A Gravitational Interior Point Method for LP

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Abstract

In [4, 1] gravitational methods for linear programming (LP) have been introduced. Several versions exist, the three main versions discussed there use a ball of **(a):** 0 radius, **(b):** small positive radius, and **(c):** the ball of largest possible radius with the given center that will completely fit within the polytope, with the option of changing its radius as the algorithm progresses.

In versions **(a), (b)**, after the first move, the center of the ball always remains very close to the boundary (because the ball hugs the boundary), and hence these versions behave like other boundary algorithms such as the simplex algorithm in terms of exponential complexity in the worst case [3].

Here we discuss a gravitational method of type **(c)** that behaves like an interior point method. To guarantee that the ball used has the largest possible radius, it uses a new centering strategy that moves any interior feasible solution x^0 to the center of the intersection of the feasible region with the objective hyperplane through x^0 before beginning each gravitational descent move. Also, using this centering strategy we discuss a method that can obtain an approximate optimum solution for an LP by a very efficient method without using any matrix inversions.

Key words: Linear programming (LP), gravitational method, interior point method, avoiding zigzagging, approximate optimum without matrix inversions.

1 The Importance of Linear Programming

Linear algebra dealing with methods for solving systems of linear equations is the classical subject that initiated the study of mathematics in prehistoric times. The most effective methods for solving systems of linear equations have been discovered over 2500 years ago, these methods are still the leading algorithms in use today.

Even though **linear equations** have been conquered thousands of years ago, systems of linear inequalities remained inaccessible to humans until the middle of the 20th century. Linear Programming (LP) is the branch of mathematics developed in the 20th century as an extension of linear algebra to solve systems of **linear inequalities**. The development of LP is a landmark event in the history of mathematics and its applications that brought our ability to solve general systems of linear constraints (including linear equations, inequalities) to a state of completion.

Soon after its development, LP has become the dominant subject in the development of efficient computational algorithms, study of convex polyhedra, and in algorithms for decision making. But for a short time in the beginning, its potential was not well recognized. **Dantzig** tells the story of how when he gave his first talk on LP and his simplex method for solving it at a professional conference, Hoteling dismissed it as unimportant since everything in the world is nonlinear. But Von Neuman defended Dantzig saying that the subject will bercome very important. See Page xxvii of [Dantzig, Thappa, 2, 1997]. The preface in this book contains an excellent account of the early history of LP from the inventor of the most successful method in OR and in the mathematical theory of polyhedra.

Von Neuman's early assessment of the importance of LP turned out to be astonishingly correct. Today, the applications of LP in almost all areas of science are so numerous, so well known and recognized that I do not have to repeat them here. Also, LP seems to be the basis for most of the efficient algorithms for many problems in other areas of mathematical programming. Many of the successful approaches in nonlinear programming, discrete optimization, and other branches of optimization are based on LP in their iterations.

2 Algorithms Used for Solving LPs Today

The simplex method developed by Dantzig in the 1940s is still the dominant algorithm in use for solving LPs. The simplex method exhibits exponential growth in the worst case, but its performance in practice has been outstanding, and is being improved continuously by developments in implementation technologies. It is a **one-dimensional boundary method** in the sense that it follows a path along the edges (one-dimensional boundary faces) of the set of feasible solutions of the LP. In each iteration it needs either updating the inverse, or computing the inverse of a matrix of order m (number of constraints in the LP not counting the bounds on individual variables).

In the late 1900s a variety of interior point methods (IPMs) have been developed for LP [7, 11]. These IPMs follow a path through the interior of the set of feasible solutions. Among them the ones that give the best performance try to follow the central path (path through a mathematically defined center of the set of feasible solutions) approximately. They are based on very elegant theory, and converge to a near optimum in polynomial time. In practice the number of iterations needed by them is much smaller than that for the simplex method, but each of these steps is much harder and more complex as it needs the inversion of a matrix of order larger than m. Also taking advantage of sparcity in their implementations is a much more complex task than that in implementations of the simplex method. The IPMs have been observed to give slightly better performance than the simplex method only on large scale sparce problems.

