

Chapter 3 α -decay

The α -particle is the nucleus of the He-atom.

$2p + 2n$. First double magic nucleus.

It is very stable.

It's mass defect $\Delta = (m - A)c^2$ ^{in (u)}

is 0.002603

while its closest neighbors

${}^3\text{He}$	has	0.016029
${}^3\text{H}$		0.016049
${}^5\text{He}$		0.0121
${}^5\text{Li}$		0.0125

	${}^5\text{Li}$	
	0.0125	
${}^3\text{He}$	${}^4\text{He}$	${}^5\text{He}$
0.01603	0.002603	0.0121
	${}^3\text{H}$	
	0.01605	

α decay is
one means by
which a
proton rich heavy
nucleus can
get closer
to the line
of stability

3.2 Basic α -decay processes

e.g. ${}^{232}_{90}\text{U} \rightarrow {}^{228}_{88}\text{Th} + \alpha$
 $Z/A = 90/232 = 0.39166$



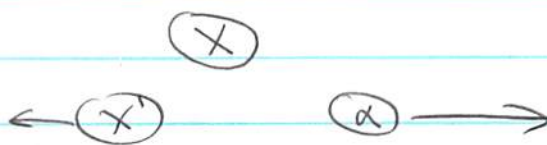
${}^{228}_{88}\text{Th}$ $Z/A = 88/228 = 0.3947$

energetics

$$m_x c^2 = m_{x'} c^2 + \overline{T}_{x'} + m_\alpha c^2 + \overline{T}_\alpha$$

$\uparrow$ recoil KE $\uparrow$ α 's KE

$Q = (m_x - m_{x'} - m_\alpha) c^2$ is the energy available to go into the KE of the resultant particles



$$p_{x'} = p_\alpha$$

$$Q = T_{x'} + T_\alpha$$

$$p_{x'} = p_\alpha \rightarrow 2m_{x'} T_{x'} = 2m_\alpha T_\alpha$$

 $\Rightarrow$


$$T_\alpha = \frac{Q}{(1 + m_\alpha/m_{x'})} \quad (1)$$

$$T_{x'} = \frac{m_\alpha}{m_{x'}} \frac{Q}{(1 + m_\alpha/m_{x'})} \quad (2)$$

Fully relativistic calculation

$$\overline{T}_\alpha = \frac{Q \left(1 + \frac{Q}{2m_\alpha c^2} \right)}{\left(1 + \frac{m_\alpha}{m_{x'}} + \frac{Q}{m_{x'} c^2} \right)} \quad (3)$$

$$\overline{T}_{x'} = \frac{\frac{m_\alpha}{m_{x'}} Q \left(1 + \frac{Q}{2m_\alpha c^2} \right)}{\left(1 + \frac{m_\alpha}{m_{x'}} + \frac{Q}{m_{x'} c^2} \right)} \quad (4)$$

(correction)
m_{x'} was posted

One can show, in the least favorable case, that the relativistic correction is about 2.5×10^{-4} (low A, getting much better as A increases.)

Back to ①, ②

$$\frac{m_\alpha}{m_{x'}} \approx \frac{4}{A}$$

$$\textcircled{5} \quad T_\alpha \approx \frac{Q}{1 + \frac{4}{A}} \approx Q \left(1 - \frac{4}{A}\right) \quad \text{KII (8.7)}$$

$$\textcircled{6} \quad T_{x'} \approx \frac{4}{A} Q$$

The recoil energy is not insignificant

For a typical α -emitter ($A=200$) $Q=5\text{MeV}$

$$T_{x'} = \frac{1}{50} 5000\text{keV} \approx 100\text{keV}$$

↗
This will break
crystal bonds

Supplemental Notes

Fermi's Golden Rule #2

Consider the Schrödinger eq'n for a particle in a static binding potential

$$H_0 \Psi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, t)$$

where $H_0 = T + V(\vec{r})$; $T = \frac{\hbar^2 \nabla^2}{2m}$; $V(\vec{r})$ is static
no time dependence

We know that such a potential has a set of orthonormal eigenstates

$$|j\rangle = \Psi_j(\vec{r}, t) = \Psi_j(\vec{r}) e^{-iE_j t/\hbar}$$

where E_j are the eigenenergies,

$$\text{Also } \langle n | j \rangle = \delta_{nj} \quad \text{orthonormal}$$

We also know, that these states, if unperturbed are stable.

To model transitions, we add a perturbation potential, that is time dependent

$$V_p(\vec{r}, t)$$

Now we solve

$$(H_0 + V_p)\psi = i\hbar \frac{\partial \psi}{\partial t} \quad (1)$$

We write ψ as a superposition of the basis eigenfunctions

$$\psi(\vec{x}, t) = \sum_j \underbrace{a_j(t)}_{\substack{\uparrow \\ \text{time dependent,} \\ \text{since } V_p \text{ is}}} \underbrace{\psi_j(\vec{x})}_{\text{solns of } H_0} \quad (2)$$

$$\frac{\partial}{\partial t} (2) \Rightarrow$$

$$\frac{\partial \psi}{\partial t} = \sum_j (\dot{a}_j - i \frac{E_j}{\hbar}) \psi_j \quad (3)$$

$$(3) \div (2) \Rightarrow (1) \Rightarrow$$

$$(H_0 + V_p) \sum_j a_j \psi_j = \sum_j (i\hbar \dot{a}_j + E_j) \psi_j, \text{ or}$$

$$(4) \quad \sum_j a_j \underbrace{(H_0 - E_j)}_{=0 \text{ for each } j \text{ (eigenfunctions)}} \psi_j + \sum_j (a_j V_p - i\hbar \dot{a}_j) \psi_j = 0$$

Thus
$$\sum_j (a_j V_j - i\hbar \dot{a}_j) \psi_j = 0$$

or,
$$\sum_j (a_j V_j - i\hbar \dot{a}_j) |j\rangle = 0 \quad (5)$$

Let $|f\rangle$ be the state that the particle is in the j th state transitions to

$$\langle f | \times (5) \Rightarrow$$

$$(6) \quad \sum_j \left\{ a_j \langle f | V_p | j \rangle e^{i(E_j - E_f)t/\hbar} - i\hbar \dot{a}_j \delta_{fj} \right\} = 0$$

$$\text{Let } \omega_{jf} \equiv \frac{E_j - E_f}{\hbar}$$

$$(6) \Rightarrow i\hbar \dot{a}_f = \sum_j a_j V_{fj} e^{i\omega_{jf}t} \quad (7)$$

where

$$V_{fj} = \langle f | V_p | j \rangle$$

(7) is a completely general solution to the problem. But getting a practical solution require a lot of calculation!

To make some headway, we assume that the a_j 's are slowly varying enough that they can be treated as constants

Small perturbation approximation

Start with initial conditions

$a_j(0) = 1$ excited state at beginning
all other a 's = 0.

Thus $i\hbar \dot{a}_f = V_{fj} e^{i\omega_j t}$

If $f \neq j$ $i\hbar \dot{a}_f = \frac{1}{i\omega_f} V_{fj} e^{i\omega_j t} \Big|_0^t$
 $= \frac{i}{\omega} V_{fj} (1 - e^{i\omega_j t})$

$a_f = \frac{V_{fj} (1 - e^{i\omega_j t})}{\hbar \omega_f} \quad f \neq j$

$i\hbar \dot{a}_j = \frac{V_{jj} t}{\hbar} + i\hbar \quad \leftarrow \text{since } a_j(0) = 1$

$a_j = 1 - \frac{i}{\hbar} V_{jj} t \quad f = j$

(8)

$a_j = 1 - \frac{i}{\hbar} V_{jj} t \quad f = j$

$a_f = \frac{V_{fj} (1 - e^{i\omega_j t})}{\hbar \omega_f}$

$P_f = |a_f|^2$ is the occupation of the final state as a fun of time

$$P_f = |a_f|^2 = \frac{|V_{fi}|^2}{\hbar^2 \omega_{if}^2} \underbrace{(1 - e^{-i\omega_{if}t})(1 - e^{i\omega_{if}t})}_{= 2 - 2\cos\omega_{if}t} \\ = 4\sin^2\omega_{if}t/2$$

$$(10) \quad P_f = |a_f|^2 = \frac{|V_{fi}|^2}{\hbar^2 \omega_{if}^2} 4\sin^2(\omega_{if}t/2)$$

(10) assumed that the energy of the initial state is precisely defined. If isn't. We know this from $\Delta E \Delta t \geq \hbar/2$.

