CONE OF RECESSION AND UNBOUNDEDNESS OF CONVEX FUNCTIONS

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Abstract:

We consider the problem of determining whether or not a convex function $f(x)$ is bounded below over $\mathbb{R}^n$. Our focus is on investigating the properties of the vectors in the cone of recession 0^+f of $f(x)$ which are related to the unboundedness of the function.

Keywords: Convex programming; Cone of recession; Unboundedness.

1. Introduction

We consider the unconstrained problem

$$\text{minimize}: \quad f(x) \tag{1}$$

$$\text{subject to}: \; x \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \to \mathbb{R}$, is a convex function, assuming finite values for all $x \in \mathbb{R}^n$. The problem is said to be **unbounded below** if the minimum value of $f(x)$ is $-\infty$.

Our focus is on the properties of vectors in the cone of recession $0^+ f$ of $f(x)$, which are related to unboundedness in (1).

The problem of checking unboundedness is as old as the problem of optimization itself. In special types of constrained problems there exist efficient methods for checking unboundedness: linear and convex quadratic programming [3,5] and quadratic programming with quadratic constraints (QCQP) [4]. In the first two types, if the problem is unbounded, these methods identify a feasible half-line along which the objective function diverges to $-\infty$. This is not the case for the third type of problem, or for more general problems with nonlinear constraints; these problems can be unbounded even if the objective function is bounded along every feasible half-line.

However it was shown in [4] that if QCQP is unbounded below, then there exists an equivalent problem in a lower dimensional space with possibly fewer constraints and the same type of objective function, which is unbounded along a half-line contained in the feasible region. However, the existence of a half-line along which the objective function diverges to $-\infty$ is not necessary for unboundedness even in unconstrained optimization problems of the form (1). An example is given below.

Example 1.1. Define the set K $\in \mathbb{R}^2$ as the region between the x_2 axis and the parabola $x_2 = x_1^2$. For $x \in \mathbb{R}^2$ define

$$f(x) = -x_1 + (g(x))^4$$

where $g(x)$ is the Euclidean distance from x to the set K. It can be verified that $f(x)$ is convex and C^2. Although $f(x)$ is unbounded below on $\mathbb{R}^2$, it is bounded below on every half-line.

□

We show however that for some classes of functions (e.g. the one satisfying conditions of Theorem 2.3) unboundedness implies the existence of a half-line along which the function is unbounded below.

The other factor which makes the problem of checking unbondedness nontrivial, is that some functions have a finite infimum, but no minimum along some half-lines, and that the level sets of these functions are unbounded. When applied on a convex function with an unbounded level set, many unconstrained minimization algorithms will only generate a stationary sequence $\{x^i\}$, that is, a sequence satisfying

$$\nabla f(x^i) \to 0.$$

In general such a sequence will not necessarily converge to a global minimum, but as was shown in [2] convergence of such a sequence to a global minimum holds for so called asymptotically well behaved convex functions, i.e., closed convex functions for which 0 belongs to the relative interior of the domain of its Fenchel conjugate. For this class of functions, the set of minima is nonempty and stationary sequences converge towards this set of minima.

The aim of the paper is to investigate relationships between unboundedness and the vectors in the cone of recession.

We prove necessary and sufficient conditions for a vector to be in the cone of recession. We also show that unboundedness of $f(x)$ from below along some half-line, implies unboundedness of the function along any half-line with the same direction, or any direction from the relative interior of 0^+f. We prove a similar property for the vectors in 0^+f along which function $f(x)$ is bounded below; to be specific, boundedness of $f(x)$ along each of several linearly independent vectors and their positive linear combination implies that all vectors in the smallest subspace spanned by these vectors are directions of boundedness.

2. Properties of the directions of recession

Throughout the paper if no additional assumptions about the differentiability of $f(x)$ are mentioned, it is implicitly assumed that $f(x)$ is a convex function finite at every point x of $\mathbb{R}^n$.

A vector $s \neq 0$ is called a **direction of recession** of $f(x)$ if for every x the function $f(x + ts)$ is a nonincreasing function of t [6]. Since f is finite throughout

$\mathbb{R}^n$, then f is a proper convex function [p. 24, 6]. Therefore, by Corollary 7.4.2 [6], f is closed and consequently by Theorem 7.1 [6], f is a lower semi-continuous function. Therefore Theorem 8.6 [6] implies that if $f(x+ts)$ is nonincreasing in t for even one $x \in \mathbb{R}^n$ then it is nonincreasing in t for every x. The symbol 0^+f denotes the set of directions of recession of $f(x)$.

