

electron - e^- negative (aka shadow family)
 positron - e^+ positive

β -decay

note: usually "electron" stands for e^-
 it can also stand for an e^+

β -particles are electrons or positrons that are emitted through nuclear decay.

"Discoverer" Henri Becquerel (photographic plates wrapped in black paper were exposed when placed near a β -emitter)

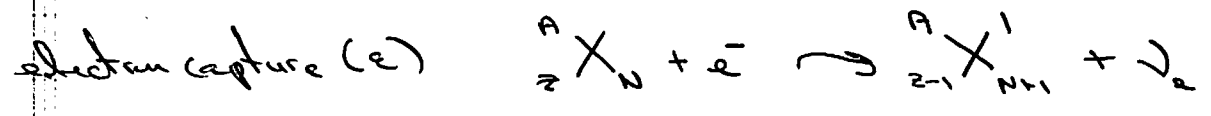
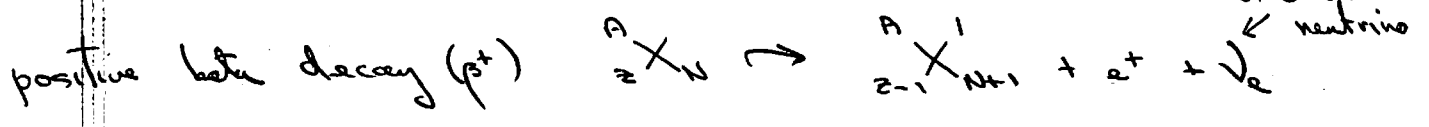
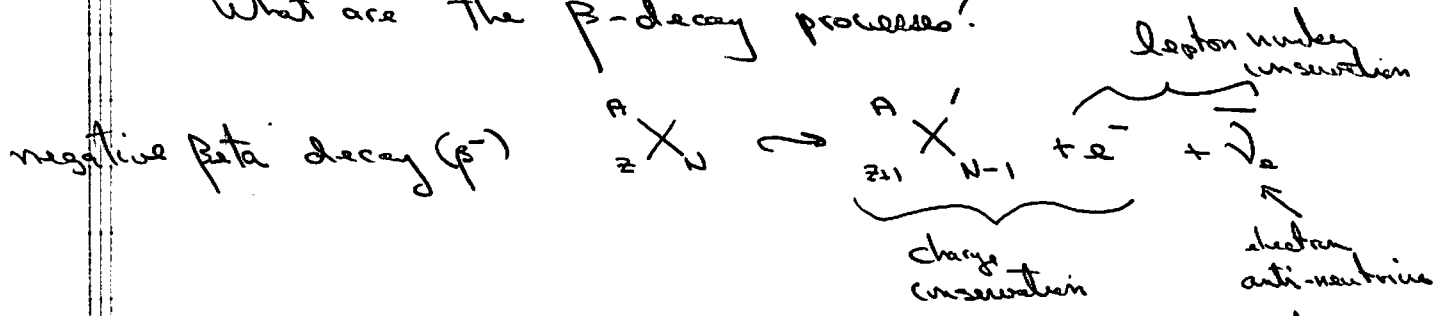
Note: This could not be caused by α 's, as they would be completely stopped by the paper

Rutherford (1897) α -particles

β -particles - about 100 times more penetrating than α -particles

electrons are spin- $1/2$ fermions

What are the β -decay processes?



The neutrino is also a spin $1/2$ fermion, but has no electric charge. It does carry a lepton quantum number

How the particles in nature are organized

Bosons ($s=0, 1, 2, \dots$)

(e.g. photons ($s=1$))

Fermions ($s=\frac{1}{2}, \frac{3}{2}, \dots$)

Baryons

(strong force)

e.g. $n, p, \bar{n}, \bar{p}$

$\bar{x} \Rightarrow \text{anti "x"}$

Leptons

(weak force)

e-family

$e^+, e^-, \nu_e, \bar{\nu}_e$

μ -family

$\mu^+, \mu^-, \nu_\mu, \bar{\nu}_\mu$

τ -family

$\tau^+, \tau^-, \nu_\tau, \bar{\nu}_\tau$

ν_e is practically massless (but proven to be non-zero)

$$0.04 \text{ eV} < m_{\nu_e} < 0.3 \text{ eV} \text{ (inferred)} \\ < 2.2 \text{ eV} \text{ (direct measurement)}$$

$$m_{\nu_\mu} < 170 \text{ keV}$$

$$m_{\nu_\tau} < 15.5 \text{ MeV}$$

β -decay at the nucleons level

$$(\beta^-) \quad n \rightarrow p + e^- + \bar{\nu}_e \quad \left(\frac{1}{\lambda} = 905.7(8)s \right)$$

$\approx 15m$

$$\lambda_{1/2} \approx 10m$$

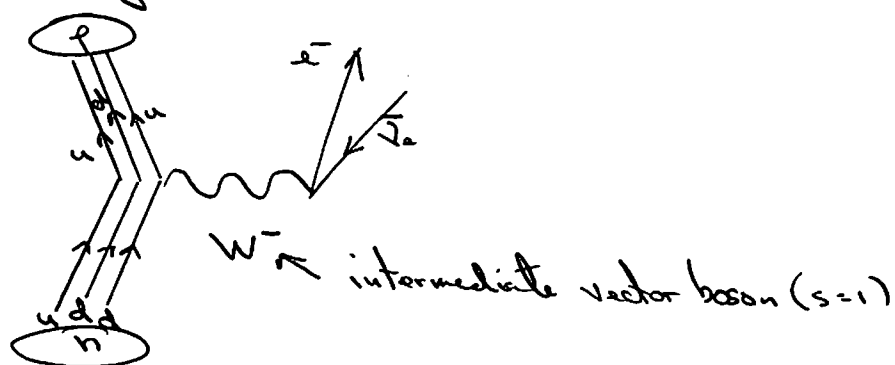
note: bound neutrons do not decay unless energetics favor it.