So, assume that it is distributed continuously about E_i . Then we rewrite (10) as follows

$$dP_f(\omega) = \frac{|V_{fi}|^2}{\hbar^2} 4 \frac{\sin^2(\omega_{if}-\omega)t/2}{(\omega_{if}-\omega)^2} d\omega g(\omega)$$

where $g(\omega)$ is the density of these states

And integrate over all ω to get

$$P_f = \frac{|V_{fi}|^2}{\hbar^2} 4 \int_{-\infty}^{\infty} d\omega \frac{\sin^2(\omega_{if}-\omega)t/2}{(\omega_{if}-\omega)^2} g(\omega)$$

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We note that $\frac{\sin^2(\omega_{if} - \omega)t/2}{[(\omega_{if} - \omega)/2]^2}$

is very tightly centered around $\omega = \omega_{if}$

In fact, as t gets large (in the realm of measurement) it becomes a δ -function.

Changing variables, we set

$$P_f = \frac{|V_{fi}|^2}{\hbar^2} g(\omega_{if}) t \underbrace{\int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2}}_{= \pi}$$

$$\text{So } P_f = \frac{2\pi}{\hbar^2} |V_{fi}|^2 g(\omega) t$$

$$g(\omega) = \frac{dn_{\omega}}{d\omega} = \hbar \frac{dn_{E_{if}}}{dE_{if}} = \hbar g(E_{if})$$

Therefore

$$P_f = \frac{2\pi}{\hbar} |V_{fi}|^2 g(E_{if}) t$$

Thus, the transition rate is

$$\lambda = \frac{dP}{dt} = \frac{2\pi}{\hbar} |V_{fi}|^2 g(E_{if})$$

Fermi's Golden Rule #2

was "quite a gus" !

$$\frac{1}{\sqrt{2\pi}} \quad \psi(\vec{r}, E) = \psi(\vec{r}) \int dE g(E) \frac{2k_i}{h - 2i(E - E_j)\tau}$$

$$RPF = \frac{1}{2\pi} \frac{1}{k^2} \int dE d\vec{r} \frac{1}{(h - 2iE\tau)(h + 2iE\tau)}$$

$$S^2(E) = S^2$$

$$\psi(\vec{r}, t) = \psi(\vec{r}) e^{-iEt/\hbar}$$

$$\int_{-\infty}^{\infty} dE \frac{1}{(E_i - E)^2 + \left(\frac{\Gamma}{2}\right)^2}$$

$$= \int_{-\infty}^{\infty} dE \frac{1}{E^2 + \left(\frac{\Gamma}{2}\right)^2}$$

$$= \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{dE}{\left(\frac{E\Gamma}{2}\right)^2 + 1} \quad \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{2}$$

One may easily verify that:

$$\int_{-\infty}^{\infty} dE |\Psi_i(\vec{x}, E)|^2 = |\Psi(\vec{x})|^2 . \quad (13.53)$$

13.3 Production and Decay of Radioactivity

Secular equilibrium

Consider a beam of radiation, with intensity I , that impinges upon a block of material containing nuclei of type "0". There are N_0 nuclei in the target to start, and the external radiation activates the material, producing nuclear species "1". We make two assumptions:

1. The target is "thin enough", so that the external radiation is not attenuated.
2. The irradiation is weak enough so that N_0 does not decline.

With these assumptions, R , the rate of production of N_1 is given by:

$$R = I\sigma N_0 , \quad (13.54)$$

where σ is the radioactivity production cross section.

The differential equation describing the production of N_1 is given by:

$$dN_1 = Rdt - \lambda_1 N_1 dt , \quad (13.55)$$

where λ_1 is the decay rate of nucleus "1". Solving (13.55):

$$\begin{aligned} \dot{N}_1 + \lambda_1 N_1 &= R \\ (\dot{N}_1 + \lambda_1 N_1)e^{\lambda_1 t} &= Re^{\lambda_1 t} \\ \frac{d}{dt}(N_1 e^{\lambda_1 t}) &= Re^{\lambda_1 t} \\ N_1 e^{\lambda_1 t} - N_1(0) &= \frac{R}{\lambda_1}(e^{\lambda_1 t} - 1) \\ N_1 - N_1(0)e^{-\lambda_1 t} &= \frac{R}{\lambda_1}(1 - e^{-\lambda_1 t}) \\ N_1 &= N_1(0)e^{-\lambda_1 t} + \frac{R}{\lambda_1}(1 - e^{-\lambda_1 t}) . \end{aligned} \quad (13.56)$$

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Usually the target contains no activity at $t = 0$. If $N_1(0) = 0$:

$$\begin{aligned} N_1 &= \frac{R}{\lambda_1}(1 - e^{-\lambda_1 t}) \\ A_1 &= R(1 - e^{-\lambda_1 t}) . \end{aligned} \quad (13.57)$$

Condition	Beam	$A_1(t)$	Description
$\lambda_1 t \ll 1$ i.e. $t \ll \tau$	ON	$R\lambda_1 t$	initial production
$\lambda_1 t \gg 1$ i.e. $t \gg \tau$	ON	R	secular equilibrium
$t \geq t_0$, t_0 is arbitrary	OFF (at $t = t_0$)	$N_1(t_0)\lambda_1 e^{-\lambda_1(t-t_0)}$	decay of activity
$t \geq t_0 \gg \tau$	OFF (at $t = t_0$)	~~$R e^{-\lambda_1(t-t_0)}$~~	decay of activity (from secular equilibrium)

During the condition of *secular equilibrium*, the rate of production is the same as the rate of decay, producing an unchanging number of radioactive daughter nuclei. (There will be some statistic fluctuation of this number.)

A depiction of a nuclide reaching secular equilibrium is show in Figure 13.3.

A real-life engineering application

A typical engineering challenge, in the area of creating radioactive sources, is to minimize the cost of producing a given amount of activity. We model this as follows:

The cost, C , per unit activity, factoring start-up costs, S_0 , (manufacture of the inactive source, delivery costs, operator start-up and take-down), and the cost of running the accelerator or reactor, per unit meanlife, R_0 , is given as follows:

$$C = \frac{S_0 + R_0 x_0}{1 - e^{-x_0}} , \quad (13.58)$$

where x_0 is the number of meanlives that the target is irradiated.

The optimization condition is given by:

$$S_0/R_0 = e^{x_0} - (1 + x_0) . \quad (13.59)$$

For small S_0/R_0 , the optimum $x_0 \approx \sqrt{2S_0/R_0}$. For large S_0/R_0 , the optimum $x_0 \approx \log(S_0/R_0)$.

The solutions for the optimum value of x_0 can be obtained from Figure 13.4.

The figure has been split into two parts, $x_0 < 1$ for which S_0/R_0 is shown, and $x_0 > 1$ for which $\log(S_0/R_0)$ is shown. The approximations discussed above, are plotted as dotted lines.

