Fast Descent Methods for LPs With Minimal or No Matrix Inversions

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Abstract

Existing software implementations for solving large scale Linear Programming (LP) models are all based on full matrix inversion operations involving every constraint in the model in every step. This **linear algebra component** in these systems makes it difficult to solve large

scale dense models using these systems, and it is also the source of accumulating roundoff errors affecting the accuracy of the output. We present new methods for LP that help reduce the need for this linear algebra component significantly, or even eliminate it altogether, and still get comparable or better results.

Key words: Linear Programming (LP), Interior point methods (IPMs) , ball centers of a polytope, solving LPs using matrix inversions sparingly, Sphere methods-1, 2, 2.1, 3, 4 for large scale LPs.

1 Introduction

For modeling decision making applications, LP is the most commonly used mathematical model. Software systems for solving LP models are based on either the simplex method, or interior point methods (IPMs, in particular the primal-dual IPM) developed during the second half of the 20th century (Dantzig and Thappa [1997], Kojima, Mizuno, Yoshishe [1989], Megiddo [1989], Mehrotra [1992], Monteiro and Adler [1989], Sonnevend, Stoer and Zhao [1989], and the books Saigal [1995], Wright [1997], and Ye [1997])) and are able to solve large scale sparse models (those involving thousands of constraints) within reasonable times by exploiting the sparcity of the models. As several real world applications lead to sparse models, these systems are very popular in practice.

But the simplex method, and these IPMs are based on matrix inversion operations involving every constraint in the model in every step. In large scale applications, these matrix inversion operations limit the ability of these algorithms to only those with very sparse coefficient matrices. Typically, the effectiveness of these algorithms fades as the density of the coefficient matrix increases.

Many application areas (e.g., models in AHP (analytic heirarchy process), and supply chain problems etc.) do lead to large scale models that are dense (sometimes 100% dense). Sphere methods (SMs) have the goal of solving LPs using matrix inversion operations sparingly, or not at all; to handle all such models and also the others.

SMs consider LPs in the form:

$$\begin{aligned} \text{Minimize} \quad & z = cx \\ \text{subject to} \quad & Ax \geq b \end{aligned} \tag{1}$$

where A is an $m \times n$ data matrix; with a known interior feasible solution x^0 (i.e., satisfying $Ax^0 > b$). Strategies for modifying any given LP into this form are discussed in (Murty[2009-2, 3]). We assume that c, and each row vector of A is normalized so that $||c|| = ||A_{i.}|| = 1$ for all $i = 1$ to m, here $A_{i.}$ denotes the i-th row vector of A. Here is the notation we will use in this paper.

K, K^0: K denotes the set of feasible solutions of (1), and $K^0 = \{x : Ax > b\}$ its interior.

FH_i: $= \{x : A_{i.}x = b_i\}$, the i-th facetal hyperplance of K for $i = 1$ to m.

$\delta(x), B(x, \delta(x))$: defined for $x \in K^0$, $\delta(x) = \text{minimum}\{A_{i.}x - b_i : i = 1, ..., m\}$ is the radius of the largest ball that can be inscribed in K with x as its center. $B(x, \delta(x)) = \{y : ||y - x|| \leq \delta(x)\}$ is that largest inscribed ball in K with x as its center.

$T(x)$: Defined for $x \in K^0$, it is the set of all indices i satisfying: $A_{i.}x - b_i = \text{Minimum}\{A_{p.}x - b_p : p = 1 \text{ to } m\} = \delta(x)$. The facetal hyperplane $FH_i = \{x : A_{i.}x = b_i\}$ is a tangent plane to $B(x, \delta(x))$ for each $i \in T(x)$, that's why $T(x)$ is called the **index set of touching constraints in (1)** at x.