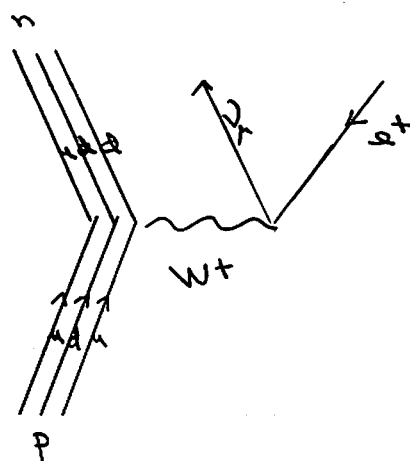
$$(\beta^+) \quad p \rightarrow n + e^+ + \nu_e$$

$$(\epsilon) \quad p + e^- \rightarrow n + \nu_e$$

↑
usually a K-shell electron (why?)

β -decay at the quark level



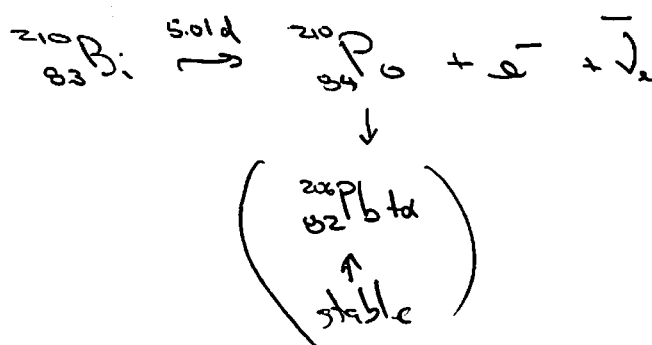


$$m_W = 80.4 \frac{\text{GeV}}{c^2}$$

The W 's and its neutral counterpart, the Z particle are the carriers of the "Weak" force just as gluons are the carriers of the strong force and photons " " " " " em force

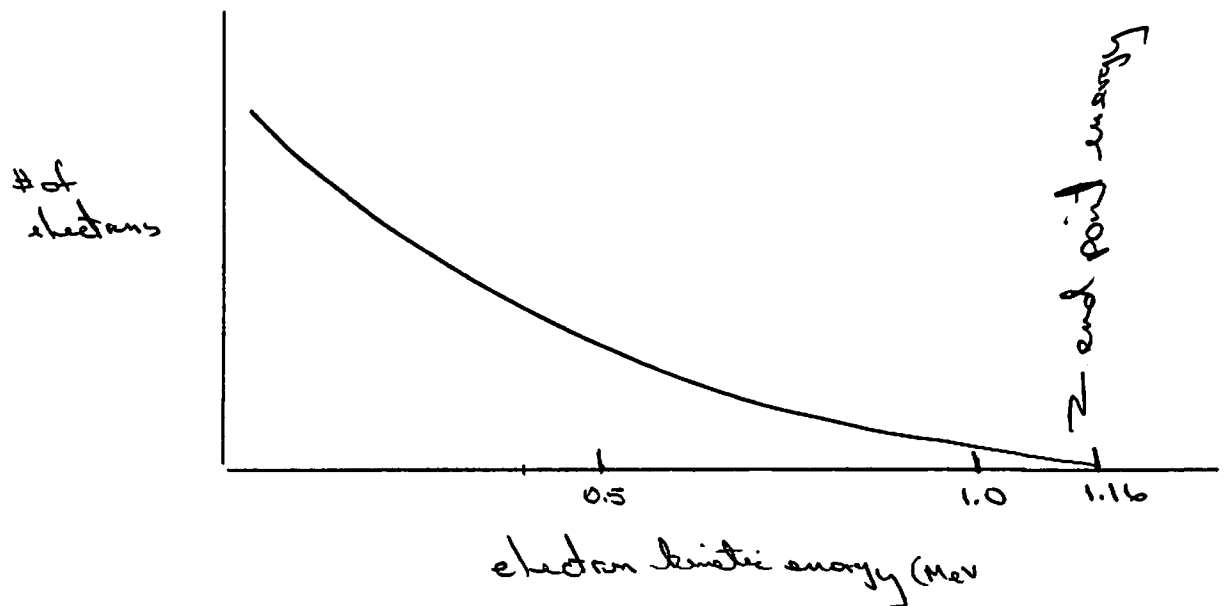
Consequences of a β -decay's 3 body final state

e.g. β decay of $^{210}_{83}\text{Bi}$



The nuclei involved are very heavy compared to the e^- & $\bar{\nu}_e$, hence, the e^- pair carry off most of the energy.

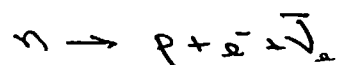
Consequently, the e^- does not have a unique decay energy, because it shares that energy with the neutrino.



The "end-point energy", (1.16 MeV) in this case, occurs when the electron has its maximum energy and $T_\nu = 0$. This allows an experimental determination of the Q -value of the reaction (accounting for nuclear recoil)

9.1 energy release in β -decay

e.g. n -decay



$$Q = (m_n - m_p - m_e - m_\nu)c^2$$

If the neutron decayed from a state of rest

$$Q = T_p + T_e + T_\nu$$

$$Q = (m_n - m_p - m_e - m_\nu)c^2$$

$$= 939.565580(81) - 938.272013(23) - 0.5109989(6) - m_\nu c^2 \text{ [MeV]}$$

$$= 0.782568(84) \text{ MeV} - m_\nu c^2$$

can be ignored

$$4 \times 10^{-8} \rightarrow 3 \times 10^{-7} \text{ MeV}$$

$$Q = 0.782568(84) \text{ MeV}$$

T_e ?

$$T_e + T_p = Q$$

(at end-point)

(electron kinematics must be treated relativistically)