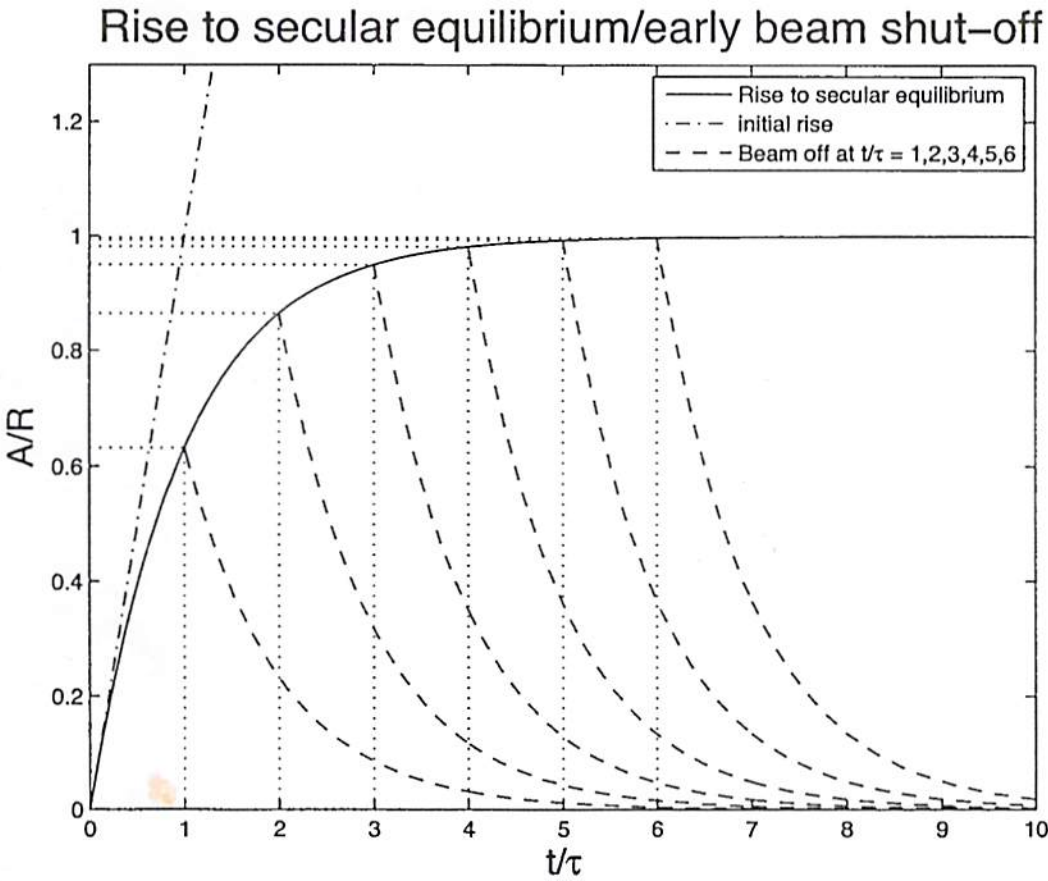


Figure 13.3: A nuclide reaching secular equilibrium is depicted. Also depicted are the decays of the source once the beam is shut off at $t/\tau = 1, 2, 3, 4, 5$.

13.3. PRODUCTION AND DECAY OF RADIOACTIVITY

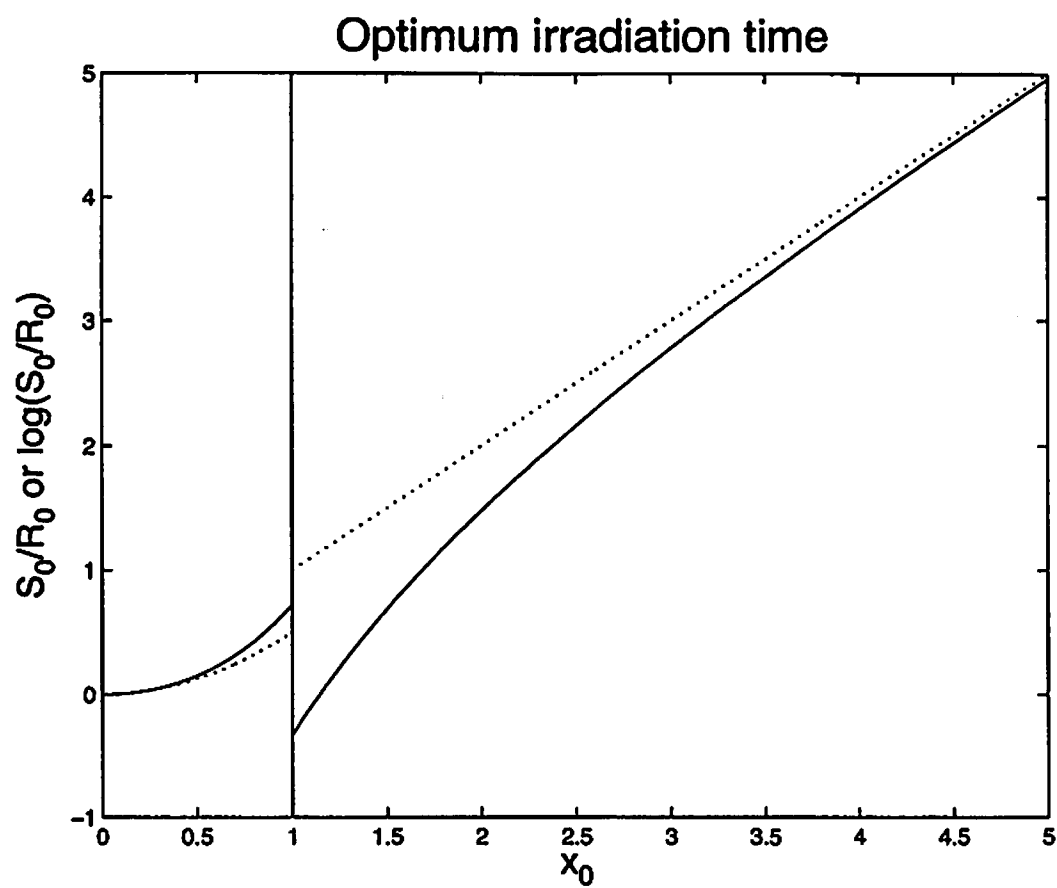
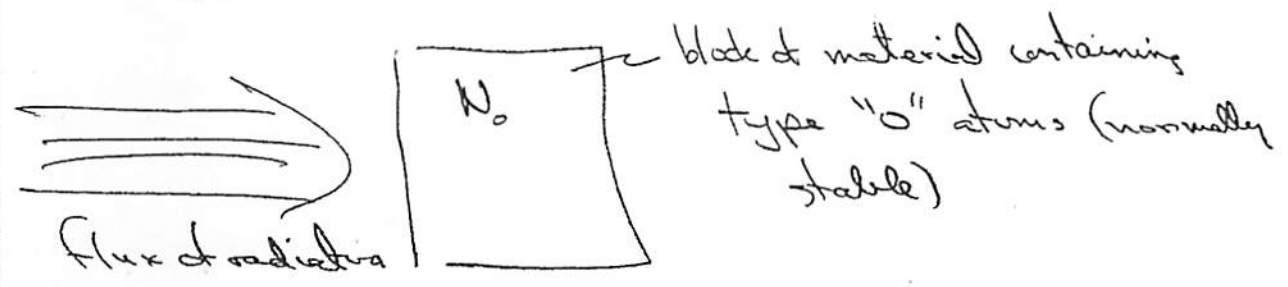


Figure 13.4: Optimization curves for determining x_0 , as a function of $\alpha = S_0/R_0$.

6.3 Production & Decay of Radioactivity



$I =$ Number of incident particles per unit area per unit time

Rate of production is

radioactivity production cross section

$$R = I \sigma N_0$$

↑ # of target nuclei irradiated, assuming no depletion (can be ignored if σ is small enough)

N_1 : radioactive atoms produced

$$dN_1 = R dt - \lambda_1 N_1 dt \quad (1)$$

$$\dot{N}_1 + \lambda_1 N_1 = R \quad (2)$$

$e^{\lambda_1 t} \times (2) \Rightarrow$ (integrating factor)

$$(N_1 + \lambda_1 N_1) e^{\lambda_1 t} = R e^{\lambda_1 t}$$

$$\frac{d}{dt}(N_1 e^{\lambda_1 t}) = R e^{\lambda_1 t}$$

integrate $N_1 e^{\lambda_1 t} - N_1(0) = \frac{R}{\lambda_1} (e^{\lambda_1 t} - 1)$