K^{r+1}: When x^r is the current interior feasible solution in the algorithm, the set

$$K^{r+1} = \{x : Ax \geq b, \quad A_{m+1.}x \geq b_{m+1}\} \tag{2}$$

where $A_{m+1.} = -c$, $b_{m+1} = -A_{m+1.}x^r - \epsilon$, and ϵ is a small positive tolerance. K^{r+1} is the set of feasible solutions of (1) updated by the current objective value in

the algorithm. The current objective value is strictly monotonic decreasing in the algorithm, and hence this **updated set of feasible solutions** keeps getting smaller during the algorithm.

GPTC (gradient projection on touching constraint) directions: Let c^i denote the orthogonal projection of c^T on $\{x : A_{i.}x = 0\}$, i.e., $c^i = (I - A_{i.}^T(A_{i.}))c^T$ for $i = 1$ to m. When the ball $B(x, \delta(x))$ is under consideration, the diections $-c^i$ for $i \in T(x)$ are called the GPTC directions at the current center x.

Ball center of K: It is the center of a largest ball in K, it maximizes $\delta(x)$ over K.

Ball center of K^{r+1}: When x^r is the current interior feasible solution in the algorithm, this is the ball center of the updated set of feasible solutions defined by (2).

We will now describe the main strategy used by the SMs to solve (1). Each iteration of the method begins with the best interior feasible solution obtained at the end of the previous iteration; and consists of two cycles; a **centering cycle**, and a **descent cycle** consisting of several descent steps. Details of both these steps are discussed next.

1.1 The Centering Cycle in Sphere Method 1 (SM-1)

SM-1, the first sphere method for LP, was developed in (Murty [2006-1, 2006-2], Murty and Oskoorouchi [2008]). In SM-1, the set of feasible solutions considered remains the original K throught the algorithm, it remains unchanged. But the current objective plane $\{x : cx = cx^r\}$ where x^r is the current interior feasible solution, keeps sliding parallelly towards decreasing values of the RHS constant in the equation defining it, from one iteration to the next. Consider the general iteration $r + 1$ with x^r as the initial interior feasible solution.

The centering cycle in this iteration has the aim of computing a **center** for this iteration which maximizes $\delta(x)$ subject to the constraint: $cx \leq cx^r$. So, it is the center of a largest ball inside K subject to this constraint on its center. In each iteration, this centering cycle is computationally the most expensive step in the algorithm. But it is not necessary to compute

this center exactly; in fact for computational efficiency in SMs we only compute this center approximately by using a series of **line search steps**. One of the great advantages of SMs over other IPMs for LP for practical implementation, is **the flexibility that they offer** in implementing them.

In SM-1, in the centering cycle when the current interior feasible solution is $\bar{x}$, the algorithm selects a direction y which is called a **profitable direction to move at $\bar{x}$ for K**, i.e., one satisfying the property that $\delta(\bar{x} + \alpha y)$ strictly increases as α increases from 0; and determines the optimum step length to maximize $\delta(\bar{x} + \alpha y)$ over $\alpha \geq 0$.

A direction y has been shown to be a profitable direction at $\bar{x}$ for K iff $A_{i.}y > 0$ for all $i \in T(\bar{x})$ [8, 9, 14], so checking a given direction for profitability is easy. Two procedures for generating profitable directions are discussed in [9, 11, 14], one is LSFN which selects a direction among those in $\Gamma_1 = \{\pm A_{i.} : i = 1, ..., m\}$. The other is LSCPD which obtains profitable directions by solving a system of linear equations with coefficient matrix consisting of only rows $A_{i.}$ for $i \in T(\bar{x})$ at the current point $\bar{x}$, for details see (Murty and Oskoorouchi [2008, 2010]).

Once a profitable direction y at the current point $\bar{x}$ has been determined, the optimum step length that maximizes $\delta(\bar{x} + \alpha y)$ is $\bar{\alpha}$, where $(\bar{\delta}, \bar{\alpha})$ is the optimum solution in the following 2-variable LP:

$$\begin{aligned} & \text{Maximize} \quad \delta \\ \text{subject to} \quad & \delta - \alpha A_{i.}y \leq A_{i.}\bar{x} - b_i \quad \text{for all } i \\ & \delta, \alpha \geq 0 \end{aligned} \tag{3}$$

and $\bar{\delta} = \delta(\bar{x} + \bar{\alpha} y)$, the optimum objective value in this 2-variable LP. We will discuss an efficient algorithm to solve this 2-variable LP later on.