Definition 2.1. (i) We say that the vector $s \in 0^+f$ is an **asymptotic direction for** f, if f has a finite infimum but not a minimum along any half-line with the direction vector s. Let D_f^a denote the set of all asymptotic directions of f.

We say that $f(x)$ is **asymptotically decreasing** along the half-line $\bar{x}(t) = \bar{x} + ts$, $t \geq 0$, if it is strictly decreasing and bounded below along this half-line.
(ii) We say that the vector $s \in 0^+f$ is a direction of **unboundedness** for the function $f(x)$ if it is unbounded along any half-line with the direction s. We let the set of all directions of unboundedness be denoted by D_f^u.

□

Lemma 2.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex, differentiable and bounded below along the half-line $\bar{x}(t) = \bar{x} + ts,\ t \geq 0$. Let $s \in 0^+f$. Then

$$\limsup_{t\to\infty} \|\nabla f(\bar{x}(t))\| < \infty.$$

Proof. Let

$$\limsup_{t\to\infty} \|\nabla f(\bar{x}(t))\| = \infty.$$

This implies that $\exists i \in \{1, ..., n\}$, $\exists\{t_j\}, (t_j < t_{j+1}, \forall j,)$ such that $\lim_{j\to\infty} t_j = \infty$, and as $j \to \infty$

$$\frac{\partial f(\bar{x} + t_j s)}{\partial x_i} \to \text{either} + \infty, \text{or} \ - \infty. \tag{2}$$

Let us assume that the first alternative in (2) holds. Let $\delta > 0$ and e_i be the ith unit vector. By convexity

$$f(\bar{x} + t_j s + \delta e_i) \geq f(\bar{x} + t_j s) + \frac{\partial f(\bar{x} + t_j s)}{\partial x_i}\delta. \tag{3}$$

As $j \to \infty$ the left hand side of (3) decreases since $s \in 0^+f$ and $t_j \to \infty$, but the right hand side of (3) $\to +\infty$, a contradiction. Let us assume now that the second alternative in (2) holds. Replacing δ by $-\delta$ yields a proof for the second alternative.

□

In Lemma 2.2 below we will state a necessary and sufficient condition for the given vector s to be a direction of recession.

Lemma 2.2. Assume that the function $f(x)$ is convex and differentiable. A nonzero vector $s \in \mathbb{R}^n$ is a direction of recession of $f(x)$ iff for arbitrary $\bar{x} \in \mathbb{R}^n$, and the half-line $\bar{x}(t) = \bar{x} + ts,\ t \geq 0$, then

$$\lim_{t \to \infty} \langle s, \nabla f(\bar{x}(t)) \rangle \leq 0. \tag{4}$$

Proof. Suppose that $s \in 0^+ f$. Then $f(\bar{x}(t))$ is nonincreasing in $t \geq 0$ for every $\bar{x} \in \mathbb{R}^n$. So $\langle s, \nabla f(\bar{x}(t)) \rangle \leq 0$ for all $t \geq 0$. Also, since $f(x)$ is convex, $\langle s, \nabla f(\bar{x}(t)) \rangle$ is monotonically increasing in $t \geq 0$. Hence (4) holds. Conversely, suppose that $\lim_{t \to \infty} \langle s, \nabla f(\bar{x}(t)) \rangle$ exists and is nonpositive. By convexity of $f(x)$ this implies that $\langle s, \nabla f(\bar{x}(t)) \rangle \leq 0$ for all $t \geq 0$. Hence $f(\bar{x}(t))$ is nonincreasing in $t \geq 0$, which implies that $s \in 0^+ f$.

□

Example 2.1. Let us consider the convex quadratic function $Q(x) = \frac{1}{2} x^T B x + a^T x$ where $a \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n}$ is symmetric positive semidefinite. It follows immediately, by using the fact that $s^T B s \leq 0$ implies $s^T B s = 0$ and $Bs = 0$, that the set (S) of vectors satisfying the condition in Lemma 2.2 for this function $Q(x)$ is given by $S = \{s : Bs = 0, a^T s \leq 0\}$. Hence, the cone of recession $0^+ Q = \{s : Bs = 0, a^T s \leq 0\}$.

