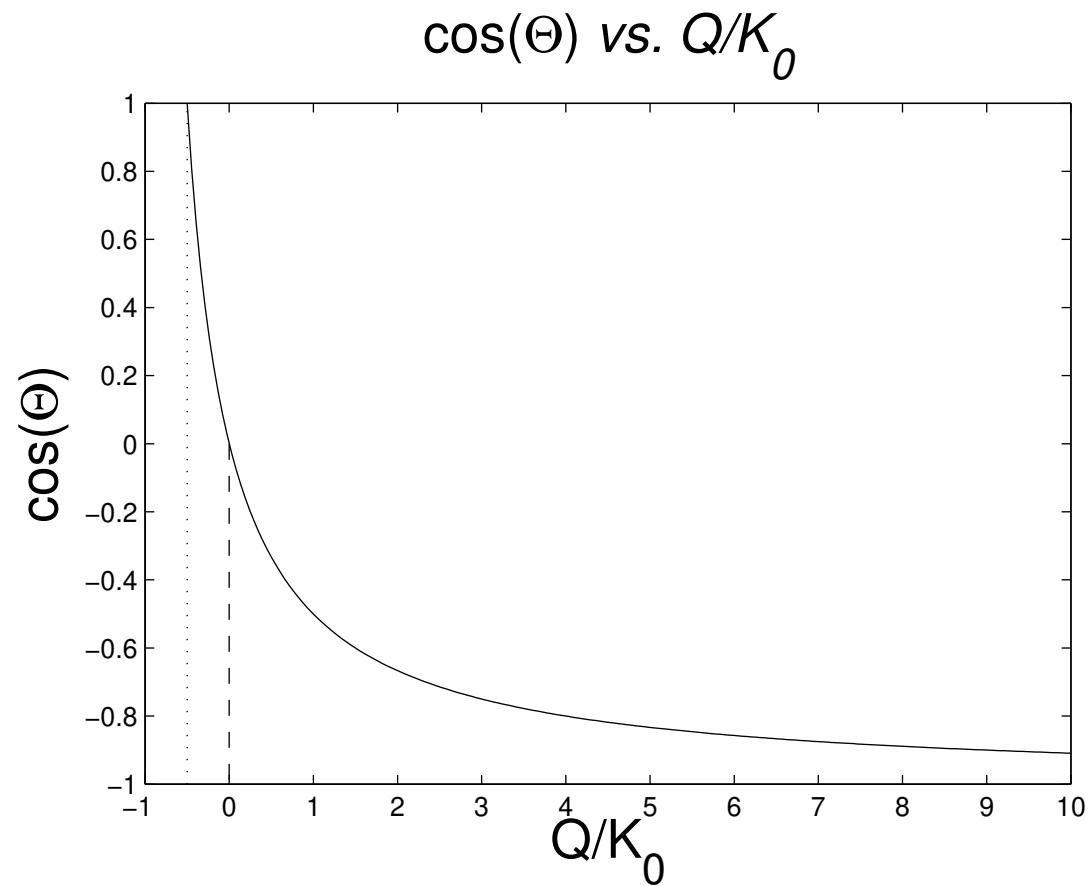
You can show, in this case

$$\cos \Theta = \frac{-Q}{K_0 + Q} \quad (1.35)$$

Does it make sense? As $Q \longrightarrow -K_0/2$, $\cos \Theta \longrightarrow 1$ (Previous case).

As $Q \longrightarrow \infty$, $\cos \Theta \longrightarrow ?$ Explain!

... Inelastic collisions, equal masses $Q \geq -K_0/2$...



Zero-Momentum Frame

Sometimes, it is advantageous to work in an inertial frame where there is no net momentum. The velocity of this frame is determined as follows:

We seek a velocity $\vec{V}$ that, when subtracted from the velocities of the initial particles, has zero resultant momentum. Thus, referring to the left hand side (LHS) of (1.11) we want:

$$m_1(\vec{v}_1 - \vec{V}) + m_2(\vec{v}_2 - \vec{V}) = 0 , \quad (1.36)$$

for which the solution is:

$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} . \quad (1.37)$$

Perform the calculation extracting the kinematic relations, in this frame. That is, $\vec{v}_1 \rightarrow \vec{v}_1 - \vec{V}$, and so on. And then, once the desired result is obtained, transform the velocity vectors in the opposite direction.