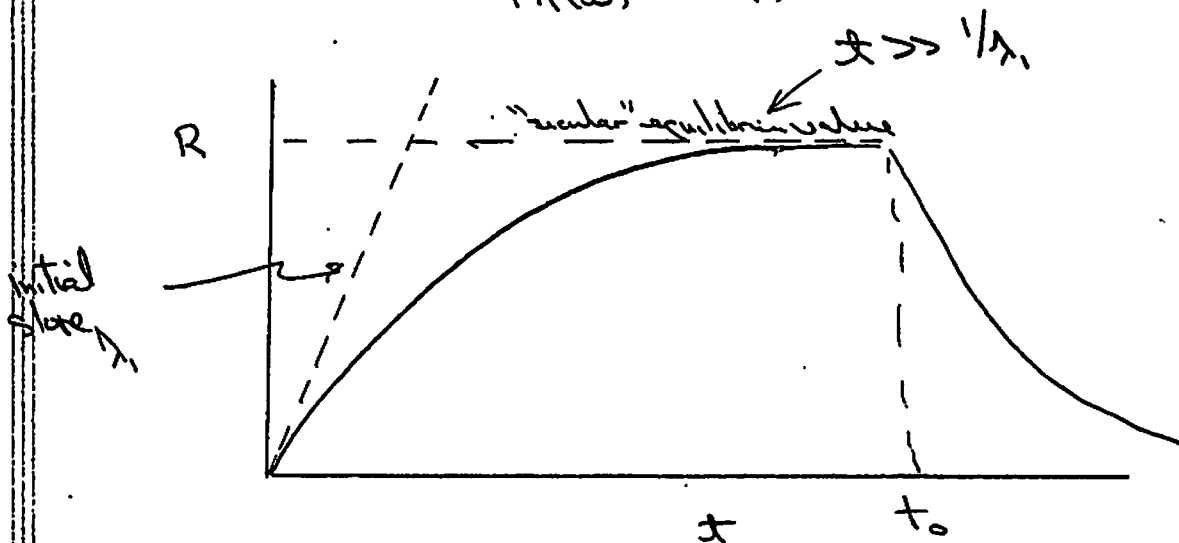
$$N_1 = \underbrace{N_1(0)}_{\substack{\uparrow \\ \text{usually } 0}} e^{-\lambda_1 t} + \frac{R}{\lambda_1} (1 - e^{-\lambda_1 t})$$

if $N_1(0) = 0$

$$N_1 = \frac{R}{\lambda_1} (1 - e^{-\lambda_1 t}) \quad (3)$$

Activity $A_1 = \lambda_1 N_1 = R (1 - e^{-\lambda_1 t}) \quad (4)$

Sketch $A_1(0) = 0$
 $A_1'(0) = \lambda_1 R$
 $A_1(\infty) = R$



"secular equilibrium" $\Rightarrow$ daughter creation rate
 — daughter destruction rate

So, at $t \gg 1/\lambda$, $A_1 = R$

If we then switch the beam off at $t = t_0$

$$A_1(t) = A_1(t_0) e^{-\lambda t}$$

$$\begin{aligned} \text{So } A_1(t) &= R(1 - e^{-\lambda t}) & t \leq t_0 \text{ beam on} \\ &= A_1(t_0) e^{-\lambda t} & t \geq t_0 \text{ beam off} \\ &= R e^{-\lambda t} & t \geq t_0 \Rightarrow 1/\lambda \end{aligned}$$

Real-life engineering-type application

$A_1(t) = R(1 - e^{-\lambda t})$ describes the build up of activity over time.

Optimize the function, $A_1(t)$ over time accounting for a "dead" time Δt to switch samples

$$1.2 \text{ Optimize } f(t_0) = \frac{R(1 - e^{-\lambda t_0})}{\Delta t + t_0}$$

where t_0 time target is in the beam.

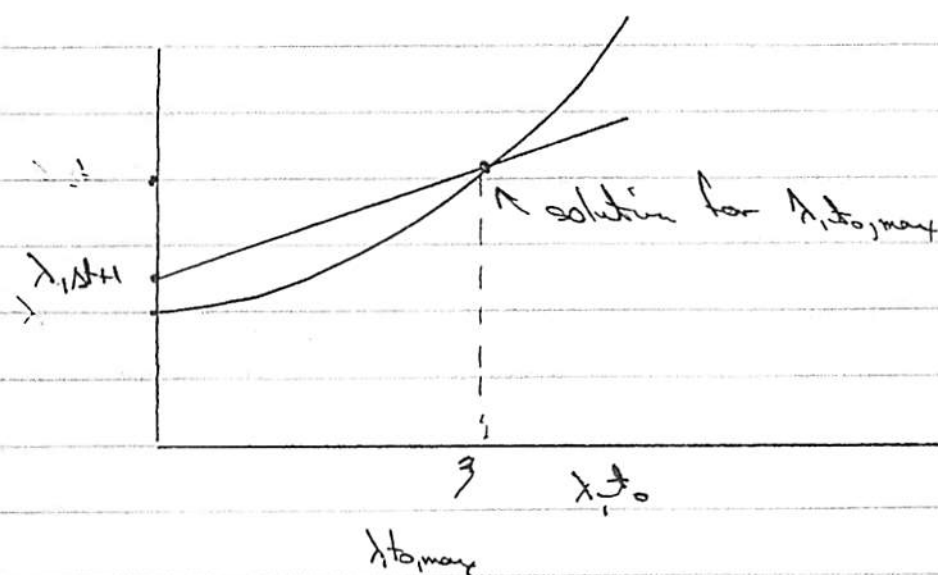
Δt = time to survey in a new, fresh target

$$f(t_0) = \frac{R(1 - e^{-\lambda t_0})}{\Delta t + t_0}$$

$$f'(t_0) = R \left\{ \frac{\lambda_1 e^{-\lambda_1 t_0}}{\Delta t + t_0} - \frac{(1 - e^{-\lambda_1 t_0})}{(\Delta t + t_0)^2} \right\}$$

$$= \frac{R}{(\Delta t + t_0)^2} \left\{ (\lambda_1 (\Delta t + t_0) + 1) e^{-\lambda_1 t_0} - 1 \right\}$$

$$f(t) = 0 \Rightarrow e^{\lambda_1 t_0} = \lambda_1 (\Delta t + t_0) + 1 \quad (6)$$



If $\Delta t \ll \lambda_1^{-1}$, Taylor expand (6) (generally this is the case)

$$1 + \lambda_1 t_0 + \frac{1}{2} \lambda_1^2 t_0^2 + \dots = \lambda_1 (\Delta t + t_0) + 1$$

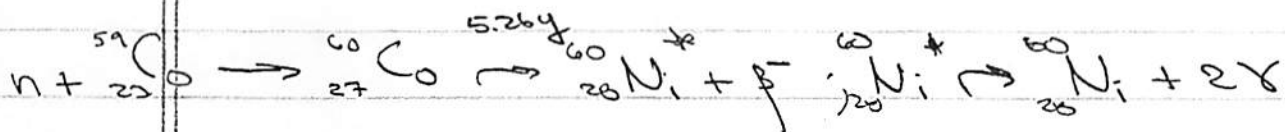
$$\Rightarrow \frac{1}{2} \lambda_1^2 t_0^2 = \lambda_1 \Delta t$$

$$\lambda_1 t_0 = \sqrt{2 \lambda_1 \Delta t} \quad (7)$$

Generally $\lambda_1 \Delta t \ll 1$

$$\text{e.g. } \lambda_1 \Delta t = 10^{-2} : \lambda_1 t_0 = 0.1$$

Real life example



t_0 typically 1 year

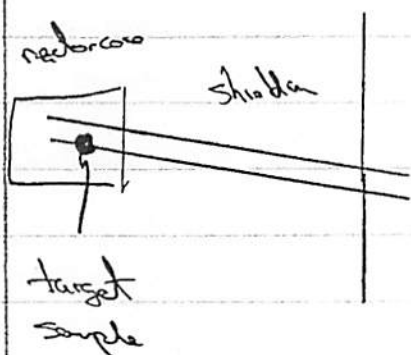
$$\lambda t_0 \approx 0.02$$

$$\Rightarrow \lambda \Delta t = \frac{1}{2}(\lambda t_0)^2$$

$$= 0.02$$

$$\Delta t = 0.02 \times 527\text{y} \\ = 34\text{ days}??$$

Δt actually? $\lesssim 15\text{ minutes}$



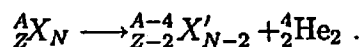
Note: In 1 yr only about 1% of the sample become radioactive

13.5 Types of Decays

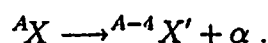
Details are covered elsewhere in this course. Here we just give a list.

α decay

This is the form employed when an accurate mass calculation is to be performed. Electron masses cancel, and the small differences in electron binding energy are ignored.



Usually the following shorthand is employed:

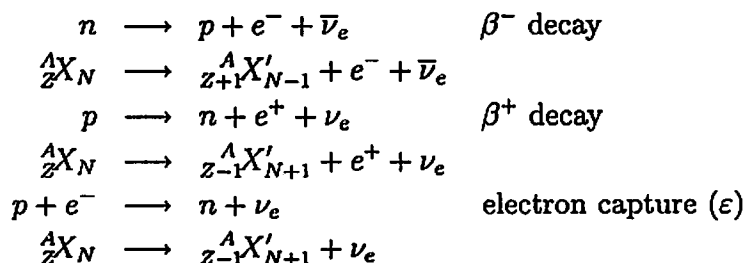


To come, more details on ...