Let $\operatorname{rint}(X)$ denote the relative interior of the set X, and the symbol $D_f^{=}$ denote the constancy space of f, where $D_f^{=} = \{y \in \mathbb{R}^n | y \in 0^+ f \wedge -y \in 0^+ f\}$ [6].

Theorem 2.1. (i) Suppose $f(x)$ is convex, and unbounded below along the half-line $a(t) = a + ts, t \geq 0$ for some $a \in \mathbb{R}^n$. Then for every vector $\bar{x} \in \mathbb{R}^n$, $f(x)$ is unbounded below along the half-line $\bar{x}(t) = \bar{x} + ts, t \geq 0$. Also $f(x)$ is unbounded below along every half-line $a(t) = a + ty, t \geq 0$ for all $y \in \operatorname{rint}\ (0^+ f)$.

(ii) If the convex function $f(x)$ is bounded below along the half-line $a(t) = a + ts,\ t \geq 0$, where $s \in 0^+ f$, then for every $\bar{x} \in \mathbb{R}^n$, the function is bounded below along the line $\bar{x}(t) = \bar{x} + ts$.

(iii) Let us assume that the convex function $f(x) \in C^\infty$. Suppose $f(x)$ asymptotically decreases along the half-line $a(t) = a + ts,\ t \geq 0$, (i.e. it is strictly decreasing and bounded below along this half-line), then it also asymptotically decreases along every half-line with the direction s.

Proof. (i) Our original proof was long and required the function to be differentiable. Here we give the simple proof without any differentiability assumption provided by C. Zalinescu.

Let c be a point such that $\bar{x} = (1-\lambda)a + \lambda c$ for some $0 < \lambda < 1$. Of course $s \in 0^+ f$, and so $f(c+ts) \leq f(c)$ for every $t \geq 0$. So, for all $t \geq 0$

$$f(\bar{x}+ts) = f((1-\lambda)(a+ts) + \lambda(c+ts)) \leq (1-\lambda)f(a+ts) + \lambda f(c+ts)$$
$$\leq (1-\lambda)f(a+ts) + \lambda f(c).$$

So $f(x)$ is unbounded below along the half-line $\bar{x}(t)$, $t \geq 0$.

Also, if $y \in \text{rint } (0^+ f)$, there exists a $z \in 0^+ f$ and $0 < \alpha < 1$ such that $y = (1-\alpha)s + \alpha z$. We also have $f(a+tz) \leq f(a)$ for every $t \geq 0$ because $z \in 0^+ f$. Whence:

$$f(a+ty) = f((1-\alpha)(a+ts) + \alpha(a+tz)) \leq (1-\alpha)f(a+ts) + \alpha f(a+tz)$$
$$\leq (1-\alpha)f(a+ts) + \alpha f(a).$$

and this implies that $f(x)$ is unbounded below along the half-line $a(t) = a+ty$, $t \geq 0$.
(ii) Proof follows directly from part (i) of the theorem.
(iii) Part (ii) of the theorem implies that for any $\bar{x} \in \mathbb{R}^n$, and $\bar{x}(t) = \bar{x} + ts,\ t \geq 0$, $f(x)$ is either asymptotically decreasing along $\bar{x}(t)$, $t \geq 0$ or constant for $t \geq t_0$ for some $t_0 \geq 0$. Since $f \in C^\infty$, if the latter case holds then the function is constant along a whole line containing the half-line $a(t)$, $t \geq 0$ (see for example [6]) and consequently $s \in D_f^=$, and $-s \in 0^+ f$. The latter conclusion contradicts the assumption that $f(x)$ is asymptotically decreasing along $a(t)$, $t \geq 0$. Therefore $f(x)$ asymptotically decreases along every half-line with the direction s.

□

In Example 2.2 we provide an illustration of Theorem 2.1. The function $f(x)$ given there was used in [1] as an example of the function for which application of the classical methods, such as the steepest descent, fails to generate a sequence convergent to the infimum of $f(x)$.

Example 2.2. Let $f(x) = x_2 + e^{-x_1-x_2}$.