Exercise: Repeat the case of a 2D elastic collision with equal masses in the zero-momentum frame, transform back to the rest frame of the target particle, and show the same result that was discussed before.

Particle-at-Rest Frame ...

At other times, it is advantageous to work in a frame where one of the particles in the initial state is at rest (the stationary target), since in fact, this is usually the case. But, had we started from the situation described in (1.11) and (1.12) we may transform to a reference frame where the target (m_2 , say) is at rest by subtracting $\vec{v}_2$ from all velocities in the problem, then rotating the new $\vec{v}_1$ into the z -direction, and the new $\vec{v}_a$ and $\vec{v}_b$ into the xz -plane, so it can be visualized on the blackboard, or paper. After the required kinematical quantities are obtained, reverse the rotations⁵, and reverse the velocity transformation and you're done.

This situation (target-at-rest) is very important since it often represents the starting physical condition.

Let us illustrate what happens to (1.13) in this circumstance.

Setting $\vec{v}_2 = 0$ in (1.13) gives us:

$$m_a^2 v_a^2 - 2m_1 m_a v_1 v_a \cos \theta_a + m_1^2 v_1^2 - m_b(m_1 v_1^2 - m_a v_a^2) - 2m_b Q = 0 , \quad (1.38)$$

where θ_a is the angle of m_a wrt (“wrt” is shorthand for “with respect to”) the z -axis, in spherical-polar coordinates.

⁵Rotations do not commute, so be careful of the order!

... Particle-at-Rest Frame ...

Or, more suggestively,

$$[m_a(m_a + m_b)]v_a^2 + [-2m_1m_av_1\cos\theta_a]v_a + [m_1(m_1 - m_b)v_1^2 - 2m_bQ] = 0 . \quad (1.39)$$

This can be regarded as a solution of θ_a in terms of v_a , or a quadratic formula of the form

$$Ax^2 + Bx + C = 0 ,$$

for which the solution of v_a in terms of θ_a is well known. (A, B, C are constants, and we seek the values of x for which the above is true.) We shall leave further exploration of this quadratic solution to the examples and problems. However, the quadratic equation has a rich variety of solutions. Sometimes, there are no solutions (kinematically restricted) or one or two solutions. We shall revisit this topic later in the course.

... Particle-at-Rest Frame ...

Illustration

Consider a collision between two identical masses, traveling with equal but opposite velocities along the x -direction, that produce two masses, m_a and m_b .

By applying the conservation laws we just discussed, we can show that the resultant velocities are:

$$v_b = \pm \sqrt{\frac{m_a}{m_b}} v ; \quad v_a = \mp \sqrt{\frac{m_b}{m_a}} v ,$$

where the final velocities are oriented in directions opposite.

This makes sense! There is no net momentum before the collision, so you expect the lighter particle to be moving faster and in the opposite direction to the heavier particle. It is also easy to verify that the initial and final kinetic energies are the same.

Potential, Kinetic, Total Energy, Angular Momentum ...

A particle (Let's think of this as a small object.) moving in one dimension can be subjected to a force $F(x)$ that causes it to move. Associated with this force is a potential, $U(x)$. External forces and potentials are related by:

$$F(x) = -\frac{dU(x)}{dx} . \quad (1.40)$$

In three dimensions, we have $F(\vec{x})$ and $U(\vec{x})$, that are similarly related:

$$F(\vec{x}) = -\vec{\nabla}U(\vec{x}) . \quad (1.41)$$

The total energy, E , of a particle subjected to an external force is, in one dimension, a constant:

$$E = K(x) + U(x) . \quad (1.42)$$

...Potential, Kinetic, Total Energy, Angular Momentum ...