All these methods in use today need either the updating of a matrix inverse, or inversion of a matrix in each step, this is the hard part of solving LPs in spate of all the improvements that have occurred so far. It seems that practitioners are quite content with obtaining a solution not necessarily optimal, but close to being so, but they want a method that can obtain such a solution much faster than existing methods. Problems in all areas of optimization can be solved much more efficiently if faster and more efficient algorithms can be developed for obtaining solutions close to the optimum for LPs.

3 The Gravitational Method for LP

Starting about 20 years ago, Murty[4, 1983], Chang and Murty[1, 1989] developed a new method for LP based on the principle of the gravitational force. We consider the LP in the form

$$\begin{array}{rrcll} \text{maximize} & \pi b & & & \\ \text{subject to} & \pi A & = & c, & \pi \geq 0 \end{array} \tag{1}$$

where A is a matrix of order $m \times n$, $\pi \in R^m$ is the row vector of primal variables. Its dual is

$$\begin{array}{rrcl} \text{minimize} & z(x) = cx & & \\ \text{subject to} & Ax & \geq & b \end{array} \tag{2}$$

where $x \in R^n$ is the column vector of dual variables.

For any matrix D, we use the symbols $D_{i.}, D_{.j}$ to denote the i-th row, j-th column of D. If D is of order $m \times n$, and $S \subset \{1, \dots, m\}$, $D_{S.}$ denotes the submatrix of D consisting of rows $D_{i.}$ for all $i \in S$. We denote the cone consisting of all the nonnegative combinations of row vectors of D by $\text{Rpos}(D)$. For any vector y, $||y||$ denotes its Euclidean norm.

The method is applied on (2) and needs an initial interior point x^0 of its feasible region K (i.e., satisfying $Ax^0 > b$). It introduces a spherical drop (we will refer to it as the *drop* or the *ball*) of small radius with center x^0 lying completely in the interior of K, and traces the path of its center as the drop falls under a gravitational force pulling it in the direction $-c^T$. The drop cannot cross the boundary of K, so after an initial move in the direction $-c^T$ it will be blocked by the face of K that it touches; after which it will start rolling down along the faces of K of varying dimensions. Hence the center of the drop will follow a piecewise linear descent path completely contained in the interior of K, but since the drop's radius is small, the center remains very close to the boundary of K after the first change in direction in its path. Therefore the method is essentially a boundary method. However, unlike the simplex method which follows a path strictly along the one dimensional boundary of K, this method is a **higher dimensional boundary method** in which the path followed remains very close to faces of K of varying dimensions.

After a finite number of changes in the direction of movement, the drop will reach the lowest possible point in the direction $-c^T$ that it can reach within K and halt. If the radius of the drop is sufficiently small, the touching constraints (i.e., those whose corresponding facets of K are touching the ball) in (2) at this final halting position will determine an actual optimum solution of the LP (1). If its radius is not small enough, the direction finding step in the method at the final halting position with center x^* yields a feasible solution $\tilde{\pi}$ of (1) and the optimum objective value in (1) lies in the interval $[\tilde{\pi}b, cx^*]$. Then the radius of the drop is reduced and the method continues the same way. In [1] finite termination of the method to find an optimum solution has been proved.

[Morin, Prabhu, and Zhang, 3, 2001] have shown that this version of the gravitational method using point drops (i.e., drops of radius 0) has exponential growth in the worst case like the simplex method.

4 How to Make the Gravitational Method Efficient?

It is clear that in order to make the gravitational method efficient, it is necessary to keep the center of the drop from hugging the boundary of K all along its path, i.e., make the method a truely interior point method. This can be achieved by making the radius of the drop as large as possible by moving its center to the center of the set of feasible solutions. For this we develop a new centering strategy that is very different from centering strategies used in other IPMs. Its biggest advantage is that it needs no matrix inversion, and hence is lot simpler than other centering strategies. We will describe this centering strategy next.

5 The Centering Strategy

We assume that $||A_{i.}|| = 1$ for all i. K denotes the set of feasible solutions of (2), and K^0 its interior. We assume that an initial interior feasible solution $x^0 \in K^0$ for (2) is available. If such an initial point is not avilable, we modify the problem using the usual big-M augmentation with one artificial variable as follows

$$\begin{array}{rrcl} \text{minimize} & cx + Mx_{n+1} & & \\ \text{subject to} & Ax + ex_{n+1} & \geq & b, \quad x_{n+1} \geq 0 \end{array}$$

where $e = (1, ..., 1)^T \in R^m$ and M is a positive number significantly larger than any other number in the problem. Let $x^0_{n+1} > \max\{0, b_1, ..., b_m\}$. Then $(0, ..., 0, x^0_{n+1})^T$ is a strict interior feasible solution of the modified problem which is in the same form as (2).