$\nabla f(x) = (-e^{-x_1-x_2}, 1-e^{-x_1-x_2})$. Expression on the left side of the inequality (6), with $\bar{x} = 0$, gives

$$\lim_{t\to\infty} \langle (-e^{-ts_1-ts_2}, 1-e^{-ts_1-ts_1}), (s_1, s_2) \rangle \leq 0.$$

We observe that the last inequality is equivalent to:

$$\langle (0,1),(s_1,s_2)\rangle \le 0, \text{if } s_1+s_2>0,$$

$$\langle (-1,0),(s_1,s_2)\rangle \le 0, \text{if } s_1+s_2=0,$$

The cone of recession is a convex cone bounded by two half-lines : $s^1(t)=(1,-1)t,\ t\ge 0$, and $s^2(t)=(1,0)t,\ t\ge 0$. The function is unbounded along a half-line $\tilde{x}^1(t)=\tilde{x}+(1,-1)t,\ t\ge 0$ and bounded along a half-line $\tilde{x}^2(t)=\tilde{x}+(1,0)t,\ t\ge 0$, for any $\tilde{x}\in\mathbb{R}^n$. By Theorem 2.1 (i) it follows that $f(x)$ is unbounded along any half-line $x^3(t)=\tilde{x}+(s_1,s_2)t,\ t\ge 0$ such that $(s_1,s_2)\in \operatorname{int} 0^+f$, that is, if (s_1,s_2) satisfies the system : $s_2<0,\ s_1+s_2>0$.

□

Lemma 2.3. Let $f:\mathbb{R}^n\to\mathbb{R}$ be convex and differentiable. If for some $a\in\mathbb{R}^n$ and $s\in\mathbb{R}^n, s\ne 0$, $\lim_{t\to\infty}\langle s,\nabla f(a+ts)\rangle<0$, then $f(x)$ is unbounded below along every half-line with direction s.

Proof. By convexity $f(a+ts)\le f(a)+t\langle s,\nabla f(a+ts)\rangle\le f(a)+t\lim_{r\to\infty}\langle s,\nabla f(a+rs)\rangle$ for $t\ge 0$. But the last expression $\to-\infty$ as $t\to\infty$. So $f(x)$ is unbounded below on the half-line $\{a+ts: t\ge 0\}$. Therefore, by Theorem 2.1, $f(x)$ is unbounded below along every half-line with direction s.

□

Corollary 2.1. Assume that $f(x)$ is convex and differentiable. If $f(x)$ remains bounded below along a half-line $\bar{x}(t)=\bar{x}+ts, t\ge 0$, where s is a direction of recession of $f(x)$, and $\nabla f(x)\ne 0$ on this half-line, then either

$$\limsup_{t\to\infty}\langle s,\frac{\nabla f(\bar{x}(t))}{\|\nabla f(\bar{x}(t))\|}\rangle=0$$

or $\lim_{t\to\infty}\|\nabla f(\bar{x}(t))\|=0$.

Proof. Assume that $f(x)$ remains bounded below on the given half-line, and that

$$\limsup_{t\to\infty}\langle s,\frac{\nabla f(\bar{x}(t))}{\|\nabla f(\bar{x}(t))\|}\rangle<0$$

and $\limsup_{t\to\infty}\|\nabla f(\bar{x}(t))\|=M>0$ where M is finite by Lemma 2.1. Then there must exist an $\epsilon>0$ and $\bar{t}>0$ such that for all $t\ge\bar{t}$

$$\langle s,\frac{\nabla f(\bar{x}(t))}{\|\nabla f(\bar{x}(t))\|}\rangle<-\epsilon.$$

Let $F(t) = f(\bar{x}(t))$. So, for all $t \geq \bar{t}$

$$F'(t) = \langle s, \nabla f(\bar{x}(t))\rangle < -\epsilon ||\nabla f(\bar{x}(t))||.$$

Since M is finite, the above implies that $\lim_{t\to\infty}\langle s, \nabla f(\bar{x}(t))\rangle \leq -\epsilon M$ which by Lemma 2.3 implies that $\lim_{t\to\infty} f(\bar{x}(t)) = -\infty$,

a contradiction to the hypothesis. Hence the corollary must hold. □

Let $Pos(\hat{s}^i,\ i = 1, ..., k)$ denote the nonnegative hull of the vectors $\hat{s}^i$, $i = 1, 2, ..., k$.