Accounting for recoil

must do a relativistic calculation

$$T_e = E_e - m_e c^2$$

$$\begin{aligned} p_e &= p_p \\ (E_e)^2 &= p_e^2 c^2 \\ E_e^2 - (m_e c^2)^2 &= 2m_p c^2 T_p \\ (E_e + m_e c^2)(E_e - m_e c^2) &= 2m_p c^2 T_p \\ (T_e + 2m_e c^2)T_e &= 2m_p c^2 T_p \quad (1) \\ T_e^2 + 2m_e c^2 T_e &= 2m_p c^2 (Q - T_e) \end{aligned}$$

$$T_e^2 - 2T_e(m_p + m_e)c^2 - 2m_e^2 Q = 0$$

$$T_e = (m_p + m_e)c^2 \pm \sqrt{[(m_p + m_e)c^2]^2 - 2m_e^2 Q}$$

↑
minus sign

$$T_e = (m_p + m_e)c^2 \left[1 - \sqrt{1 - \frac{2Q}{[(m_p + m_e)c^2]^2}} \right]$$

$$\approx Q \left[\frac{m_e}{m_p + m_e} \right]$$

$$T_e \approx Q \left(\frac{1}{1 + m_e/m_p} \right)$$

This case

$$T_e = 0.782142(84) \text{ MeV}$$

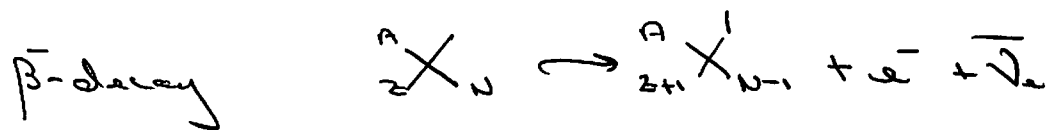
The measured max $T_e = 0.782(13) \text{ MeV}$

$$\therefore T_p = Q - T_e$$

$$= Q \left[1 - \frac{1}{1 + \frac{m_e}{m_p}} \right]$$

$$\approx \frac{m_e}{m_p} Q = 0.000426(84)$$

Generally speaking



$$Q_{\beta^-} = \left[\underset{\substack{\uparrow \\ \text{nucleus} \\ \text{mass}}}{m_N}({}^A_Z X_N) - \underset{\substack{\uparrow \\ \text{nucleus} \\ \text{mass}}}{m_N}({}^A_{Z+1} X'_{N-1}) - m_e \right] c^2$$

(ignore the neutrino)

$$\underset{\substack{\uparrow \\ \text{atomic mass}}}{m}({}^A_Z X) c^2 = m_N({}^A_Z X) c^2 + \underbrace{Z m_e c^2}_{\substack{Z \times m_e c^2 \\ \text{rest mass} \\ \text{energy}}} - \underbrace{\sum_{i=1}^Z B_i}_{\substack{\text{Binding} \\ \text{energy of} \\ \text{all the} \\ \text{electrons}}}$$

$$Q_{\beta^-} = \left\{ \left[m({}^A_Z X) - Z m_e \right] - \left[m({}^A_{Z+1} X') - (Z+1) m_e \right] - m_e \right\} c^2 + \left(\sum_{i=1}^Z B_i - \sum_{i=1}^{Z+1} B'_i \right)$$

all electron masses cancel

(160)

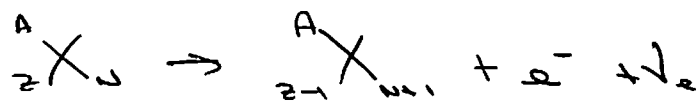
Thus,

$$Q_{\beta^-} = \{m({}_Z^AX) - m({}_{Z+1}^AX')\}c^2 \\ + \left(\sum_{i=1}^Z (B_i - B'_i) - B'_{Z+1} \right)$$

If we can ignore the B terms

$$Q_{\beta^-} \approx (m({}_Z^AX) - m({}_{Z+1}^AX'))c^2$$

notes: For large A, the B terms can add up to as much as about 10 keV.
So, for accurate work, it can not be ignored

 β^+ -decay

$$Q_{\beta^+} = (m({}_Z^AX) - m({}_{Z-1}^AX) + 2m_e)c^2 \\ + \left\{ \sum_{i=1}^Z B_i - \sum_{i=1}^{Z-1} B'_i \right\} \quad \text{same discussion}$$

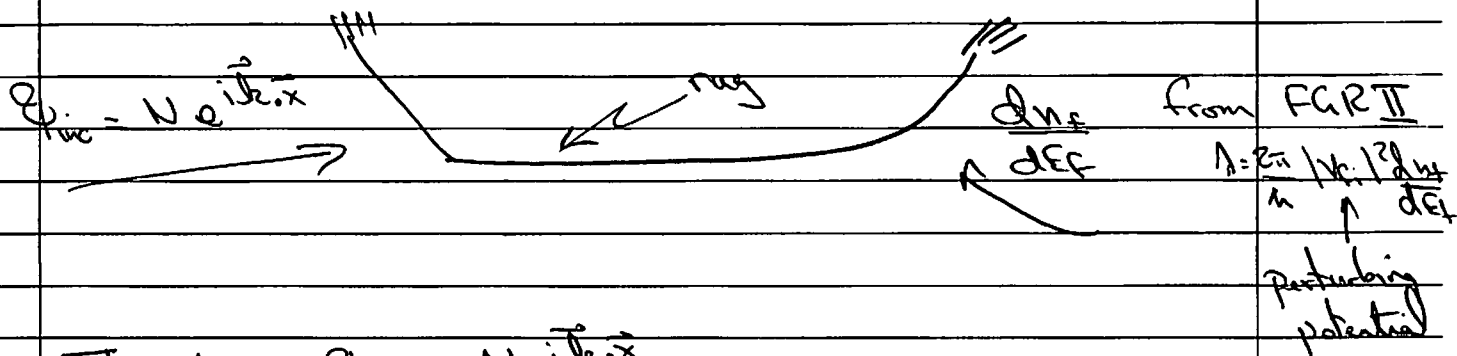
(161)

Electron Capture

BE of
the captured
electron
cannot ignore ↓

$$Q_E = \{m({}_Z^AX_N) - m({}_{Z-1}^{A'}X'_N)\}c^2 - B_N \\ + \left(\sum_{i=1}^Z B_i - \sum_{i=1}^{Z-1} B'_i \right) \left. \vphantom{\sum_{i=1}^Z B_i} \right\} \text{Kraus ignores}$$

Two details about QM have been swept under the rug. Today, it's time to do some spring cleaning and clear them up.



Thing 1: $\psi_{inc} = N e^{i\vec{k} \cdot \vec{x}}$

So far we have not normalized the incident (or outgoing) wave. Today we do!

We will put the universe in a square box, L on a side, and take $L \rightarrow \infty$ at the end without deleterious effects.

Thing 2: FGR II $\lambda = \frac{2\pi}{h} |\langle \psi_f | V | \psi_i \rangle|^2 \frac{dn_f}{dE_f}$
 transition rate $\left[\frac{1}{T} \right]$

$\frac{dn_f}{dE_f}$ is the number of "exit doors" available for the energy end-point E_f of the system.