1.2 The Centering cycle in Sphere Method 2 (SM-2)

Sphere method 2 (SM-2) discussed in [15] consists of several improvements over SM-1. As mentioned above, in SM-1 the set of feasible solutions considered, remains the original K throughout the algorithm. In contrast, in SM-2, in iteration $r + 1$ with x^r as the initial interior feasible

solution, K is replaced by K^{r+1}, the set of feasible solutions updated by the current objective value cx^r (defined in (2)) that keeps getting smaller as r increases.

The center, the output of the centering cycle in this iteration in SM-2, is the ball center of K^{r+1}, computed approximately by the corresponding line search steps discussed above, applied to K^{r+1} and not K. Also, in SM-2 in this iteration $r+1$, the touching set of constraints $T(x)$ at an interior feasible solution x of K^{r+1} refers to those in (2) which are touching.

1.3 The Various Descent Steps Used in SMs

Considering the genral iteration $r+1$, suppose the center obtained in the centering step in this iteration is $\bar{x}^r$. From this center, the descent cycle in this iteration carries out various descent steps. In this section we describe all the descent steps used in various SMs.

In a general descent step from an interior feasible solution x^* in descent direction d (i.e., d satisfying $cd < 0$), we move from x^* in this direction, the maximum distance possible while still remaining at a distance ϵ from the boundary. This gives the step length to be γ, where

$$\gamma = \text{Minimum}\{(-A_{i.}x^* + b_i + \epsilon)/(A_{i.}d) : \quad \text{over } i \text{ satisfying } A_{i.}d < 0\} \tag{4}$$

and the output of this descent step is $x^* + \gamma d$. Here are the various descent steps used in SM-1 in the descent cycle when the center is $\bar{x}^r$.

D1.1: Descent step from $\bar{x}^r$ in the direction $d^1 = -c^T$.

D1.2: Let $S = \{(A_{i.})^T : i \in T(\bar{x}^r) \text{ such that } c(A_{i.})^T < 0\} \cup \{(-A_{i.})^T : i \in T(\bar{x}^r) \text{ such that } c(A_{i.})^T > 0\}$. Take a descent step from the center $\bar{x}^r$ in the direction which is the average of all the directions in S.

D2: Descent step from $\bar{x}^r$ in the direction $d^2 = \bar{x}^r - \bar{x}^{r-1}$, direction of the path of centers being generated, here $\bar{x}^{r-1}$ is the center obtained in the previous iteration.

D3: Descent step from $\bar{x}^r$ in each of the directions $-c^i$ for $i \in T(\bar{x}^r)$.

D4: Descent step from $\bar{x}^r$ in the average of the directions in D3.

D5.1 For each $i \in T(\bar{x}^r)$, let x^{ir} be the orthogonal projection of $\bar{x}^r$ on FH_i. It is $\bar{x}^r + (A_{i.})^T(b_i - A_{i.}\bar{x}^r)$.

Let $\hat{x}^{ir} = (1-\epsilon)x^{ir} + \epsilon\bar{x}^r$, the point on the line segment joining x^{ir} and $\bar{x}^r$ at a distance of ϵ from x^{ir}. $\hat{x}^{ir}$ is called the NTP (**near touching point**) of $B(\bar{x}^r, \delta(\bar{x}^r))$ with its tangent plane FH_i.

For each $i \in T(\bar{x}^r)$, take a descent step from the NTP $\hat{x}^{ir}$ in the direction $-c^i$.

Now we describe additional descent steps D5.2, D5.3 used in this general iteration $r+1$ in SM-2 [15].

D5.2, Descent Step 5.2: Let $\tilde{x}^{r1}$ denote the best point (by objective value) obtained in descent steps D1 to D5.1 in this iteration. This $\tilde{x}^{r1}$ is the initial interior feasible solution for Descent Step 5.2 (D5.2).