Applications of α -emitters:

- "Gadget" initiators (α -decay followed by n -emission.)
- Search for superheavy elements. See <http://t2.lanl.gov/tour/shn.html>
- Smoke detectors.
- Power generation in space probes and artificial hearts.
- Unsealed source radiotherapy.
- Can be used to reduce static cling. (Really! I'm not kidding! The real question is "How?")

β decay



A free neutron will β^- decay with a meanlife of 886.7(8) s. A neutron in a nucleus will β^- decay, but only when that process is favorable energetically.

Free protons do not decay, that is, it has never been observed. Proton decay is predicted by Grand Unified Theories (GUTs). However, the predicted probability of decay is exceedingly small. A lower bound for proton decay has been established experimentally, setting the half-life at greater than 6.6×10^{33} years. This is interesting, but of little consequence for Nuclear Engineering. Protons in a nucleus, if favored energetically, do β^+ decay.

Finally, there is a process called *electron capture*, (ϵ), or *K-capture*, whereby a proton in a nucleus captures an orbital electron (usually from a 1s atomic orbital, and converts itself to a neutron.

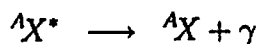
All these process result in a *electron neutrino*, ν_e , or an *electron antineutrino*, $\bar{\nu}_e$. By convention, antiparticles, like the antiproton, $\bar{p}$, and the antineutron, $\bar{n}$, are written with an overline $\bar{}$ or an "overtilde" $\bar{}$. There are exceptions to this rule. The positron, e^+ is the e^- 's antiparticle. However, it is never written as $\bar{e}^-$.

To come, more details on ...

Applications of β -emitters:

- Betavoltaics (non-thermal) (long-life, low-power batteries).
- Radiotherapy (brachytherapy).
- PET (Positron Emission Tomography.)
- Radiopharmaceuticals.
- Quality assurance in large-scale paper production.
- Irradiation of domestic ruminant (cattle, goats, sheep, bison, deer, camels, alpacas, llamas) behinds to cure the effects of "fly strike". (I'm not kidding about this one either.) (And I'd really rather not delve into the details of "fly strike". Kindly google this one on your own.)

γ decay



Here, a nucleus in an excited state, denoted by the asterisk, decays via the γ process, to a lower excited state, or the ground state. All nuclei that are observed to have excited states, ($A > 5$), have γ transitions.

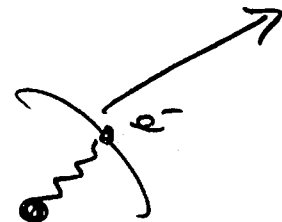
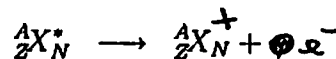
13.5. TYPES OF DECAYS

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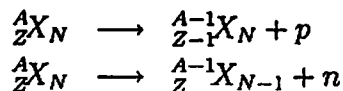
To come, more details on ...

Applications of γ -emitters:

- Basic physics: Nuclear structure, astrophysics.
- Radiotherapy, ^{60}Co , ^{137}Cs , brachytherapy..
- Sterilization of pharmaceutical products, food.
- Imaging vehicles for National Security Administration purposes.
- Industrial quality assurance.
- Discovery of oil. (Oil-well logging.)

Internal conversion

Here, a nucleus in an excited state, de-excites by exchanging a virtual photon with a K -shell electron in a "close encounter" with the nucleus. The electron acquires the de-excitation energy and exits the nucleus.

Nucleon emission

To come, more details on ...

Applications of nucleon emitters:

- Fission.

Spontaneous fission

$${}^A_Z X_N \longrightarrow \sum_{i,j} c_{i,j} [{}^{i+j}_i X_j + {}^{A-i-j}_{Z-i} X_{N-j}]$$

Here a nucleus fractures into 2 (equation shown) or more (equation not shown) nuclei. This is similar to "normal" fission, except that it occurs spontaneously. Generally a spectrum of nuclei result, with the probabilities given by the c coefficients.

Cluster decay

$${}^A_Z X_N \longrightarrow {}^{i+j}_i X_j + {}^{A-i-j}_{Z-i} X_{N-j}$$

where $i, j > 2$.

From:

http://en.wikipedia.org/wiki/Cluster_decay!...

Cluster decay is a type of nuclear decay in which a radioactive atom emits a cluster of neutrons and protons heavier than an alpha particle. This type of decay happens only in nuclides which decay predominantly by alpha decay, and occurs only a small percentage of the time in all cases. Cluster decay is limited to heavy atoms which have enough nuclear energy to expel a portion of its nucleus.

13.6 Natural Radioactivity

Not covered in 312

13.7 Radioactive Dating

Not covered in 312

13.8 Units for Measuring Radiation

Some useful radiometric quantities are listed in Table 13.1, along their traditional (outdated but still in some use), along with their new, almost universally adopted, SI².

Quantity	Measures...	Old Units	New (SI) Units
Activity ($\mathcal{A}$)	decay rate	curie (Ci)	becquerel (Bq)
Exposure (X)	ionization in air	röntgen (R)	C/kg
Absorbed Dose (D)	Energy absorption	rad	gray (Gy = J/kg)
Dose Equivalent (DE)	Radiological effectiveness	rem	sievert (Sv = J/kg)

Table 13.1: Units for the measurement of radiation

SI units	Definition	Notes:
Bq	1 "decay"/s	derived
Gy	1 J/kg	derived
Sv	1 J/kg	derived
Traditional units	Conversion	Notes:
Ci	3.7×10^{10} Bq (exactly)	
R	1 esu/(0.001293 g) = 2.58×10^{-4} C/kg	Charge/(1g) of dry, STP air
rad	1 ergs/g = 10^{-2} Gy	
rem	1 rem = 10^{-2} Sv	

Table 13.2: Conversion factors

A derived unit is one that is based upon the seven base units in SI, namely: m, kg, s, A, K, mol (mole), cd (candela, luminosity).

Activity

Activity ($\mathcal{A}$) has been covered already. However, the units of measurement were not discussed. The traditional unit, the *curie*, (Ci), was named in honor of Marie Curie, and the modern unit in honor of Henri Becquerel.

²SI stands for *le Système International d'unités*, that was adopted universally (almost) in 1960. For more information, see: http://en.wikipedia.org/wiki/SI_units. SI is an abbreviation of a French-language phrase. France also houses the international metrology, the BIPM (*le Bureau International des Poids et Mesures*.) See, as well, <http://en.wikipedia.org/wiki/BIPM>.

Exposure

Exposure, given the symbol X , is defined as

$$X = \lim_{\Delta m \rightarrow 0} \frac{\Delta Q}{\Delta m}, \quad (13.66)$$

where ΔQ is the amount of charge of one sign produced in, dry, STP air. (NIST definition: STP at STP: 20°C (293.15 K, 68°F, and an absolute pressure of 101.325 kPa (14.696 psi, 1 atm).) There are no accepted derived SI unit for exposure. Exposure measurements are probably the most accurately measured radiometric quantity.

Absorbed Dose

Absorbed dose measures the energy absorbed in matter, due to radiation. The traditional unit, the rad, has been supplanted by the Gy.

Dose Equivalent

The tradition unit, the rem (for röntgen equivalent man) and its modern counterpart, the Sv, attempts to accounts for the radiological damage from different "qualities" (species of particle imparting the dose) of radiation. Quality factors dependent on the energy of the radiation and are given in Table 13.2. The conversion from D to DE is given by:

$$DE = \sum_i \int dE D_i(E) \times QF_i(E), \quad (13.67)$$

where we sum over radiation types, and integrate over the energy of the radiation that imparts dose.