So, we assume that a strict interior feasible solution x^0 of (2) is available. We also assume that $c \neq 0$, as otherwise 0 is already an optimum solution of (1). We normalize c, so that $||c|| = 1$.

Without any loss of generality we assume that each constraint in (2) determines a facet of K. Let $H_i = \{x : A_{i.}x = b_i\}$ be the i-th facetal hyperplane for K. Let $H^0 = \{x : cx = cx^0\}$ be the objective hyperplane through the current point x^0.

Since x^0 is in the interior of K, $A_{i.}x^0 > b_i$ for all $i = 1$ to m. Then $\delta^0{}_i = A_{i.}x^0 - b_i$ is the distance (Euclidean) of x^0 from H_i. With x^0 as center, the largest sphere we can construct within K has a radius $\min\{\delta^0_i : i = 1$ to $m\}$. This may be too small. To construct even larger drops inside K, we need to move the center of the drop from x^0 to a better interior point. Starting with x^0, the centering strategy tries to find a new position for the center of the drop inside K^0 that maximizes the radius of the drop that can be constructed within K. It does this while keeping the objective value at the new center the same as that at x^0, by including $cx = cx^0$ as a constraint that the new center x has to satisfy.

So the new center x is chosen from $K^0 \cap H^0$ and maximizes $\min\{A_{i.}x - b_i : i = 1$ to $m\}$. The model for this choice is:

$$\begin{array}{rrcll} \text{Maximize} & \delta & & & \\ \text{subject to} & \delta & \leq & A_{i.}x - b_i, & i = 1 \text{ to } m \\ & cx & = & cx^0 & \end{array} \qquad (3)$$

This is another LP, and if $(\bar{x}, \bar{\delta})$ is its optimum solution, $\bar{x}$ is the new center, and $\bar{\delta}$ is the maximum radius for the drop within K^0 subject to the constraint that its center lie on $K^0 \cap H^0$.

But this itself is another LP, perhaps as difficult to solve as the original LP (1), or (2) itself. Also, this type of model may have to be solved several times before we get a solution for our original LP, so solving this model (3) exactly will be counterproductive. For this reason we use the following procedure to get an approximate solution for (3).

Procedure for Getting an Approximate Solution for (3)

In this procedure, for finding the new center $x \in K^0 \cap H^0$, we only consider moves in directions perpendicular to the facetal hyperplanes of K, since our goal is to increase the minimum distance of x from a facetal hyperplane. These directions (with positive or negative step lengths) are: $A_{i.}^T$ for $i = 1$ to m. However since x has to lie on H^0, the actual directions of movement considered are: $P_{.i} = A_{i.}^T + c^T(cx^0 - cA_{i.}^T)$, where $P_{.i}$ is the orthogonal projection of $A_{i.}^T$ on H^0, for $i = 1$ to m.

So, this procedure consists of a series of moves beginning with x^0, generating a sequence of points x^r in $K^0 \cap H^0$. When at x^r look for an i between 1 to m such that $f_{ir}(\alpha) = \min\{A_{t.}(x^r + \alpha P_{.i}) - b_t : t = 1, \ldots, m\}$ is increasing as α changes from 0 to positive or negative values.

If such an i does not exist, terminate this step with x^r as the position for the center of the next ball, and fix that ball to be the one with x^r as the center and radius as $\min\{A_{i.}x^r - b_i : i = 1, \ldots, m\}$.

If such indices i exist, select one of them. Keep changing α in the direction from 0 that keeps on increasing $f_{ir}(\alpha)$, until it reaches a value α_r where $f_{ir}(\alpha)$ stops increasing as α changes beyond α_r. If $\theta_r = f_{ir}(\alpha_r)$, then (θ_r, α_r) is the optimum solution of the following 2-variable LP in which the variables are θ, α.

$$\begin{array}{rrcll} & \text{Maximize} \quad \theta & & & \\ \text{subject to} & \theta - \alpha A_{t.}P_{.i} & \leq & A_{t.}x^r - b_t & \quad t = 1, \ldots, m \end{array}$$

which can be found with at most $O(m)$ effort.