For each $i \in T(\tilde{x}^{r1})$, from $\tilde{x}^{r1}$ take a descent step in the GPTC direction $-c^i$. Also, from $\tilde{x}^{r1}$ take a descent step in the direction which is the average of $-c^i$ for $i \in T(\tilde{x}^{r1})$. Let $\tilde{x}^{r2}$ denote the best point obtained in all these descent steps, by objective value. If $c\tilde{x}^{r1} - c\tilde{x}^{r2}$ is:

> $>$ the selected tolerance ϵ for objective value reduction, with $\tilde{x}^{r2}$ as the initial interior feasible solution repeat this Descent Step 5.2; and continue the same way.
>
> $\leq \epsilon$, take $\tilde{x}^{r2}$ as the output of this D5.2, with this point go to D5.3.

D5.3, Descent Step 5.3: We come to this step from the output point of D5.2, let us denote it by x^s. Clearly $\delta(x^s) \leq \epsilon$ from the manner it is obtained.

For each $i \in T(x^s)$, define $x^{is} = x^s + (A_{i.})^T(b_i - A_{i.}x^s)$, the orthogonal projection of x^s on facetal hyperplane FH_i . Define $\bar{x} = [\sum_{i \in T(x^s)} x^{si}]/|T(x^s)|$. Typically, a move from x^s in the direction $x^s - \bar{x}$ goes through the central portion of K^{r+1}, so a step in this direction at this stage can be expected to lead to good improvement in objective value. We have 2 cases to consider.

Case 1: If $c(x^s - \bar{x}) < 0$ carry out a descent step at x^s in the descent direction $(x^s - \bar{x})$, and make the output of this descent step the new current point (new x^s) and repeat this step with it, as long as the improvement in objective value is greater than the selected tolerance.

Case 2: If $c(x^s - \bar{x}) \geq 0$, let y be the orthogonal projection of $(x^s - \bar{x})$ on the hyperplane $\{x : cx = 0\}$, $y = (I - c^T c)(x^s - \bar{x})$.

Solve the 2-variable LP: max δ subject to $\delta - \alpha A_{i.} y \leq A_{i.} x^s - b_i$ for all i, and $\delta, \alpha \geq 0$. Let $\bar{\delta}, \bar{\alpha}$ be the optimum solution of this 2-variable LP. The point $x^s + \bar{\alpha} y$ has objective value $= cx^s$ because $cy = 0$, from this point take all descent steps D1 up to D5.2. Call the final output point of these descent steps as the new current point (new x^s), and with it repeat this D5.3 until the improvement in objective value becomes less than the selected tolerance.

Next we will describe a new descent steps D5.4, D5.5 that can be used in the descent cycle in every iteration of SM-2 , and also in improved versions Sphere methods 3, 4 (SM-3, 4) to be discussed later in this paper.

D5.4, Descent Step 5.4: This descent step is carried out in the descent cycle after all the descent steps D1 to D5.3 have been carried out in this cycle. . Let $\bar{K}$ denote the current updated set of feasible solutions.

Let $x^1, ..., x^s$ be all the points obtained at the end of all the descent steps carried out in the latest D5.1 above in this iteration; and suppose x^s is the best among all these by objective value. Let $H = \{x : cx = cx^s\}$, the objective plane through x^s, called **the current objective plane**. Let ϵ_1 be a small positive number, e.g. $\epsilon_1 = 0.1$ or smaller. Here $s =$ the number of touching constraints at the center using which this D5.1 was carried out.

For each $t \in \{1, ..., s-1\}$, let $\tilde{x}^t$ be the orthogonal projection of $x_s + \epsilon_1(x^t - x^s)$ on H. For all t such that $\tilde{x}^t \in K$, leave $\tilde{x}^t$ as it is.