Radiation	Energy	QF
X -rays, γ , $e^{\pm}$, $\mu^{\pm}$	all	1
p (non-recoil)	> 1 MeV	5
n	< 10 keV	5
	10–100 keV	10
	10 keV – 2 MeV	20
	2–20 MeV	10
	> 20 MeV	5
α , fission fragments, heavy nuclei	all	20

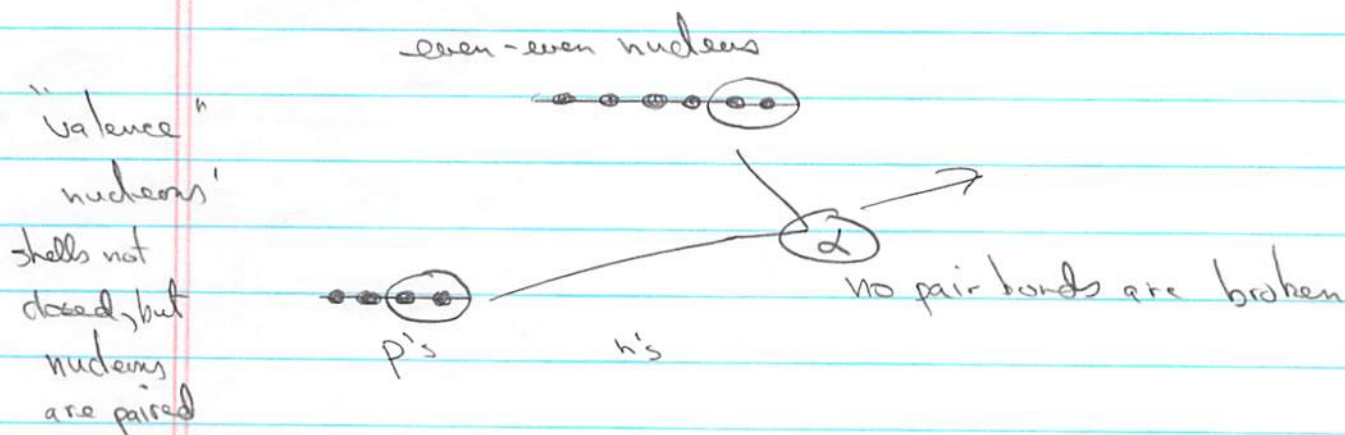
Table 13.3: Units for the measurement of radiation

8.3 α -decay systematics

As Q increases, $t_{1/2}$ decreases. (Geiger-Nuttall rule)

[See KII, fig. 8.1 p. 249]

The smoothest, systematic behavior is observed in the decay of even-even nuclei. These systems are less sensitive to shell model effects, since no pair bonds are being broken.



odd/even, even/odd, odd/odd nuclei

$t_{1/2}$ is $2 \rightarrow 1000$ times greater, since pair bonds need to be broken. Plots of $t_{1/2}$ vs Q are also much more scattered.

Geiger-Nuttall law (empirical)

$$\log_{10} \lambda = C - \frac{D}{\sqrt{E_\alpha}} \quad \text{for} \quad \log_{10} t_{1/2} = -C' + \frac{D}{\sqrt{E_\alpha}}$$

As n increases in a given isotopic line, Q decreases
(except in the vicinity of a closed shell)

$$Q = B(^4\text{He}) + B(Z-2, A-2) - B(Z, A)$$

$$\approx 28.3 - 4a_v + \frac{8}{3}a_s A^{-1/3} + 4a_c Z A^{-1/3} (1 - Z/3A) \\ - 4a_{\text{sym}} \left(1 - \frac{2Z}{A}\right)^2 + 3a_p A^{-7/4}$$

assumed $Z, A \gg 1$

$$\frac{\partial Q}{\partial A} = -\frac{8}{9}a_s A^{-1/4} - \frac{4a_c Z A^{-4/3} \left(1 - \frac{4}{3}\frac{Z}{A}\right)}{3} \\ + 16a_{\text{sym}} \frac{Z}{A^2} \left(1 - \frac{2Z}{A}\right) - \frac{7}{4}a_p A^{-11/4}$$

$$Z \approx \frac{2}{3}A$$

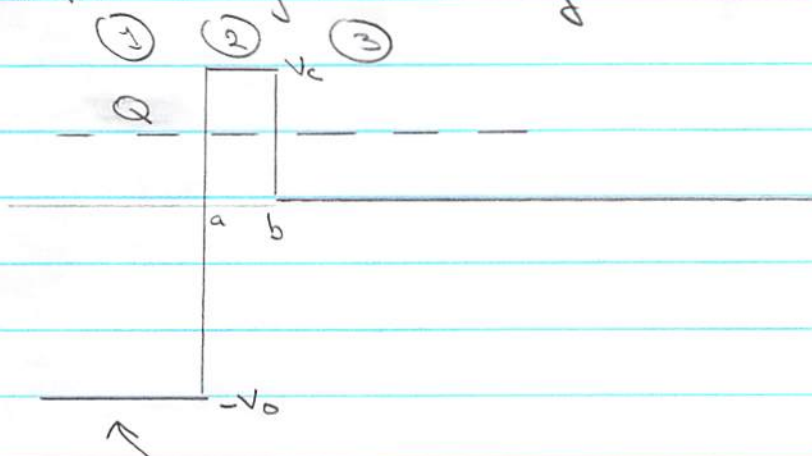
sums
all other
terms

$$a_s = 16.8 \quad a_c = 0.72 \quad a_{\text{sym}} = 23 \\ a_p = 34$$

$$\frac{\partial Q}{\partial A} = -\frac{8}{9}a_s - \frac{56}{225}a_c A^{-1/3} + \frac{32}{25}a_{\text{sym}} A^{-1} \\ - \frac{7}{4}a_p A^{-11/4}$$

See XII fig. 8.2 p.250

Observe $\frac{\partial Q}{\partial A}$ is about the same.

Simple Theory of α -decay 14.4.1

represents the binding potential of the nucleus,
attractive, depth $-V_0$ $V_0 = 35 \text{ MeV}$

V_C is the Coulomb barrier, repulsive

a = nuclear radius $R_0 A^{1/3}$ usually used
 b = outside the Coulomb barrier

Q = Q -value of the reaction

1-D formalism [note 3D $l=0$ formalism same as 1D]

Solution	Zone 1	$u_1 = A e^{ik_1 x} + B e^{-ik_1 x}$
	Zone 2	$u_2 = C e^{k_2 x} + D e^{-k_2 x}$
	Zone 3	$u_3 = F e^{ik_3 x}$, only $\rightarrow$

$$k_1 = \left[\frac{2m}{\hbar^2} (V_0 + Q) \right]^{1/2}$$

$$k_2 = \left[\frac{2m}{\hbar^2} (V_C - Q) \right]^{1/2}$$

$$k_3 = \left[\frac{2m}{\hbar^2} Q \right]^{1/2}$$

Apply u & u' continuity at $x=a, b$

$\Rightarrow 4$ equations. Solve for B, C, D, F in terms of A

Transmission coefficient

$$T = \frac{k_3 |F|^2}{k_1 |A|^2}$$

Grind the crack...

$$T = \left[\frac{1}{4} \left(2 + \frac{k_1}{k_3} + \frac{k_3}{k_1} \right) + \frac{\sinh^2 k_2(b-a)}{4} \left(\frac{k_1 k_3 + k_3^2 + k_1 + k_2}{k_1^2 k_2 k_3 k_2 k_1} \right) \right]^{-1}$$

[Last year we solved the case $k_1 = k_3$ (i.e. $V_0 = 0$)]

$$k_2(b-a) = \sqrt{\frac{2m(V_0 - Q)}{\hbar^2}} (b-a)$$

$$= \left[\frac{2m^2(V_0 - Q)(b-a)^2}{(\hbar c)^2} \right]^{1/2}$$

$$\approx \left[\frac{\overbrace{3000 \text{ MeV}}^{2mc^2} (5 \text{ MeV}) (35)^2 \text{ fm}^2}{(200)^2 \text{ MeV}^2 \text{ fm}^2} \right]^{1/2}$$

$$\approx 35$$

$$T_\alpha = \left[\frac{1}{4} \left(2 + \frac{k_1}{k_2} + \frac{k_3}{k_1} \right) + \frac{e^{2k_2(b-a)}}{16} \left(\frac{k_1 k_3}{k_2^2} + \frac{k_2^2}{k_1 k_3} + \frac{k_3}{k_1} + \frac{k_1}{k_3} \right) \right]^{-1}$$

$$\text{Decay rate} = \frac{16}{e^{-2k_2(b-a)} \left(\frac{k_1 k_3}{k_2^2} + \frac{k_2^2}{k_1 k_3} + \frac{k_3}{k_1} + \frac{k_1}{k_3} \right)}$$

$$\lambda = f T_\alpha$$

↑ frequency that the α encounters the boundary

Krane	$f = \frac{v_\alpha}{a}$	$\sim 10^{21} \text{ s}^{-1}$	} α 1-2 orders of magnitude too low
Others	$f = \frac{v_\alpha}{2a}$	$\sim 5 \times 10^{20} \text{ s}^{-1}$	