We will do $\frac{dn_f}{dE_f}$ first and then ψ_{inc} later

within the context of Fermi's Theory of β -decay. But the discussion is much more general than just β -decay

Fermi's Theory of β -decay

The transition rate is given by

$$\lambda = \frac{2\pi}{\hbar} |V_{fi}|^2 \frac{dN_{Ef}}{dE_f} \quad (1)$$

density
of final
states

note: this calculation is valid for
relativistic & non-relativistic
systems.

Density of final states

Given a final energy of the system, how many quantum states fall within E_f & $E_f + dE_f$. (The decay rate is proportional to the number of ways it can decay)

Example 1

$\frac{dN_e}{dE_e}$ for an α -particle following
and α -decay

This is how it's done

Construct a cubical box, L on a side
 $V=0$ inside $V=\infty$ outside, L defines
the size of our "universe". At the end
we take $L \rightarrow \infty$

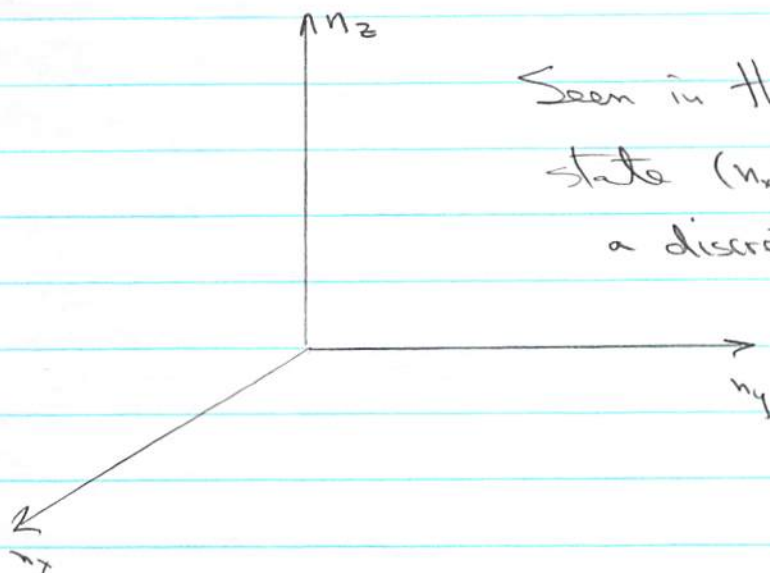
$$\textcircled{2} \quad \psi_{n_x, n_y, n_z}(x, y, z) = \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right)$$

(VERS 311 stuff)

$$\textcircled{3} \quad p_x = \frac{n_x \pi \hbar}{L}, \quad p_y = \frac{n_y \pi \hbar}{L}, \quad p_z = \frac{n_z \pi \hbar}{L}$$

$$\textcircled{4} \quad E = \frac{\hbar^2 \pi^2}{2mL^2} (n_x^2 + n_y^2 + n_z^2) \quad \left. \vphantom{\frac{\hbar^2 \pi^2}{2mL^2}} \right\} \text{non-relativistic}$$

$$\left. \begin{array}{l} 1 \leq n_x < \infty \\ 1 \leq n_y < \infty \\ 1 \leq n_z < \infty \end{array} \right\} n_i \text{ are integral}$$



Seen in this way, a quantum state (n_x, n_y, n_z) occupies a discrete "lattice" point in this octant

However L is big!

So, we'll treat the n_i as continuous

Let $(n_x, n_y, n_z) = \vec{n}$, a vector

How many states from $E \rightarrow E + dE$?

$$dN = \frac{1}{8} \frac{4\pi}{3} \left\{ (n+dn)^3 - n^3 \right\}$$

$n_i > 0$

just one

octant

$$= \frac{1}{8} \frac{4\pi}{3} \left\{ \cancel{n^3} + 3n^2 dn + 3n dn^2 + \cancel{dn^3} - \cancel{n^3} \right\}$$

$$\xrightarrow{dn \rightarrow 0} \frac{\pi n^2 dn}{2}$$

$$dN = \frac{\pi}{2} n^2 dn \quad (5)$$

$$E = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \quad \text{vector, remember}$$

$$\Rightarrow n^2 = \frac{2mL^2 E}{\hbar^2 \pi^2} \quad (6)$$

$$n = \frac{(2m)^{1/2} L \sqrt{E}}{\hbar \pi}$$

$$dn = \frac{1}{2} \frac{(2m)^{1/2}}{\hbar \pi} \frac{1}{\sqrt{E}} \quad (7)$$

(6) & (7) $\rightarrow$ (5) $\Rightarrow$

$$dN = \frac{\pi}{2} \frac{2mL^2 E}{\hbar^2 \pi^2} \frac{1}{2} \frac{(2m)^{1/2}}{\hbar \pi} \frac{1}{\sqrt{E}}$$

Gathering terms

$$dN = \frac{(2m)^{3/2}}{4\pi^2} \frac{L^3}{\hbar^3} E^{1/2} dE$$

$$\boxed{\frac{dN}{dE} = \frac{(2m)^{3/2}}{4\pi^2} \frac{L^3}{\hbar^3} E^{1/2}} \quad (8)$$

So,

$$\lambda_d = \left(\frac{2\pi}{h} \right) |V_{ci}|^2 \frac{(2m)^{3/2} E_d^{1/2}}{4\pi h^3} \quad (9)$$

Now look at $|V_{ci}|^2$

$$|V_{ci}|^2 = | \langle \psi_f | V_p | \psi_i \rangle |^2 \quad (10)$$

$\uparrow$ final $\uparrow$ initial
 states of the system

 $\psi_i = \psi_x$ ground state of the system

$$\psi_f = \psi_{x'} \frac{1}{L^{3/2}} e^{i\vec{k}_x \cdot \vec{x}}$$

$\uparrow$ daughter nucleus plane wave in 3D

properly normalized in the box

Note, this is the correct normalization since

$$\begin{aligned} \langle \psi_x | \psi_x \rangle &= \frac{1}{L^3} \int d\vec{x} e^{-i\vec{k}_x \cdot \vec{x}} e^{i\vec{k}_x \cdot \vec{x}} \\ &= \frac{1}{L^3} L^3 = 1 \quad \checkmark \end{aligned}$$