For any $t \in \{1, ..., s-1\}$ such that $\tilde{x}^t \notin \bar{K}$, do the following: Compute $\theta =$ Maximum$\{(-(A_{i.}\tilde{x}^t - b_i)/(A_{i.}x^s - A_{i.}\tilde{x}^t))$: over all constraints i defining $\bar{K}$ and satisfying $A_{i.}\tilde{x}^t - b_i < 0\}$; and then replace $\tilde{x}^t$ by $(\theta)x^s + (1 - \theta)\tilde{x}^t$. It can be verified that after this change all $\tilde{x}^t \in \bar{K}$ for $t = 1$ to

$s-1$.

Now define the direction y as the average of $\{(\tilde{x}^t - x_s)/||\tilde{x}^t - x^s|| : \ \ t = 1 \text{ to } s-1\}$. All the $\tilde{x}^t$ for $t = 1$ to $s-1$, are spread out in different directions all around $\bar{K} \cap H$. So the half-line from x_s in the direction y will be in the central portion of $\bar{K} \cap H$, and hence the point which maximizes $\delta(x^s + \alpha y)$ over $\alpha \geq 0$ for the current set of feasible solutions $\bar{K}$ may be a reasonable approximation to the ball center of $\bar{K}$ on H.

Solve the 2-variable LP (of the form (3)) to find the point $x^s + \alpha y$, $\alpha \geq 0$ which maximizes $\delta =$ the radius of the largest ball inscribed inside $\bar{K}$ with $x^s + \alpha y$ as center, for the current set of feasible solutions $\bar{K}$. Let $\bar{\bar{x}}^2$ be the resulting point.

Let $S(\bar{\bar{x}}^2) = \{(A_{i.})^T : i \in T(\bar{\bar{x}}^2)$ such that $c(A_{i.})^T < 0\} \cup \{(-A_{i.})^T : i \in T(\bar{\bar{x}}^2)$ such that $c(A_{i.})^T > 0\}$, and let y be the average of all the directions in $S(\bar{\bar{x}}^2)$. Redefine $\bar{K} = \{x : Ax \geq b$ and $A_{m+1.}x \geq b_{m+1}\}$ where $A_{m+1.} = -c$ as defined earlier, and $b_{m+1} = -c\bar{\bar{x}}^2 - \epsilon$ (here ϵ is a small positive number), as the current set of feasible solutions. Solve the 2-variable LPs to maximize the radius of the largest ball inscribed inside the current set of feasible solutions with its center on each of the half-lines $\{\bar{\bar{x}}^2 + \alpha(-c^T) : \alpha \geq 0\}$ and $\{\bar{\bar{x}}^2 + \beta y : \beta \geq 0\}$; and let $\bar{\bar{x}}^3$ be the point among the outputs which corresponds to the maximum radius of the inscribed ball.

With $\bar{\bar{x}}^3$ as the center carry out the descent cycle with all descent steps D1 to D5.3, and after these repeat D5.4 again with the points obtained at the end of this recent descent cycle. Continue this way repeating D5.4 as long as good reductions in objective value are obtained

If the reduction in objective value in two successive applications of D5.4 is less than the selected tolerance, the best point among the outputs of all the descent steps carried out in this iteration is the output of this iteration. With that point the method goes to D5.5.

Descent step 5.4 (D5.4) can also be used in SM-1, where it will be the same as the above with the exception that $\bar{K} = K$ in all the iterations, and only the first of the 2-variable LPs is solved, and the center to begin the Descent Cycle is taken as $\bar{\bar{x}}^2$ defined above.

D5.5: Let $x^1, ..., x^s$ be all the points obtained at the end of all the descent steps carried out in the latest D5.1 above in this iteration; and suppose x^s is the best among all these by objective value. For $i = 1$ to $s-1$, define $x^i(\alpha) = x^s + \alpha(x^i - x^s)$.

For $i = 1$ to $s-1$, carry out the following step.

Step: Take $\alpha = 2^{-p}$, start with $p = 1$. Take a descent step from $x^i(\alpha)$ in the direction $-c^T$.