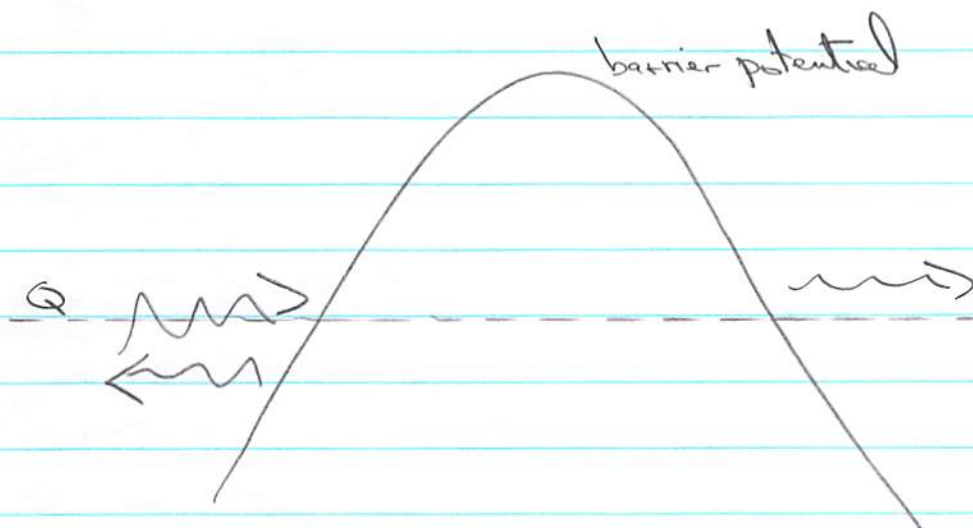
Fermi's Golden Rule #2	$3 \times 10^{24} \text{ s}^{-1}$	} α 1-2 orders of magnitude too high
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The correct answer (experiment) lies somewhere in between. There is no known theory for this frequency factor, yet.

So, our only hope is to somehow understand something about the relative trends.

Gamow's theory of α -emission

Approximate solution of the SE



$$T = \exp \left\{ -2 \left(\frac{2m}{\hbar^2} \right)^{1/2} \int_a^b dx [V(x) - Q]^{1/2} \right\}$$

Some comments are in order.

Compare with the previous result

$$\text{Let } V(x) = V_c$$

$$T_{\text{Gamow}} = \exp \left\{ -2 \sqrt{\frac{2m}{\hbar^2}} (V_0 - E) (b-a) \right\}$$

$$T_{\text{exact}} = \frac{16}{\left(\frac{k_1 k_3}{k_2^2} + \frac{k_2^2}{k_1 k_3} + \frac{k_2}{k_1} + \frac{k_1}{k_3} \right)} T_{\text{Gamow}}$$

This is a factor 2-3
for most applications

$\Rightarrow$ Gamow theory is more general (arbitrary potential)

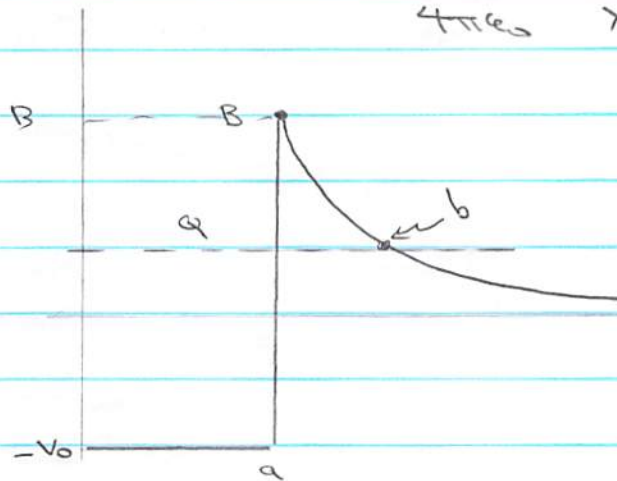
but, it can't be expected to get exact results, off by a factor of 2 or 3

This is inconsequential compared to the f , which is unknown to several orders of magnitude

Specialization to the Coulomb potential
(single form) (Krane 8.4) (

$$V(x) = -V_0 \quad x < a \quad \text{inside the nucleus}$$

$$= \overset{\alpha}{\text{daughter}} \frac{2(Z-2)e^2}{4\pi\epsilon_0} \frac{1}{x} \quad x > a$$



height of Barrier $B = \frac{2(Z-2)e^2}{4\pi\epsilon_0} \frac{1}{a}$

At b , $Q = V(b)$

i.e. $Q = \frac{2(Z-2)e^2}{4\pi\epsilon_0} \frac{1}{b}$

note $\frac{Q}{B} = \frac{a}{b}$

Or, $b = \frac{2(Z-2)e^2}{4\pi\epsilon_0} \frac{1}{Q} \quad (30-60 \text{ fm})$
 extends well beyond the nucleus

$$\left[\frac{2.99 \text{ MeV} \cdot (Z-2)}{Q} \right]$$

$$T_c = \exp \left\{ -2 \left(\frac{2m}{\hbar^2} \right)^{1/2} \int_a^b dx \left[\frac{2(z-2)e^2}{4\pi\epsilon_0} \frac{1}{x} - Q \right] \right\}$$

}

$$= \exp \left\{ -2 \sqrt{\frac{2mc^2 a^3 B}{(\hbar c)^2}} \left(\frac{\cos^{-1}(\sqrt{\alpha}) - \sqrt{\alpha(1-\alpha)}}{\sqrt{\alpha}} \right) \right\}$$

$$\alpha = \frac{Q}{B} = \frac{a}{b}$$

This is extremely sensitive to Q and b

Numerical results

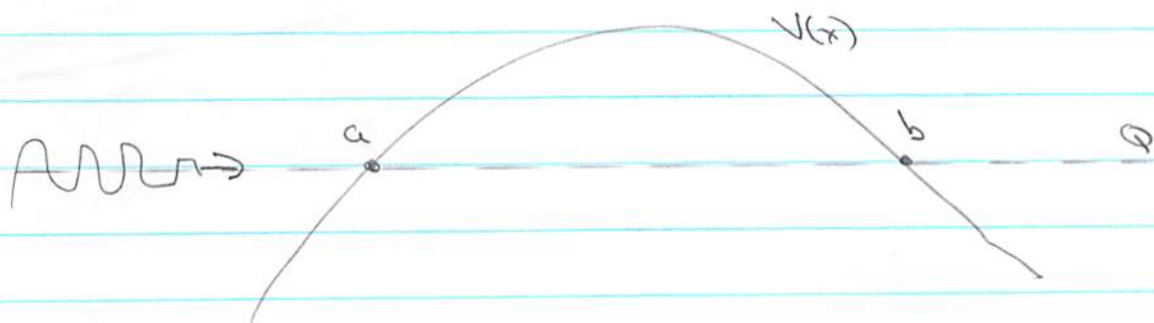
$$I_{1/2} = \frac{\log(2)}{f T_c}$$

8.4 Kramers discussion of α -emission theory

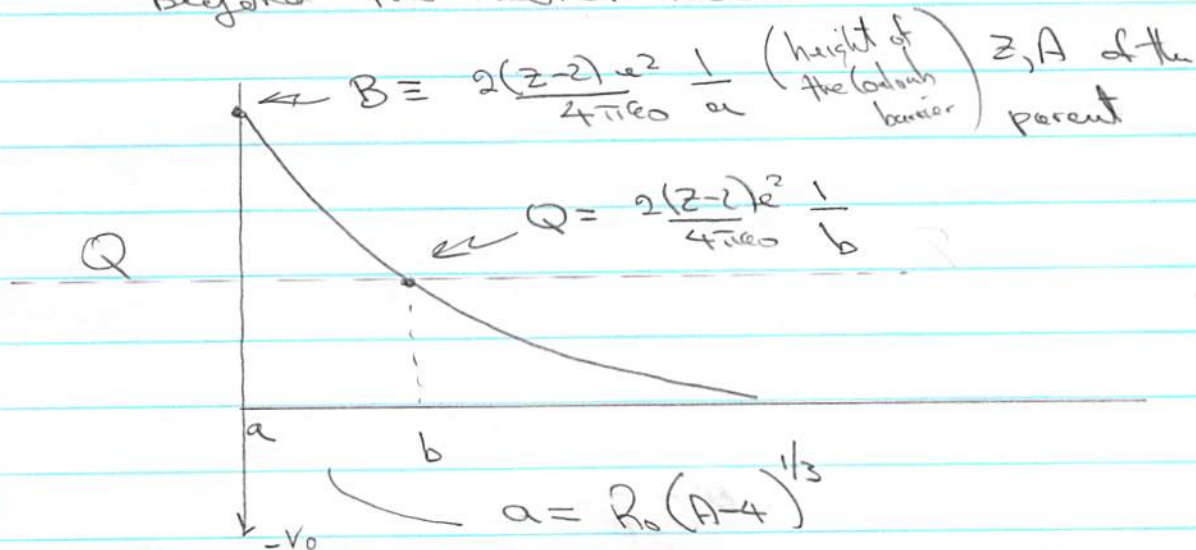
Use Gamow's transmission theory

$$T = \exp\left\{-2\left(\frac{2m}{\hbar^2}\right) \int_a^b dx [V(x) - Q]^{1/2}\right\}$$

a, b are the classical 'turning points'