Theorem 2.2. Suppose that $f(x)$ is convex and $\hat{s}^i \in 0^+ f$, $i = 1, 2, ..., k$. Then
(i) If the function f is bounded below along the half-lines with directions $\hat{s}^i, i = 1, 2, ...k$, and $\hat{s} \in$ rint (Pos($\hat{s}^i, i = 1, 2, ..., k$)), then it is bounded below along any half-line with the direction vector in Pos ($\hat{s}^i, i = 1, ..., k$).
(ii) If the function $f(x)$ is bounded below along the directions $\hat{s}^1, \hat{s}^2, \ldots, \hat{s}^k$, and $\sum_{i=1}^k \lambda_i \hat{s}^i$, $\lambda_i > 0$, $i = 1, ..., k$, then the function is bounded below along any direction in the subspace spanned by $\hat{s}^1, \hat{s}^2 \ldots, \hat{s}^k$.

Proof. (i) Suppose that there exists a vector $\bar{s} \in$ Pos($\hat{s}^i, i = 1, 2, ..., k$) which is a direction of unboundedness. It follows that there exists a vector $\tilde{s} \in$ rint Pos($\hat{s}^i, i = 1, 2, ..., k$) such that $\hat{s} = \beta\bar{s} + \gamma\tilde{s},\ \beta, \alpha > 0$. Therefore for $x \in \mathbb{R}^n, t \geq 0$, we have

$$f(x + t\hat{s}) = f(x + t\beta\bar{s} + t\gamma\tilde{s}) \leq f(x + t\beta\bar{s})$$

which contradicts the assumption that $\hat{s}$ is also a direction of boundedness.
(ii) From part (i) it follows that $f(x)$ is bounded below along any half-line with direction vector from Pos($\hat{s}^i$, $i = 1, ..., k$). Now suppose that f is unbounded along the direction vector

$$\tilde{s} = \sum_{i=1}^{k} \alpha_i \hat{s}^i,$$

where some α_i are negative. We have that

$$\text{Pos}(\hat{s}^i, \quad i = 1, \ldots, k) \subset 0^+ f,$$

and $\tilde{s} \in 0^+ f$. We will show that there exists vector $\bar{s} \in$ rint Pos($\hat{s}^i$,$i = 1, \ldots, k$), that can be represented as a positive combination of the vectors in Pos($\hat{s}^i$, $i = 1, \ldots$,k) and the vector s̃ (with positive coefficient corresponding to s̃). Let $\bar{K} = \{j|\alpha_j < 0\}$ and define

$$\check{s} = \sum_{i=1}^{k} \bar{\alpha}_i \hat{s}^i$$

with $\bar{\alpha}_j > |\alpha_j|$ for $j \in \bar{K}$ and $\bar{\alpha}_i > 0$ for $i \notin \bar{K}$. Thus

$$\bar{s} = \tilde{s} + \check{s} \in \text{rint Pos}(\hat{s}^i, \quad i = 1, \ldots, k).$$

Because $\check{s}$ is a vector of recession, we get

$$f(t\tilde{s}) \geq f(t\bar{s}), \quad t \geq 0. \tag{5}$$

From the assumption that $f(x)$ is bounded along the vector $\sum_{i=1}^{k} \lambda_i \hat{s}^i$ and part (i) of the theorem it follows that $f(x)$ is bounded along direction $\bar{s}$. This leads to the contradiction with the inequality (5). □

The example given below illustrates Lemma 2.2 and the Theorems 2.1(i) and 2.2.

Example 2.3. Let $f(x) = \sum_{i=1}^{n-1} e^{-x_i} - x_n$, and $\bar{x} = 0$. The inequality (4) has a form

$$\limsup_{t\to\infty} \langle(-e^{-ts_1}, -e^{-ts_2}, ..., -e^{-ts_{n-1}}, -1), (s_1, s_2, ..., s_n)\rangle \leq 0. \tag{6}$$

Let us define $s_K = \min\{s_1, s_2, ..., s_{n-1}\}$, and consider three cases:

1) $s_K > 0$,

2) $s_K < 0$, and

3) $s_K = 0$.

In case 1), inequality (6) is equivalent to : $s_n \geq 0$.

If condition 2) is satisfied we divide the expression in the limit (6) by e^{-ts_K}, which yields

$$\langle -(\chi^K), s\rangle \leq 0, \tag{7}$$

where $\chi^K = (\chi_1, \chi_2, ..., \chi_n)$, with $\chi_j = 1$, if $j \in$ argmin $\{s_1, ..., s_{n-1}\}$ and $\chi_j = 0$ otherwise.