If the output point corresponds to an objective value $< cx^s$, call this point $\tilde{x}^i$, then go to the next value of i. If the output point corresponds to an objective value $> cx^s$, keep the value of i the same but increment p by 1 and repeat the above step.

Let $\tilde{x}$ denote the best by objective value among the $\tilde{x}^i$. Take $\tilde{x}$ as the initial interior feasible solution for carrying out D5.2 followed by D5.3.

2 Method Used for Solving 2-variable LPs of the Form (3)

In SMs, we solve 2-variable LPs in variables (δ, α) of the form (3) in various stages. All these problems arise in finding the optimum step length (value of α that maximizes δ) from an interior feasible solution $\bar{x}$ of the current set of feasible solutions, with $\delta(\bar{x}) = \bar{\delta}$, in a profitable direction y. So in all such instances we have an initial feasible solution $(\delta, \alpha) = (\bar{\delta}, 0)$ for the instance of (3) being solved. We use the following method to solve this instance.

Let Γ denote the set of feasible solutions of the instance of (3) in the 2-dimensional space of (δ, α) with α plotted on the horizontal axis, and δ plotted on the vertical axis. The method performs a series of iterations. The first iteration begins with $(\bar{\delta}, 0)$ on the boundary of Γ. Each iteration begins with a feasible solution on the boundary of Γ, performs a (horizontal move + a vertical move) twice, and finally a diagonal move. We will now discuss a general iteration beginning with the intial solution (δ_0, α_0).

The first horizontal move: Keeping $\delta = \delta_0$, find $\alpha_1 =$ the value of α at the mid-point of the line segment $\{(\delta_0, \alpha) \in \Gamma\}$. Given $\delta = \delta_0$, we know from the constraints in (3) that $\theta_1(\delta_0) \leq \alpha \leq \theta_2(\delta_0)$ where

$\theta_1(\delta_0) =$ Maximum$\{0, (A_{i.}\bar{x} - b_i - \delta_0)/(-A_{i.}y)$: over i such that $A_{i.}y > 0\}$
$\theta_2(\delta_0) =$ Minimum$\{(A_{i.}\bar{x} - b_i - \delta_0)/(-A_{i.}y)$: over i such that $A_{i.}y < 0\}$.

So we know that the α_1 mentioned above is $(\theta_1(\delta_0)+\theta_2(\delta_0))/2$. We will call the corresponding point (δ_0, α_1) in Γ as the *Center of* Γ on $\delta = \delta_0$.

The vertical move: In this move α is held constant at present value α_1, and the maximum value of δ subject to the constraint that $(\delta, \alpha_1) \in \Gamma$ is computed. This is equal to:

$\delta_1 = \gamma(\alpha_1) = \text{Minimum}\{A_{i.}\bar{x} - b_i + \alpha_1 A_{i.}y : i = 1 \text{ to } m\}$.

and the point in Γ achieving this value of δ is (δ_1, α_1).

The 2nd (horizontal + vertical) moves: Find the center (δ_1, α_2) of Γ on $\delta = \delta_1$ as described above. Then the 2nd vertical move on $\alpha = \alpha_2$ finds the maximum value δ_2 of δ keeping $\alpha = \alpha_2$, attained at the point $(\delta_2, \alpha_2) \in \Gamma$.

The diagonal move: This move involves finding the maximum value of δ for points along the line joining the two centers of Γ obtained in the two horizontal moves in this iteration. The two centers are $(\delta_0, \alpha_1), (\delta_1, \alpha_2)$ where $\delta_1 > \delta_0$. Let L denote the line joining these two centers. From the coordianates of these two centers we know that that L is defined by the equation

$$\delta \quad = \quad \delta_0 + s(\alpha - \alpha_1)$$

where $s = (\delta_1 - \delta_0)/(\alpha_2 - \alpha_1)$. Let

$\beta_1 =$ minimum value of α in $L \cap \Gamma$ is $=$ maximum$\{(0, A_{i.}\bar{x} - b_i - \delta_0 + s\alpha_1)/(s - A_{i.}y)$: over i such that $s - A_{i.}y < 0\}$.