Then take the next point in the sequence to be $x^{r+1} = x^r + \alpha_r P_{.i}$. Repeat the same process, until moves of this type are not possible any more, or when improvements in the values of $f_{ir}(\alpha)$ become small.

If x^* is the final point, take the ball to be the one with x^* as center and $\min\{A_{i.}x^* - b_i : i = 1, \ldots, m\}$ as the radius. It is the ball of largest possible radius that can be fixed inside K with its center on the current objective plane by the simple process used in this step.

As can be seen, the procedure used in this centering strategy does not need any matrix inversion, and only solves a series of 2-variable LPs which can be solved very efficiently. Hence this centering strategy can be expected to be efficient.

We define the **central path** to be the path of the center of the drop in its descent to the optimal face of (2).

6 Stage 1, Repititions of Iteration 1 in the Gravitational Interior Point Method to Solve (1), (2)

By fixing the initial drop as the largest possible ball with its center in $K^0 \cap H^0$, the centering strategy pushes the center of the ball close to the center of $K^0 \cap H^0$. Stage 1 of the overall method consists of repitions of a special iteration that exploits this property to get as much reduction in the objective value of (2) as possible using cheap effort cosisting of the following two steps repeatedly. These steps in this iteration are described below, some changes in the second step to accelerate convergence will be discussed later. This iteration begins with x^0 as the initial interior feasible solution.

Iteration 1

BEGIN

Step 1.1: Centering: Let $H^0 = \{x : cx = cx^0\}$. Starting with x^0 apply the centering strategy of Section 5 to get the largest ball $B(x^*, \delta)$ with $x^* \in K^0 \cap H^0$ as center and δ as radius. Go to Step 1.2.

Step 1.2: Descent Move Following Centering: This move does not use the ball $B(x^*, \delta)$ constructed in Step 1.1 at all, it only uses its center x^* and its property of being close to the center of $K^0 \cap H^0$. It takes a maximum possible step from x^* in a descent direction for cx.

If this is the first time this step is being carried out, the only descent direction that we have readily available at this time is $-c^T$, and we use that as the direction to move from x^*.

If this is not the first time this step is being carried out, besides $-c^T$ we have another descent direction for cx, the direction of the central path at the current point x^*, which can be approximated by $x^* - \tilde{x}$ where $\tilde{x}$ is the center of the drop in the previous time Step 1.2 is carried out.

If $d \in \{-c^T, x^* - \tilde{x}\}$ is the direction selected for moving from x^*, we will move in this direction the maximum distance possible while still remaining inside K^0.

A small tolerance $\epsilon > 0$, is selected and a move to the farthest point satisfying $A_{i.}x \geq b_i + \epsilon$ for $i = 1$ to m is made. This gives

$$\begin{aligned} \bar{x} &= x^* + \gamma d \\ \gamma &= \min\{\frac{-A_{i.}x^* + b_i + \epsilon}{A_{i.}d} : i \text{ such that } \ A_{i.}d < 0\} \end{aligned}$$

The decrease in the objective value in this move is $|\gamma cd|$. Select the direction d from $\{-c^T, x^* - \tilde{x}\}$ to be the one which yields the maximum decrease in the objective value in this move. Make $\bar{x}$ obtained after the move the new x^0, and go back to Step 1.1 for another repeat of this Iteration 1.

END

Zigzagging

Repitions of this Iteration 1 may encounter **zigzagging** (a phenomenon commonly discussed in nonlinear programming) if the direction to move from x^* is always taken to be $-c^T$. Ziczagging occurs when the center of the ball gets trapped in a narrow cone like region, with successive balls having the same touching set of constraints repeatedly. If this occurs (likely when the center nears the optimal face, if the optimal face is of low dimension) each successive repitition of Iteration 1 makes progressively decreasing improvements. Taking the direction to move in each repitition of Iteration 1 to be the best one among $\{-c^T, x^* - \tilde{x}\}$ helps to prevent zigzagging from occurring.

We continue repeating applications of Iteration 1 until the improvement in the objective value in each application becomes small. Let $\bar{x}$ denote the interior feasible solution in this final repition of Iteration 1 in Stage 1.