The system (7) is equivalent to $-s_K \leq 0$, which contradicts the assumption in 2). Therefore there does not exist vector s satisfying inequality (6) and condition 2). Now consider case 3). Then the system (6) reduces to

$$-(\Gamma^K)^T s \leq 0, \tag{8}$$

where $\Gamma^K = (\Gamma_1, ..., \Gamma_n)$, with $\Gamma_i = 1$, if $i = n$ or $i \in$ argmin$\{s_1, ..., s_{n-1}\}$, and $\Gamma_i = 0$ otherwise. The inequality (8) along with the assumption in 3) implies $s_i \geq 0, i = 1, ..., n$. We finally conclude that $0^+ f = \{s | s_i \geq 0, i = 1, ..., n\}$. In order to illustrate Theorem 2.2 we take $\hat{s}^i = e_i,\ i = 1, ..., n-1$, where e_i are unit vectors in $\mathbb{R}^n$. Clearly, $e_i \in 0^+f$, and $f(x)$ is bounded along $e_i,\ i = 1, ..., n-1$, as well as it is bounded along $\sum_{i=1}^{n-1} e_i$. It is also straightforward to show that $f(x)$ is bounded along any vector in the subspace spanned by the vectors $e_i,\ i = 1, ..., n-1$. On the other hand the function is unbounded below along the vector e_n as well as along $e_\alpha = \sum_{i=1}^{n} \alpha_i e_i$, where $\alpha_i \geq 0,\ i = 1, ..., n-1$ and $\alpha_n > 0$. The vector e_α is in $\mathrm{rint}(0^+f)$ and unboundedness along the vector e_α verifies that Theorem 2.1(i) holds in this example.

□

Definition 2.2. Let $\partial 0^+f = 0^+f \setminus \mathrm{rint}\ 0^+f$. The set $F \subset \partial 0^+f$ is called a **face** of 0^+f if $s^1, s^2 \in F \Rightarrow \alpha_1 s^1 + \alpha_2 s^2 \in F,\ \forall \alpha_1, \alpha_2 \geq 0$.

□

Corollary 2.2. If $f(x)$ is convex, then any face F of 0^+f has a property that either all vectors $s \in F$ are directions of boundedness, or all vectors $s \in \mathrm{rint}\ (F)$ are directions of unboundedness.

Proof. Let us suppose that there exists $s \in D_f^u \cap F$. Then if $y \in \mathrm{rint}(F)$, then there exists $z \in F$, and $0 < \alpha < 1$, such that $y = (1-\alpha)s + \alpha z$. We have

$$f(ty) = f((1-\alpha)ts + \alpha tz) \leq (1-\alpha)f(ts) + \alpha f(tz) \leq (1-\alpha)f(ts) + \alpha f(0),\ \forall t \geq 0,$$

which proves that $y \in D_f^u$.

□

Lemma 2.4. Let $f(x)$ be convex and differentiable. Then, for any $\hat{y} \in \mathbb{R}^n$, we have

$$0^+f \subset \{y | \langle c, y \rangle \leq 0\},$$

where c is an arbitrary accumulation point of the sequence

$$\{\frac{\nabla f(t_j \hat{y})}{\|\nabla f(t_j \hat{y})\|}\}.$$

where the sequence $\{t_j\}$ satisfies $\lim_{j\to\infty} t_j = \infty$.

Proof. Let us suppose that the opposite is true, that is $\exists y^0 \in 0^+f, \exists \hat{y} \in \mathbb{R}^n$ such that $c^T y^0 > 0$, where $c = \lim_{j\to\infty}\{\frac{\nabla f(t_j \hat{y})}{\|\nabla f(t_j \hat{y})\|}\}$ for some $t_j \to \infty$. Therefore $\exists \bar{t} > 0, \langle \nabla f(\bar{t}\hat{y}), y^0 \rangle > 0$, which implies that $f(x)$ is increasing along the half-line $\hat{x}(t) = \bar{t}\hat{y} + ty^0,\ t \geq 0$. This contradicts the earlier assumption that $y^0 \in 0^+f$.

□

We will show in Theorem 2.3 below that under some conditions, the existence of the direction of uboundedness is necessary for the function $f(x)$ to be unbounded. The convex function $f(x)$ is called faithfully convex if it is constant along some segment only if it is constant along whole line containing this segment. Convex analytic functions belong to this class. In Theorem 2.3 below we will prove in particular that under assumption that $D_f^a = \emptyset$, an analytic convex function can be unbounded only if it is unbounded along a half-line.