So, let

$$|V_{fi}|^2 = |\langle \psi_f | V_P | \psi_i \rangle|^2$$

$$|V_{fi}|^2 = \frac{1}{L^3} \underbrace{|\langle \psi_f | e^{i\vec{k} \cdot \vec{x}} | V_P | \psi_i \rangle|^2}_{\text{call this } |M_{fi}|^2} \quad (11)$$

So (11) $\Rightarrow$ (9)

$$\lambda_d = \left(\frac{2\pi}{h} \right) \frac{1}{L^3} |M_{fi}|^2 \frac{(2m)^{1/2}}{4\pi^2 h^3} L^3 E_d^{1/2}$$

or, gathering terms L^3 's cancel

$$\lambda_d = |M_{fi}|^2 \frac{(2m)^{1/2}}{2\pi h^3} E_d^{1/2}$$

or, if you like

$$\boxed{\lambda_d = \frac{\sqrt{2}}{\pi} e \frac{(mc^2)^{3/2}}{(h c)^4} |M_{fi}|^2 E_d^{1/2}} \quad (12)$$

Leaving off point for a more refined calculation of α -decay.

Example 2 Gamma Decay

G₀ back to (5)

$$dN = \frac{\pi n^2 dn}{2} \quad (5)$$

$$\vec{p} = \frac{\pi \hbar n}{L}$$

$$p^2 = \frac{\pi^2 \hbar^2 n^2}{L^2}$$

$$n^2 = \frac{p^2 L^2}{\hbar^2 \pi^2} \quad (13)$$

$$n = \frac{pL}{\hbar \pi}$$

$$dn = \frac{L}{\hbar \pi} dp \quad (14)$$

$$(13) \div (14) \rightarrow (5)$$

$$dN = \frac{\pi}{2} \frac{p^2 L^2}{\hbar^2 \pi^2} \frac{L}{\hbar \pi} dp$$

$$dN = \frac{1}{2\pi^2} \frac{L^3}{\hbar^3} p^2 dp \quad (15)$$

(fully relativistic)

For photons $p = \frac{E_\gamma}{c}$

$$\left| \frac{dN}{dE_\gamma} = \frac{1}{2\pi^2} \frac{L^3}{(\hbar c)^3} E_\gamma^2 \right| \quad (16)$$

Appropos for

single γ
emission

(17)

$$\lambda_x = \frac{2\pi}{h} |V_{fi}|^2 \frac{1}{2\pi^2} \frac{L^3}{(hc)^3} E_x^2$$

$$\text{Let } |V_{fi}|^2 = \frac{1}{L^3} \underbrace{\left| \langle \psi_{x'} | e^{i\vec{k}\cdot\vec{r}} | \psi_p \rangle \right|^2}_{|M_{fi}|^2}$$

$$\lambda_x = \frac{c}{11} \frac{|M_{fi}|^2 E_x^2}{(hc)^4}$$

(17)

Particles that have mass and are relativistic

Go back to (15)

$$dN = \frac{1}{2\pi^2} \frac{L^3}{h^3} p^2 dp \quad (15)$$

$$\text{or } dN = \frac{1}{2\pi^2} \frac{L^3}{(hc)^3} (cp)^2 d(cp) \quad (18)$$

$$\begin{aligned} \text{Recall } (cp)^2 &= E^2 - (mc^2)^2 = (E+mc^2)(E-mc^2) \\ &= (T+2mc^2)T \end{aligned} \quad (19)$$

$$\frac{d}{dT} (19) \Rightarrow 2(cp) d(cp) = T + 2mc^2 + T = 2(T+mc^2)$$

$$\text{So } (cp) d(cp) = (T+mc^2) dT \quad (20)$$

$$(20) + \sqrt{19} \rightarrow (18) \Rightarrow$$

$$dN = \frac{1}{2\pi^2} \frac{L^3}{(\hbar c)^3} \sqrt{T(T+2mc^2)} (T+mc^2) dT$$

$$(21) \quad \left| \frac{dN}{dT} = \frac{1}{2\pi^2} \frac{L^3}{(\hbar c)^3} \sqrt{T(T+2mc^2)} (T+mc^2) \right|$$

We must use this expression in β -decay

β -decay

β -decay has 2 free particles in the final state, the relativistic electron and the (nearly) massless neutrino.

What we start with is

$$\lambda_\beta = \frac{2\pi}{\hbar} |V_{fi}|^2 \frac{dn_{ef}}{dE_f} \quad (22)$$

What we wish to obtain is the electron spectrum. We do this as follows

$$d\lambda = \frac{2\pi}{\hbar^2} |V_{fi}|^2 dN_e \frac{dN_\nu}{dE_f} \quad (23)$$

independent densities
but coupled through
Conservation of
Energy

$$E_f = T_\beta + E_\nu \quad (24)$$

Let p be the electron momentum
 q " " neutrino "

From (15)

$$dN_e = \frac{1}{2\pi^2} \frac{L^3}{\hbar^3} p^2 dp \quad (25)$$

$$dN_v = \frac{1}{2\pi^2} \frac{L^3}{\hbar^3} q^2 dq \quad (26)$$

$$E_v = cq, \text{ so}$$

$$E_s = T_p + E_v = T_p + cq$$

$$q = \frac{E_s - T_p}{c} \quad (27)$$

$$dE_s = d(T_p + cq) = cdq \quad (28)$$

with T_p held constant

$$(28), (27), (26), (25) \Rightarrow (23) \Rightarrow$$

$$(29) \quad d\Omega = \frac{2\pi}{\hbar} |Mc_i|^2 \underbrace{\left(\frac{1}{2\pi^2} \frac{L^3}{\hbar^3} p^2 dp \right)}_{N_e} \underbrace{\left(\frac{1}{2\pi^2} \frac{L^3}{\hbar^3} \left(\frac{E_s - T_p}{c} \right)^2 \right)}_{N_v} \frac{1}{c} dE_s$$

$$* \frac{1}{dE_s} \delta(E_s - (T_p + E_v))$$

$$\frac{d\lambda}{dp} = \frac{|M_{fi}|^2}{2\pi^3 \hbar^7 c^3} p^2 (E_f - T_p)^2 \delta(E_f - (T_p + E_\nu))$$

Integrate over E_ν to get

$$\frac{d\lambda}{dp} = \frac{|M_{fi}|^2}{2\pi^3 \hbar^7 c^3} p^2 (E_f - T_p)^2$$

Generally nuclear recoil is ignored.