If the centering strategy performs well, this final solution $\bar{x}$ in Stage 1 can be expected to be such that $c\bar{x}$ is quite close to the minimum objective value in (2) (i.e., $\bar{x}$ can be expected to be a near optimum to (2)). Several efficient strategies developed in LP theory are available to get an approximate optiumum to (1) (2) from $\bar{x}$.

Stage 1 has the aim of getting as close to the optimum as possible without matrix inversions. The final point obtained in Stage 1 may itself be a reasonable approximation to the optimum in some practical applications.

If a true optimum solution of (1), (2) is needed, strating from the final point $\bar{x}$ obtained at the end of Stage 1 as the current point we go to Stage 2 which carries out applications of the general iteration, Iteration 2, that consists of the following steps besides the centering step: Gravitational direction finding step, Step length determination and the main move, Additional move of the center, What to do if the ball halts. There are several possible options for selecting the gravitational direction along which the ball will move, we will first discuss these in detail.

7 The Gravitational Direction Finding Step

In the versions of the gravitational methods discussed in [1, 4], the initial ball is always selected to have a very small radius so that it is completely inside K^0 without any of the boundary faces of K touching it, so the initial move in the method always takes place in the direction $-c^T$, and could be of very short length depending on the location of the center of the initial ball. But here, the centering step in Section 5 used for selecting the initial ball makes sure that it is already touching some facets of K, these are called the *touching facets*, and the constraints that define them are called the *touching constraints*. Let:

$B(x^*, \delta)$ = the current ball inside K with center x^* and radius δ

$J(x^*, \delta)$ = $\{i : A_{i.}x^* = b_i + \delta\}$, the index set of touching constraints for $B(x^*, \delta)$

$G(x^*, \delta)$ = $\{y : cy < 0,\ A_{i.}y \geq 0 \text{ for all } i \in J(x^*, \delta)\}$, the set of descent feasible directions for $B(x^*, \delta)$.

The gravitational direction at x^* is a direction selected from $G(x^*, \delta)$ along which the current ball $B(x^*, \delta)$ will be moved. Various options for selecting this direction are given below.

The Steepest Descent Gravitational Direction (SDGD)

Defined in [1] and used in the SDGM (Steepest Descent Gravitational Method) discussed there, this is the steepest descent direction among all those in $G(x^*, \delta)$. So, the SDGD is the optimum solution of

$$\begin{array}{rrcl} & \text{Minimize} \quad cy & & \\ \text{subject to} & A_{J(x^*,\delta).}y & \geq & 0 \\ & 1 - y^T y & \geq & 0 \end{array} \tag{4}$$

In [1] it has been proved that this problem is equivalent to the problem

$$\begin{array}{rl} \text{Minimize} & (c - \eta A_{J(x^*,\delta).})(c - \eta A_{J(x^*,\delta).})^T \\ & \text{subject to} \quad \eta \ \geq \ 0 \end{array} \tag{5}$$

which is a nearest point problem (finding nearest point to c in Rpos($A_{J(x^*,\delta).}$) = the nonnegative hull of row vectors in the matrix $A_{J(x^*,\delta).}$).

If $\bar{\eta}$ is an optimum solution of (5), then $\bar{y} = 0$ if $\bar{\xi} = (c - \bar{\eta} A_{J(x^*,\delta).}) = 0$, or $-\bar{\xi}^T / ||\bar{\xi}||$ otherwise, is an optimum solution of (4).

Also, if $\bar{\xi} = 0$, $G(x^*, \delta) = \emptyset$, i.e., the ball $B(x^*, \delta)$ cannot move from its present position in gravitational descent, hence it halts in its present position. If this happens, let $\bar{\pi} = (\bar{\pi}_i)$ where $\bar{\pi}_i = \bar{\eta}_i$ if $i \in J(x^*, \delta)$, 0 otherwise. Then $\bar{\pi}$ is feasible to (1), and the optimum objective value in (1) lies in the interval $[\bar{\pi} b, cx^*]$. In this case the method goes to the step to carry out when the ball halts, discussed in Section 8.

If $\bar{\xi} \neq 0$, go to the Main move step with $\bar{y}$ as the gravitational direction for the move. Now the method goes to the main move discussed in Step 2.5 of Section 8.