Theorem 2.3. Let us assume that $f(x)$ is a faithfully convex function. Then
(i) $0^+f = D_f^= \cup D_f^u \cup D_f^a$, and $D_f^u \cap D_f^a = \emptyset$.
(ii) If $\text{rint}(0^+f) \cap D_f^a \neq \emptyset$, then every vector $s \in \text{rint}(0^+f)$ is an asymptotic direction.
(iii) If $0^+f \neq D_f^=$, then $\dim(0^+f) > \dim(D_f^=)$.
(iv) If $D_f^a = \emptyset$, then $f(x)$ is unbounded below only if it is unbounded below along some half-line.

Proof. The proof follows directly from the Theorem 2.1 and the assumption that $f(x)$ is a faithfully convex function.
(ii) Let $\hat{s} \in \text{rint}(0^+f) \cap D_f^a$. Let us suppose that there exists $\tilde{s} \in \text{rint}(0^+f) \cap D_f^=$. Since $\tilde{s} \in \text{rint}(0^+f)$ then there exists $\epsilon > 0$ such that $s_\epsilon = \tilde{s} + \epsilon(\tilde{s} - \hat{s}) \in 0^+f$. It follows that there exists $\tau \in (0,1)$ and $t > 0$, such that $\tau(-\tilde{s}) + (1-\tau)s_\epsilon = -t\hat{s}$. We clearly have $-\hat{s} \notin 0^+f$, but $\tau(-\tilde{s}) + (1-\tau)s_\epsilon \in 0^+f$, which leads to contradiction and completes the proof.
(iii) Let us assume that

$$D_f^= \subsetneq 0^+f \tag{9}$$

and $\dim(0^+f) = \dim(D_f^=)$. The latter assumption implies that $D_f^= = L(0^+f)$, (where $L(0^+f)$ denotes the smallest linear subspace containing the cone of recession of f), which along with the inclusion $0^+f \subset L(0^+f)$ contradicts assumption (9).
(iv) Consider $X_0 = D_f^=$ and $X_1 = X_0^\perp$. Then $f(x + x_0) = f(x)$ for all $x_0 \in X_0$ and arbitrary $x \in \mathbb{R}^n$. Suppose that f is not bounded below. Then there exists $\{x^i\} \subset X_1$ such that $\|x^i\| \to \infty$ and $f(x^i) \to -\infty$. Taking u an accumulation point of $\{\frac{x^i}{\|x^i\|}\}$, $u \in 0^+f$. Of course $u \in X_1$, and so $u \notin X_0$. As $D_f^a = \emptyset$, we have that $u \in D_f^u$. This completes the proof of the part (ii) of the lemma.

□

Remark 1: The difficulty with the Corollary 2.1 and Lemma 2.4, is that they require to determine the limit

$$\limsup_{t\to\infty}\langle s, \frac{\nabla f(ts)}{\|\nabla f(ts)\|}\rangle, \tag{10}$$

and the accumulation point of the sequnce

$$\{\frac{\nabla f(t_j s)}{\|\nabla f(t_j s)\|}\}$$

respectively, which may cause some computational difficulties.

The functions in both Example 2.2 and 2.3, indicate that the expression

$$\lim_{t\to\infty}\frac{\nabla f(ts)}{\|\nabla f(ts)\|} \tag{11}$$

can be discontinuous (although piecewise continuous) function of s, while the limit (10) exists. Nevertheless it can shown that the limit (11) exists for some types of functions: polynomial, and functions being combination of polynomials, rational functions, exponential and logarithmic functions. This means, that if a convex function $f(x)$ is in particular of the form

$$f(x) = \sum_{i=1}^{k}\beta_i ln\ \frac{P_i(x)}{Q_i(x)} \quad + \sum_{i=k+1}^{m}\frac{P_i(x)}{Q_i(x)}\ e^{R_i(x)},$$

where $P_i(x)$, $Q_i(x)$ and $R_i(x)$ are polynomials of n-variables, $\beta_i \in \mathbb{R}$, $Q_i(x) \neq 0, \forall x \in \mathbb{R}^n$, and $\frac{P_i(x)}{Q_i(x)} > 0$, $i = 1, 2, ..., k$, then the limit (11) exists.

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