Let $V(r)$ be the charge of a ^{spherical} nucleus, treated as if the α -particle were already beyond the nuclear radius



b can be expressed in terms of Q as follows

$$b = \frac{2(z-2)e^2}{4\pi\epsilon_0 Q}$$

$$\text{So, } T_g = \exp \left\{ -2 \left(\frac{2m}{\hbar^2} \right)^{1/2} \left[\int_a^b dx \left(\frac{2(z-2)e^2}{4\pi\epsilon_0 x} - Q \right)^{1/2} \right] \right\}$$

Note that b is quite large 30-60 fm or so, using typical values. Hence $b-a$ is very much dependent on Q .
 $a = R_0(A-4)^{1/3}$ is also affected by A .

$$\text{Note also that } \frac{a}{b} = \frac{Q}{B}$$

$$\textcircled{1} \quad T_g = \exp \left\{ -2 \left(\frac{2m c^2 a^2 B}{(\hbar c)^2} \right)^{1/2} \left[\frac{\cos^{-1}(\sqrt{a}) - \sqrt{a(1-a)}}{\sqrt{a}} \right] \right\}$$

The numerical results quoted by Krane

$$A_{1/2} = \frac{\log(z)}{f T_g}$$

$$f = \frac{\sqrt{a}}{a}$$

Some notes:

Since the potential is spherically symmetric

$$m_{\alpha}' \rightarrow \frac{m_{\alpha} m_{X'}}{m_{\alpha} + m_{X'}} \doteq \frac{m_{\alpha} A'}{4 + A'}$$

$$m_{\alpha}' = \frac{m_{\alpha}}{1 + 4/A'}$$

small!

This changes the result by about a factor of 2-3! (Even for heavy nuclei)

Question: If α -decay can occur, surely

${}^8\text{Be}$ ${}^{12}\text{C}$ emission can occur?

Answer: Anything that can happen, will happen.

But it's all a matter of rate

$$\alpha = \frac{Q}{B} \quad B = \frac{\overset{\text{escape}}{3(Z-2)}}{4\pi\epsilon_0 a} \quad \text{changes}$$

$$m_{\alpha} \rightarrow m_{{}^8\text{Be}} \text{ or } m_{{}^{12}\text{C}}$$

Question: Simple estimates of f , the boundary encounter frequency, are off by $1 \rightarrow 2$ orders of magnitude. Should we even bother with refined estimates of T_0 ?

Answer: Yes! The T_0 theory is quite robust. If we can refine it, measurements can isolate the systematic behavior of f , and this might lead to more insight into nuclear structure.

More refined $V(r)$?

The nuclear binding force is thought to be proportional to the nuclear density, namely

$$V(r) = \frac{-V_0}{1 + \exp[(r-R)/a]}$$

$$r = R, \quad V(r) = -V_0/2$$

$a \equiv \text{skin depth}$

The approximate behavior is

$$T_3 = (\overline{T}_\alpha)^{3/2}$$

So for ${}^8\text{Be}$ $z=4$

$$\begin{aligned} \overline{T}_{\text{Be}} &\cong (\overline{T}_\alpha)^2 \\ \overline{T}_{\text{He}} &\cong (\overline{T}_\alpha)^3 \end{aligned} \quad \left. \vphantom{\begin{aligned} \overline{T}_{\text{Be}} &\cong (\overline{T}_\alpha)^2 \\ \overline{T}_{\text{He}} &\cong (\overline{T}_\alpha)^3 \end{aligned}} \right\} \begin{array}{l} \text{these are} \\ \text{vanishingly} \\ \text{small} \end{array}$$

There are some observations
of this, however

Finally

Krone likes to approximate

$$(\alpha^{-1}\sqrt{\alpha} - \sqrt{\alpha}(\alpha)) \xrightarrow{\text{small } \alpha} \frac{\pi}{2} - 2\sqrt{\alpha}$$

bad idea! Typically $\alpha \approx 0.3$ $\sqrt{\alpha} \approx 0.55$

If one wants to use a Taylor expansion,
we must push it to many more terms.

Let's assume that $\rho_r = \frac{\rho_0}{1 + \exp[(r-r)/a]}$

Using Gauss's Law

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\int_0^r r'^2 dr' \rho(r')}{\int_0^\infty r'^2 dr' \rho(r')}$$

From this, get $V(r) = \int dr E(r) + C$

Where C is determined by setting the

$\frac{1}{r}$ form as $r \rightarrow \infty$

Then

$$T_a = \exp\left\{-2\left(\frac{2m}{\hbar^2}\right)^{1/2} \int_a^b [V_s(r) + V_c(r) - Q]^{1/2} dr\right\}$$

good extra credit project, if anyone is interested

What about angular momentum?

$$V_c(r) \rightarrow \underbrace{V_{c,l=0}(r)}_{\substack{\text{what} \\ \text{we} \\ \text{have} \\ \text{been} \\ \text{using} \\ \text{so far}}} + \underbrace{\frac{l(l+1)\hbar^2}{2mr^2}}_{\substack{\text{true,} \\ \text{additional} \\ \text{barrier} \\ \text{to penetrate}}}$$

$$Z=90 \\ Q=4.5 \text{ MeV}$$

	0	1	2	3	4	5	6
T_l/T_0	1	0.84	0.60	0.36	0.18	0.075	0.028

$l \uparrow$, probability of transmission $\downarrow$

α -decay from O^+ nuclei

$$I_\alpha = 0 \text{ hence}$$

$$\underbrace{\vec{I}_i}_0 + \underbrace{\vec{I}_\alpha}_0 + \vec{I}_\alpha = \vec{I}_f$$

$$\text{Hence } I_f = I_\alpha$$

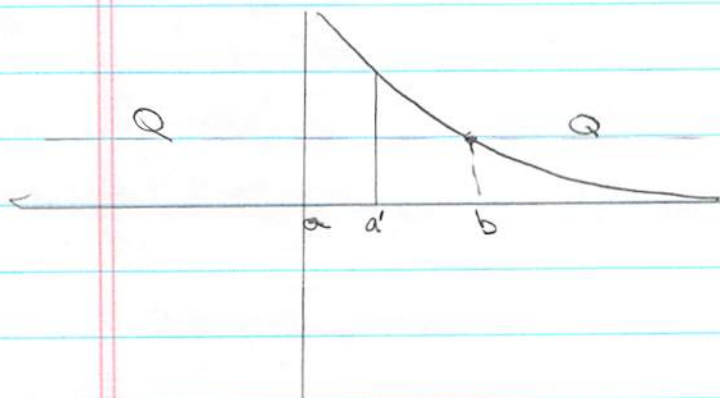
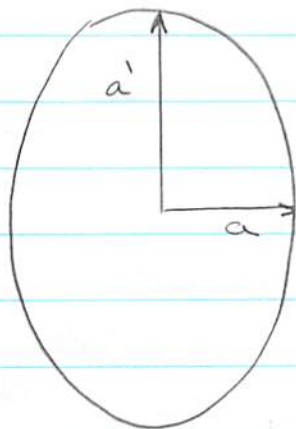
$$\text{Also } \pi = (-1)^l$$

Possible final states are

allowed	$0^+, 2^+, 4^+, 6^+ \dots$	$1^-, 3^-, 5^- \dots$
forbidden	$0^-, 2^-, 4^-, 6^- \dots$	$1^+, 3^+, 5^+ \dots$

Show graph 9.7 on page 258

Ellipsoidal nuclei



In the probe case, those α 's leaving through the polar regions are more probable since the barrier is thinner,