Modified Gravitational Directions MGD1, MGD2

Computing the SDGD becomes simplified if the cone Rpos($A_{J(x^*,\delta).}$) is simplicial, i.e., if $A_{J(x^*,\delta).}$ is of full row rank, which may not be the case always. So, in [1], simplified versions of gravitational directions MGD1, MGD2 are discussed. In these versions, the nearest point problem (5) for finding the gravitational direction is modified by replacing the matrix $A_{J(x^*,\delta).}$ by a submatrix D of it consisting of a maximal linearly independent subset of row vectors of $A_{J(x^*,\delta).}$. So, computing MGD1 requires the solution of the nearest point problem

$$\begin{array}{rrcl} \text{Minimize} & (c - \eta D)(c - \eta D)^T & & \\ & \text{subject to} \quad \eta & \geq & 0 \end{array} \tag{6}$$

which is the problem finding the nearest point in Rpos(D) to c.

So, MGD1 is the direction obtained as in the SDGD, with $\bar{\eta}$ being the optimum solution of (6) instead of (5). (6) can be solved very efficiently by geometric methods discussed in [5, 6, 8, 9, 10] using the concept of projection faces of the simplicial cone Rpos(D).

MGD2 simplifies the effort needed to find the gravitational direction even further by taking the vector $\bar{\eta}$ to be not the optimum solution of (6), but the one corresponding to a projection face of Rpos(D) that is closer than the initial one.

The Gradient Projection Direction (GPD)

Define $y^0 = -c^T$, and $T = J(x^*, \delta)$. Computing this direction defined in [4] involves the following steps:

Define $J(y^0) = \{i : i \in T, A_{i.}y^0 < 0\}$. Each of the constraints $A_{i.}x \geq b_i$ for $i \in J(y^0)$ is currently blocking the movement of the ball in the direction y^0, so $J(y^0)$ is called the index set of blocking constraints.

If $J(y^0) = \emptyset$, take the GPD to be y^0.

If $J(y^0) \neq \emptyset$, . Let E be a submatrix of A consisting of rows $A_{i.}$ which form a maximal linearly independent subset of $\{A_{i.} : i \in J(y^0)\}$, and let $P \subset J(y^0)$ be $\{i : A_{i.}$ is a row of $E\}$. Let ξ be the orthogonal projection of y^0 in the subspace $\{x : Ex = 0\}$, so $\xi = ((I - E^T(EE^T)E)y^0$.

If $\xi \neq 0$, the GPD is ξ.

If $\xi = 0$, let $\mu = -(EE^T)^{-1}Ey^0 = (\mu_i : i \in P)$. Then $\mu^T E = c$. So if $\mu \geq 0$, then $\bar{\pi} = (\bar{\pi}_i)$ defined by $\bar{\pi}_i = 0$ if $i \notin P$, $= \mu_i$ if $i \in P$, is a BFS of (1). In this case the ball $B(x^*, \delta)$ halts in its present position and cannot move any further under the gravitational force. In this case the method goes to the step to carry out when the drop halts (Step 2.5), discussed in Section 8.

If $\xi = 0$ and $\mu \not\geq 0$, then delete the i corresponding to the most negative μ_i from the set T, and repeat all this work with the new T.

8 The Gravitational Interior Point Method for LP

Here we state the whole method.

Stage 1: This stage consists of repeated applications of Iteration 1.

Iteration 1: Initial Iteration: Starting with the initial interior feasible solution x^0, apply Iteration 1 (Steps 1.1, 1.2) described in Section 6 repeatedly until the improvement in the objective value cx in each application becomes small.

Let $\bar{x}$ denote the interior feasible solution obtained at the end of Stage 1. With $\bar{x}$ go to Stage 2.

Stage 2: This stage consists of repeated applications of Iteration 2 starting with the point obtained at the end of Stage 1.

Iteration 2: General Iteration: The first application of this iteration begins with $\bar{x}$, the interior feasible solution obtained at the end of Stage 1.

BEGIN

Step 2.1: Centering: Let $\bar{H} = \{x : cx = c\bar{x}\}$. Starting with $\bar{x}$, apply the centering strategy discussed in Section 5 to get the largest ball $B(x^*, \delta)$ with

$x^* \in K^0 \cap \bar{H}$ as center and δ as radius. Go to Step 2.2.