Thus we can set $E_f = Q$

$$\text{So } \frac{d\lambda}{dp} = \frac{|M_{fi}|^2}{2\pi^3 \hbar^7 c^3} p^2 (Q - T_p)^2 \quad (30)$$

↑
note the endpoint energy.

β -decay

β -decay has 2 free particles in the final state, the relativistic electron and the (nearly) massless neutrino.

What we start with is

$$\lambda_{\beta} = \frac{2\pi}{\hbar} |V_{fi}|^2 \frac{dn_e}{dE_e} \quad (22)$$

What we wish to obtain is the electron spectrum. We do this as follows

$$d\lambda = \frac{2\pi}{\hbar^2} |V_{fi}|^2 dN_e \frac{dN_{\nu}}{dE_e} \quad (23)$$

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$$dN_e = \frac{1}{2\pi^2} \frac{L^3}{h^3} p^2 dp \quad (25)$$

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$$E_v = cq, \text{ so}$$

$$E_e = T_p + E_v = T_p + cq$$

$$q = \frac{E_e - T_p}{c} \quad (27)$$

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with T_p held constant

$$(28), (27), (26), (25) \rightarrow (23) \Rightarrow$$

$$(29) \quad d\Lambda = \frac{2\pi}{h} |Mc_i|^2 \underbrace{\left(\frac{1}{2\pi^2} \frac{L^3}{h^3} p^2 dp \right)}_{N_e} \underbrace{\left(\frac{1}{2\pi^2} \frac{L^3}{h^3} \left(\frac{E_e - T_p}{c} \right)^2 \right)}_{N_v} \frac{1}{c} dE_e$$

$$* \frac{1}{dE_e} \delta(E_e - (T_p + E_v))$$

$$\frac{d\lambda}{dp} = \frac{|M_{fi}|^2}{2\pi^3 \hbar^7 c^3} p^2 (E_f - T_p)^2 \delta(E_f - (T_p + E_\nu))$$

Integrate over E_ν to get

$$\frac{d\lambda}{dp} = \frac{|M_{fi}|^2}{2\pi^3 \hbar^7 c^3} p^2 (E_f - T_p)^2$$

Generally nuclear recoil is ignored.

Thus we can set $E_f = Q$

$$\text{So } \frac{d\lambda}{dp} = \frac{|M_{fi}|^2}{2\pi^3 \hbar^7 c^3} p^2 (Q - T_p)^2 \quad (30)$$

↑
note the endpoint energy.

β -decay (cont)

$$\frac{d\lambda}{dp} = \frac{1}{2\pi^3 k^7 c^5} p^2 q^2 \quad (1)$$

$\nearrow$ where $p = e^\pm$ momentum
 $q = \nu$ or $\bar{\nu}$ momentum

Recall that the electron and neutrino wave functions were given by

$$\psi_e = e^{i\vec{p}\cdot\vec{x}/\hbar} ; \psi_\nu = e^{i\vec{q}\cdot\vec{x}/\hbar}$$

If we expand the wave functions in a Taylor series

$$\psi_e = \left\{ 1 + i\frac{\vec{p}\cdot\vec{x}}{\hbar} - \frac{(\vec{p}\cdot\vec{x})^2}{\hbar^2} \dots \right\} \quad (2)$$

$$= \left\{ 1 + i\frac{\vec{q}\cdot\vec{x}}{\hbar} - \frac{\vec{q}\cdot\vec{x}}{\hbar^2} \dots \right\}$$

Recall

$$M_{fi} = \langle e^{i\vec{p}\cdot\vec{x}/\hbar} e^{i\vec{q}\cdot\vec{x}/\hbar} \psi_x | V_p | \psi_x \rangle \quad (3)$$

$\uparrow$ daughter $\uparrow$ parent

If we substitute $e^{i\vec{p}\cdot\vec{x}/\hbar} e^{i\vec{q}\cdot\vec{x}/\hbar} = 1$

and $M_{fi}^0 = \langle \psi_x | V_p | \psi_x \rangle \quad (4)$ this is called the "allowed transition" mode of β -decay.

If $M_{fi}^0 \neq 0$ then we have an "allowed" transition and this is the most probable channel for β -decay.

Thus, for allowed transitions

$$|M_{fi}|^2 = |M_{fi}^0|^2 \text{ is a constant,}$$

not depending on p or q

$$\text{Thus } \frac{dN^0}{dp} = \frac{|M_{fi}^0|^2}{2\pi^3 \hbar^2 c^5} p^2 q^2 \quad (5)$$

$\underbrace{\frac{|M_{fi}^0|^2}{2\pi^3 \hbar^2 c^5}}$ a constant, called C , by Krane for allowed transitions

Thus, $\frac{dN^0}{dp} = C^0 p^2 q^2 = N(p)$ (6)

So, if you were making a measurement of the β -decay spectrum, you would be measuring a distribution of the form

$$N^0(p) dp = C^0 p^2 q^2 dp \quad (7)$$

Although this is the simplest form to write it in, it mixes the kinematic variables p, q in an odd way.

Using conservation of energy $q = \frac{Q - T_\beta}{c}$, we get

$$\text{Thus } N^0(p) dp = C^0 c^2 p^2 (Q - T_\beta)^2 dp \quad (8)$$

which is the simplest way of showing the existence of the end-point energy.

If we want to be consistent and just write this expression in terms of p , using

$$T_\beta = E_\beta - m_e c^2 = \sqrt{c^2 p^2 + (m_e c^2)^2} - m_e c^2 \quad (9)$$

we have

$$(10) \quad \dot{N}(p) dp = \frac{C^0}{c^2} p^2 \left[Q - \sqrt{c^2 p^2 + (m c^2)^2} + m c^2 \right]^2 dp$$

which is Krane's (9.24).