$\beta_2 =$ maximum value of α in $L \cap \Gamma$ is $=$ minimum$\{(A_{i.}\bar{x} - b_i - \delta_0 + s\alpha_1)/(s - A_{i.}y)$: over i such that $s - A_{i.}y > 0\}$.

So, the maximum value of δ on $L \cap \Gamma$ is δ_3, where

$\delta_3 = \delta_0 + (\beta_2 - \alpha_1)s$ attained at the point (δ_3, β_2) if $\alpha_2 > \alpha_1$, or

$\delta_3 = \delta_0 + (\beta_1 - \alpha_1)s$ attained at the point (δ_3, β_1) if $\alpha_2 < \alpha_1$.

Let $\delta_4 = \text{maximum}\{\delta_2, \delta_3\}$; and denote the associated value of α for it given above by α_4. Then (δ_4, α_4) is the output of this iteration. With this point go to the next iteration.

Terminate the method with the output in an iteration when the improvement in the value of δ becomes small.

3 Sphere methods 1, 2, 2.1

In SM-1 , in each iteration the centering cycle is carried out for the original set of feasible solutions K as discussed above (also, see [14] for additional details), and uses descent steps D1 to D5.1 in the descent cycle in each iteration.

In SM-2, in each iteration the set of feasible solutions is updated by the current objective value as described above (also see [15] for details), and uses descent steps D1 to D5.3 in the descent cycle in each iteration.

Sphere method 2.1 (SM-2.1) is a modidfied version of SM-2 with the centering cycle replaced by a light centering cycle (the same centering cycle, but with the number of line search steps in profitable directions is limited to a preselected upper bound), for details see [15]).

Denoting the output of the general iteration $r+1$ by $\bar{x}^{r+1,1}$ in any of these SMs, the method is terminated with the conclusion that $\bar{x}^{r+1,1}$ is a near optimum solution of (1), if $||\bar{x}^{r+1,1} - \bar{x}^{r,1}||/||\bar{x}^{r+1,1}|| < \epsilon$.

4 Sphere Methods 3, 4 (SM-3, 4)

These are versions of sphere methods using no matrix inversions.

Sphere Method −3 (SM-3)

SM-3 is a version of SM-1 to solve the LP (1) beginning with an initial interior feasible solution x^0 for it, in which the Centering Cycle is replaced by a new routine not using any matrix inversions. We now describe the General iteration in SM-3. Let x^r denote the initial interior feasible solution for this iteration.

GENERAL ITERATION IN SM-3:

Centering Cycle: This step selects the center for carrying out the Descent cycle in this iteration in SM-3

There are two cases to consider.

Case 1: There is at least one $i \in T(x^r)$ satisfying $cA_{i.}^T < 0$.

For each $i \in T(x^r)$ satisfying $cA_{i.}^T < 0$, solve the 2-variable LP to maximize $\delta(x^r + \lambda A_{i.}^T)$ over $\lambda \geq 0$, and let $x(i)$ be an optimum solution obtained.

Also solve another 2-variable LP to maximize $\delta(x^r + \lambda y)$ over $\lambda \geq 0$, where y = average of $(A_{i.})^T$ over i satisfying $c(A_{i.})^T < 0$.

Among the optimum solutions of all these 2-variable LPs, take the one corresponding to the least value of cx as the initial center in the algorithm in this iteration.

CASE 2: For all $i \in T(x^r)$, $\quad cA_{i.}^T \geq 0$.

For each $i \in T(x^r)$, find the orthogonal projection y^i of $A_{i.}^T$ on the hyperplane $\{x : cx = cx^r\}$.

Solve the 2-variable LP to maximize $\delta(x^r + \lambda y^i)$ over $\lambda \geq 0$, and let $x(i)$ be the resulting point obtained. Notice that $cx(i) = cx^r$ for all $i \in T(x^r)$. Among these $x(i)$, take the one corresponding to the highest value for $\delta(x(i))$ = the radius of the largest inscribed ball inside K with $x(i)$ as center, as the initial center in the algorithm in this Iteration.