In three dimensions, a system of N particles has its total energy conserved as well, and may be written:

$$E = \sum_{i=1}^N [K(\vec{x}_i) + U(\vec{x}_i)] . \quad (1.43)$$

Angular momentum $\vec{L}$ for a single particle is also conserved. It is expressed in terms of the cross product of the position and momentum as:

$$\vec{L} = \vec{x} \times \vec{p} . \quad (1.44)$$

In three dimensions, a system of N particles has its total angular momentum conserved as well, and may be written:

$$\vec{L} = \sum_{i=1}^N \vec{x}_i \times \vec{p}_i . \quad (1.45)$$

1.6. Significant Figures ...

Physical things are measured two ways. There are integral quantities, *e.g.* 3 electrons, 4 photons, *etc.* It doesn't make sense to say "A tritium nucleus is made up of 1.0 protons and 2.0 neutrons.", because 1.0 and 2.0 are not integers, and there should be no ambiguity in the number of discrete particles. The correct statement would be, "A tritium nucleus is made up of 1 proton and 2 neutrons."

Other quantities are not discrete. There *is* ambiguity in the statement, "The α -particle has an energy of 5 MeV.", because there is no statement as to how good the "5" is. The best we can do, is to give to a precision, the precision of that participant, with the least precision. That's not the best situation, but it is what we have to live with, without more information.

For example, consider:

$$r = a \blacksquare b ,$$

where $\blacksquare$ represents one of the operators, $\times$, $\div$, $+$, or $-$. Define a "precision operator", $\mathcal{P}()$, that returns the precision of its argument. For example, $\mathcal{P}(1.37) = \mathcal{P}(0.000137) = 3$. Then,

$$\mathcal{P}(r) = \min \mathcal{P}(a), \mathcal{P}(b) .$$

... Significant Figures

If you come across a statement like, "The α -particle has an energy of 5 MeV.", it is likely an error, because no educator wishes to be ambiguous. If he or she does, and it's on purpose, then the question was likely designed to make you think. It is valid to say, "The α -particle has an energy of 5 MeV (exactly).", in which case, you are expected to treat the "5" as an exact real number, with infinite precision.

In some situations, you will be told, "The α -particle has an energy of 5.02 ± 0.13 MeV.", in which case you will be expected to estimate the error on the final quantity, if you use it in a computation. This is discussed later in this chapter.

†1.8. Basic Error Estimation ...

Accounting for Estimated Error for Independent Quantities

Suppose M is a measured quantity, the result of some calculation or experiment, that depends on two other quantities, a and b , that each have their own estimated uncertainties σ_a and σ_b . Assuming that the relative errors in a and b are small (*i.e.* we propagate first-order errors only), and a and b are independent (the result of two independent measurements), then the estimated error in M , namely σ_M , is determined as follows:

$$\sigma_M = \sqrt{\left(\frac{\partial M(a, b)}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial M(a, b)}{\partial b}\right)^2 \sigma_b^2}, \quad (1.46)$$

and the fractional error is:

$$\frac{\sigma_M}{M} = \frac{1}{M(a, b)} \sqrt{\left(\frac{\partial M(a, b)}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial M(a, b)}{\partial b}\right)^2 \sigma_b^2}. \quad (1.47)$$

The extension to measurements that have three or more independent inputs, is straightforward.

... Basic Error Estimation ...

Example 1

Question: After a collision reaction involving two protons, the two protons have kinetic energies 2.8 ± 0.6 and 10.0 ± 1.0 MeV. What is the total combined resultant kinetic energy and its estimated error?

Answer: The total energy is

$$K = K_1 + K_2 .$$

Therefore, by the rule stated above,

$$\sigma_K = \sqrt{\sigma_{K_1}^2 + \sigma_{K_2}^2} .$$

Therefore,

$$K = 12.8 \pm 1.2 \text{ MeV} .$$

Food for thought: How should you state the result if the energy of the second proton was 10 ± 1 MeV?

... Basic Error Estimation ...

Example 2

Question: $M(a, b, c) = ab^2c^3$. What is the relative estimated error in M ?

Answer: You can show that

$$\frac{\sigma_M}{M} = \sqrt{\left(\frac{\sigma_a}{a}\right)^2 + 4\left(\frac{\sigma_b}{b}\right)^2 + 9\left(\frac{\sigma_c}{c}\right)^2}.$$

Note: Don't expect this very neat expression for all calculations of relative error. It is really a fortuitous result of the given form of M .