Usually though, the spectrum is written in terms of the kinetic energy, T_β .

$$\text{Since } \dot{N}(T_\beta) dT_\beta = \dot{N}(p) dp$$

$$\dot{N}(T_\beta) = \dot{N}(p) \frac{dp}{dT_\beta} \quad (11)$$

$$(cp)^2 = T_\beta (T_\beta + 2m c^2)$$

$$\begin{aligned} 2(cp) \frac{d(cp)}{dT_\beta} &= T_\beta + 2m c^2 + T_\beta \\ &= 2(T_\beta + m c^2) \end{aligned}$$

$$\dot{N}(T_\beta) = \frac{C^0}{c^5} \sqrt{T_\beta (T_\beta + 2m c^2)} (Q - T_\beta)^2 (T_\beta + m c^2) \quad (12)$$

Krane's (9.25)

What if $\langle \psi_x' | N_p | \psi_x \rangle = 0$ i.e. $M_i = 0$?

These are the so-called "forbidden decay".

One can show that these decay can be characterized as "first", "second", "third" and so on forbidden transitions associated with more and more angular momentum carried away by the $e\bar{e}$ pair.

These are listed and adjusted as follows.

$$N(T_p) = N^0(T_p) S^n(pq) \quad (13)$$

Forbiddenness	n	$S^n(pq)$
allowed	0	1
first	1	$(p^2 + q^2)/(mc)^2$
second	2	$(p^4 + q^4 + \frac{10}{3} p^2 q^2)/(mc)^4$
third	3	$(p^6 + q^6 + 7p^2 q^2(p^2 + q^2))/(mc)^6$
and so on		

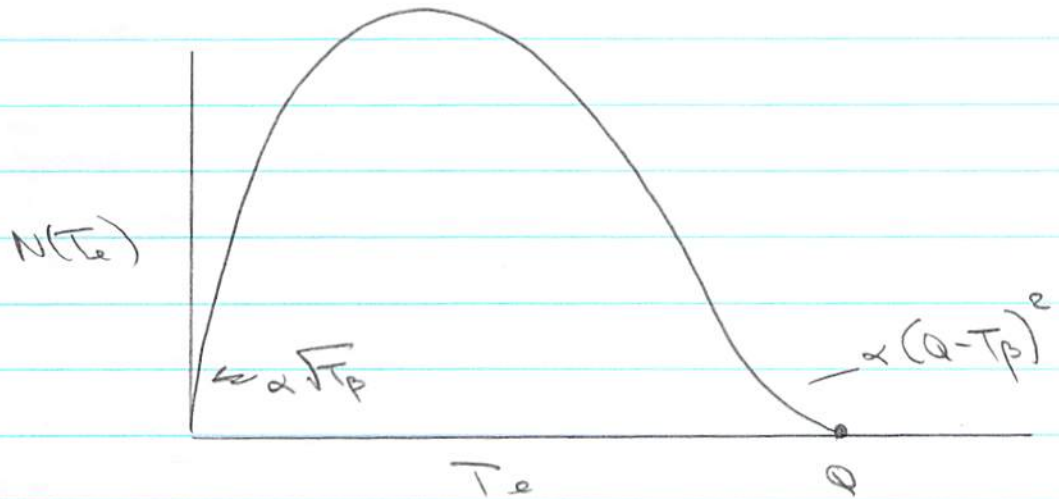
Thus, in general

$$(14) \quad N(T_p) = \frac{|M_i^n|^2}{2\pi^3 h^7 c^{10}} (T_p(T_p + 2mc^2))^{\frac{n}{2}} (Q - T_p)^2 (T_p + mc^2) S^n(pq)$$

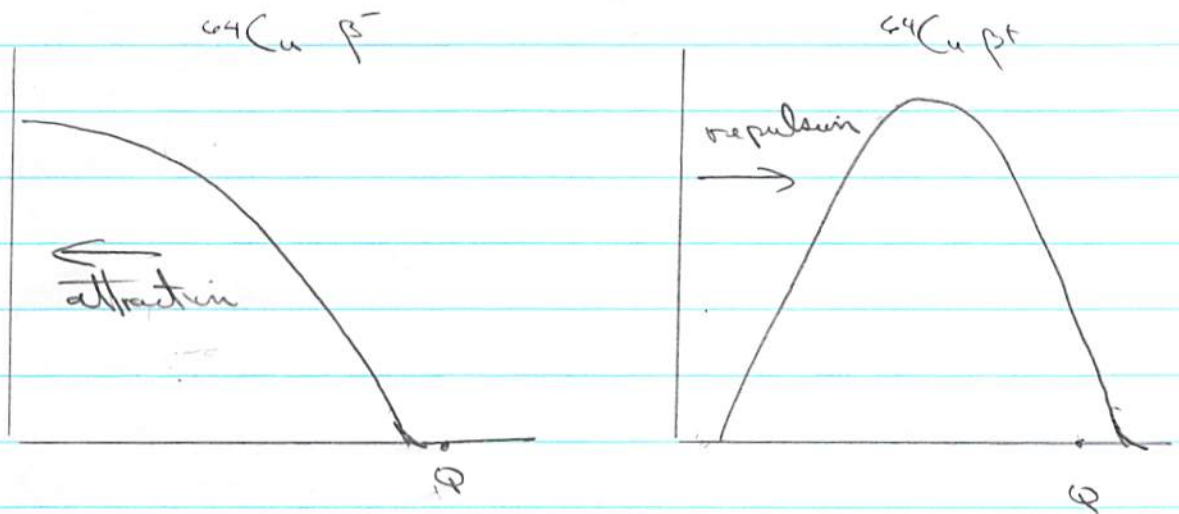
Note: The kinematic factors are drawn out to the right, but the M_i^n are strongly affected by forbiddenness

shape of the β, T_e spectrum

A plot of (12) would look like



effect of Coulomb attraction / repulsion



The Fermi Function $F(z', p)$ accounts for this. Thus

$$N(T_p) = \frac{|M_{fi}|^2}{2\pi^3 \hbar^7 c^{10}} [T_p(T_p + 2mc^2)]^{1/2} (Q - T_p)(T_p + mc^2) S^n(pq)$$

$$\text{or } N(p) = \frac{|M_{fi}|^2}{2\pi^3 \hbar^7 c^3} p^2 (Q - T_p)^2 F(z', p) S^n(pq) \quad (15)$$

which accounts for kinematic, coulomb effects and forbiddenness.