Descent Cycle: Descent steps in this iteration in SM-3

Let $\bar{x}$ be the center for initiating these steps.

Perform Descent cycle (descent steps D1 to D5.5) and with the best point obtained from these descent steps, keep repeating D5.5 as long as improvement in objective value is being realized. When improvement becomes less than the tolerance, with the best point obtained at that stage as the initial interior feasible solution go to the next iteration.

Sphere Method 4 (SM-4)

SM-4 provides a way of improving performance even further by using SM-3 for carrying out the centering cycle in SM-2 without using any matrix inversions.

This method is exactly Sphere Method 2 (SM-2) that we discussed earlier, with the centering routine replaced by a new one based on SM-3. It uses no matrix inversion operations like SM-3.

Each iteration in this method consists of two cycles. In this method, the set of feasible solutions considered keeps getting smaller after each iteration like in SM-2. Iteration 1 begins with the initial interior feasible solution x^0 given in the input.

General iteration $r+1$

Let x^r be the initial interior feasible solution to start this iteration. This iteration consists of the Centering cycle, and the Descent cycle.

Centering cycle: K^r, the set of feasible solutions considered in this iteration is the set of feasible solutions of: $Ax \geq b$, and the additional constraint $A_{m+1.}x \geq b_{m+1}$ where $A_{m+1.} = -c$ and $b_{m+1} = -cx^r - \epsilon$, here ϵ is a small positive tolerance.. The LP formulation of the problem of finding the "ball center" of K^r is:

$$\begin{array}{lllll} \text{Maximize} & \delta & & & \\ \text{subject to} & \delta & \leq & A_{i.}x - b_i & i = 1, \ldots, m+1 \end{array} \tag{5}$$

Since x^r is an interior feasible solution for the original LP (1), we know that $A_{i.}x^r - b_i > 0$ for all $i = 1$ to $m+1$. Select δ^r satisfying $0 < \delta^r <$ Minimum$\{A_{i.}x^r - b_i : i = 1$ to $m+1\}$. Then (δ^r, x^r) is an interior feasible solution of (5). Using this as the initial interior feasible solution, apply SM-3 to find an optimum solution this LP (5). Let $(\bar{x}^r, \bar{\delta}^r)$, be the optimum solution for (5) obtained by SM-3. Go to the Descent cycle with $\bar{x}^r$ as the center.

Descent Cycle: With $\bar{x}^r$ as the center apply descent steps D1 to D5.4. Repeat D5.4 as long as improvement is being obtained. Let x^{r+1} be the best point obtained (least objective value) from all these descent steps.

Go to the next iteration with x^{r+1} as the initial interior feasible solution.

Computational results with SM-3, 4

Advantages of SM-3, 4

Since both SM-3, 4 do not need any matrix inversion operations, these algorithms are easier to implement than existing methods, and implementing these new methods does not need any complicated roundoff error precautions, and other linear algebra routines. Also these methods are directly amenable to parallel implementations, which makes it possible to solve large models easily.

5 Conclusion

We presented some preliminary computational results on implementing sphere Methods 3, 4 by solving each step in these methods using MATLAB 7.0 routines separately; and compared this performance with that of MATLABs finished LP code "linprog" based on the simplex method. The results show that even this implementation of the sphere methods performs much better than "linprog".

To compare the sphere methods with existing IPMs will require developing a low-level programming language code for them using advanced techniques of numerical linear algebra and updating the basis inverse in LSCPD steps as the matrix grows by a row and column as described above (for SM-1, 2, 2.1); and parallel implementations of SM-3, 4; which we have not done in these preliminary experiments. But these preliminary results, and the fact that the work in each iteration of SM-2, 2.1, and particularly SM-3, 4 is much simpler than an iteration of other existing IPMs indicates that these sphere methods will have advantage over them for solving large scale models, in particular when the models may have redundant constraints, or a coefficient matrix that is not very sparse.

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