Step 2.2: Gravitational Direction Finding: Find the gravitational direction at x^* along which the current ball will be moved, using one of the options described in Section 7 (the options discussed there are: SDGD, MGD 1, MGD 2, and GPD). If the ball halts, go to Step 2.5. Otherwise denote the gravitational direction selected by $\bar{y}$, go to Step 2.3 with it.

Step 2.3: Step Length Determination and the Main Move: Let $\bar{y}$ denote the gravitational direction selected for the move of the ball $B(x^*, \delta)$. The step length is the maximum distance the ball can move in this direction while still remaining completely within K. So, this step length in the direction $\bar{y}$ is:

$$\theta = \min\{\frac{A_{i.}x^* - b_i - \delta}{-A_{i.}y} : i \in J(y)\}$$

where $J(y) = \{i : A_{i.}y < 0\}$, the blocking set of constraints corresponding to the direction y.

If $\theta = \infty$, the objective function is unbounded below in (2), and (1) is infeasible, terminate the algorithm.

If θ is finite, move the present ball $B(x^*, \delta)$ to $B(x^* + \theta y, \delta)$ and go to Step 2.4.

Step 2.4: The Addional Move: Suppose the main move in this iteration has moved the ball to the new position $B(x^* + \theta\bar{y}, \delta)$. The center of this ball $x^* + \theta\bar{y}$, is strictly in the interior of K. Now get rid of this ball, and take a maximum possible step from its center $x^* + \theta\bar{y}$ in the direction $-c^T$ while still remining inside K^0.

Let $\epsilon > 0$, a small positive value, denote a selected tolerance for interiorness. Then the maximum step length in this move is:

$$\gamma = \min\{\frac{A_{i.}(x^* + \theta\bar{y}) - b_i - \epsilon}{A_{i.}c^T} : \quad i \quad \text{such that} \quad A_{i.}c^T > 0\}$$

So, now we move from $x^* + \theta\bar{y}$ to the point $\bar{x} = x^* + \theta\bar{y} - \gamma c^T$.

With this $\bar{x}$ as the new interior point solution, go back to Step 2.1 to repeat this Iteration 2.

Step 2.5: When the Ball Halts: Since the ball $B(x^*, \delta)$ has halted, it cannot move from its present position under the gravitational force, because its movement is blocked by the blocking constraints among the touching constraints. In this case we obtain a feasible solution $\bar{\pi}$ for (1) as described in Section 7. Let $F = \{i : \bar{\pi}_i > 0\}$, and $E \subset F$ such that $\{A_{i.} : i \in E\}$ is a maximal linearly independent subset of $\{A_{i.} : i \in F\}$, and $d = (b_i : i \in E)$.

Let $\hat{x} = \bar{x} + E^T(EE^T)^{-1}(d - E\bar{x})$, the orthogonal projection of $\bar{x}$ on the flat $\{x : A_{i.}x = b_i, i \in E\}$. If $\hat{x}$ is feasible to (2), then it is optimal to (2), and $\bar{\pi}$ is optimal to (1), terminate the algorithm.

Suppose $\hat{x}$ is not feasible to (2). Here the center x^* of $B(x^*, \delta)$ is at the center of K on the present objective plane. Make the additional move at x^* as in Step 2.4, and continue.

END

Finite convergence of this algorithm follows from the results in [1]. However, being a truely interior point method, it is expected to have superior computational performance. We are planning to carry out a computational experiment comparing this algorithm with other methods on test problems available in the literature.

9 Intelligent Modeling

If an LP model is appropriate in a real world application, practitioners may be able to use the many flexible options usually available in real world applications, to model the problem in the form (2) directly with a feasible set of full dimension. In this case it may be possible to get an approximate optimum for it using just Stage 1 described in Section 6, which may be satisfactory for the application.

Even if there are some equality constraints, by using each equation to eliminate a variable from the problem, it may be possible to transform it easily into form (2), and then use the approach in Section 6 to obtain an approximate optimum for it.

If the centering strategy performs well, the approximate method discussed in Section 6 combined with intelligent modeling offers many potential benefits for practical problem solving.

10 References

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