Note that we can write

$$(Q - T_p) = \sqrt{\frac{N(p)}{p^2 F(z', p) S^n(pq) \left(\frac{|M_{fi}|^2}{2\pi^3 \hbar^7 c^3} \right)}}$$

Thus, if we measure $N(p)$ and divide by $p^2 F(z', p) S^n(pq)$ and $\sqrt{\quad}$, we can extract, from the slope

$|M_{fi}|$ the unknown

may have to try several different S^n 's before we get this correct.

(This type of plot is called a Kurie plot)

Total decay rate

$$\lambda = \frac{|M_{fi}|^2}{2\pi^3 \hbar^3 c^3} \int_0^{P_{\max}} dp F(Z', p) p^2 (Q - T_e)^2 dp$$

$$F(Z', Q) \equiv \frac{1}{m_e^5 c^7} \int_0^{P_{\max}} F(Z', p) p^2 (Q - T_e)^2 dp$$

is called the Fermi integral

With $\lambda = \log(2) / t_{1/2}$

$$ft_{1/2} = \log(2) \frac{2\pi^3 \hbar^7}{m_e^5 c^4 |M_{fi}|^2}$$

is a characteristic of a certain decay

$\log_{10}(ft_{1/2})$ "log eff- $t_{1/2}$ "

is often used

Classification of transitions in β -decay

The e and $\bar{\nu}$ each have spin $1/2$ and can carry off angular momentum

$$\text{Thus } \vec{J} = \vec{L} + \vec{S}$$

$\uparrow$ $\uparrow$
 orbital sp.
 angular spin
 momentum

$$\therefore \Delta I \text{ of the nucleus} = \pm L \text{ or } \pm(L+1)$$

$\Delta I = \pm L \Rightarrow e$ and $\bar{\nu}$ intrinsic spins are antiparallel.

This is called a "Fermi" β decay

$\Delta I = \pm(L+1) \Rightarrow e$ and $\bar{\nu}$ intrinsic spins are parallel.

This is called a "Gamow-Teller" β decay

As L increases, $\lambda \downarrow$ & $A \uparrow$ because there is much less overlap of the e wave functions, with the nuclear wave functions

Classification

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	<u>Type of Transition</u>	<u>Selection Rules</u>	<u>$\Delta\pi$</u>	<u>f_t</u>
$L_{ev}=0$	super allowed	$\Delta I = 0, \pm 1^*$	no	1000-4000
	allowed	$\Delta I = 0, \pm 1$	no	$2 \times 10^4 - 10^6$
$L_{ev}=1$	first forbidden	$\Delta I = 0, \pm 1$	yes	$10^6 - 10^8$
	unique** first forbidden	$\Delta I = \pm 2$	yes	$10^8 - 10^9$
$L_{ev}=2$	2nd forbidden	$\Delta I = \pm 2$	no	$2 \times 10^{10} \rightarrow 2 \times 10^{13}$
	unique 2nd	$\Delta I = \pm 3$	no	10^{12}
$L_{ev}=3$	3rd forbidden	$\Delta I = \pm 3$	yes	10^{18}
	unique 3rd	$\Delta I = \pm 4$	yes	4×10^{15}
$L_{ev}=4$	4th forbidden	$\Delta I = \pm 4$	no	10^{23}
	unique 4th	$\Delta I = \pm 5$	no	10^{19}

* $0^+ \rightarrow 0^+$ transitions can only occur via Fermi decay
 ** in unique forbidden transitions the orbital & spin angular momenta are aligned. The shape factors have a very simple form in this case

recall $S^n(p, q) = (p^2 + q^2) / (mc)^2 \quad n=1$

These simple forms are only for unique transitions

Examples of allowed transitions

$$0^+ \rightarrow 0^+$$



must be pure Fermi

other examples $^{35}\text{Cl} \rightarrow ^{35}\text{S}$, $^{10}\text{C} \rightarrow ^{10}\text{B}^*$

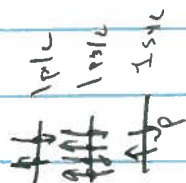
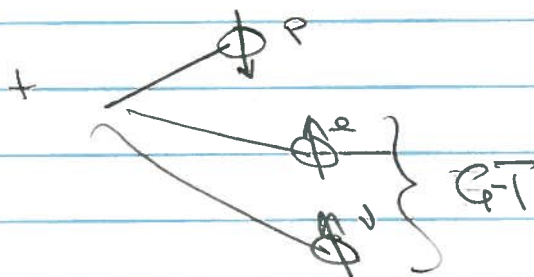
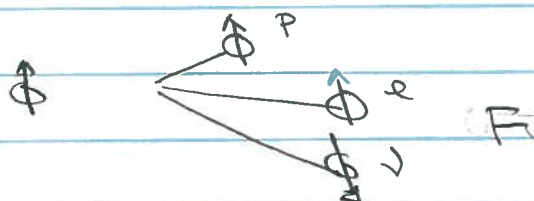
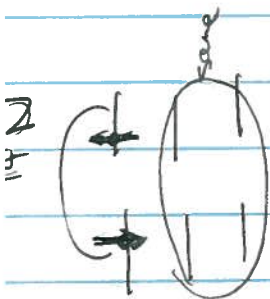
Note: in the superallowed transitions, the nucleus that transforms, does not change its single particle wave function



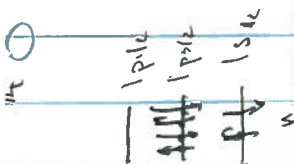
must be pure

Gamow Teller

other examples



So the n decay has both modes.



$$M_{fi} \rightarrow g_F M_F + g_{GT} M_{GT}$$

$$|M_{fi}|^2 \rightarrow g_F^2 |M_F|^2 + g_{GT}^2 |M_{GT}|^2$$

$$y = \frac{g_F M_F}{g_{GT} M_{GT}}$$

For neutron decay $|M_F| = 1$

$$\begin{aligned} |M_{fi}|^2 &= g_F^2 |M_F|^2 + g_{GT}^2 |M_{GT}|^2 \\ &= g_F^2 |M_F|^2 \left(1 + \frac{1}{y^2}\right) \end{aligned}$$

y in this case = 0.467(3) (exp'l result)

$$\%F \quad \frac{g_F^2 |M_F|^2}{|M_{fi}|^2} = (1 + \frac{1}{y^2})^{-1} = 0.179(2)$$

$$\%GT \quad = 0.821(2)$$