A New Practically Efficient Interior Point Method for Quadratic Programming

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Abstract

In [4] a new interior point method has been developed for linear programming (LP), based on a new centering strategy that moves any interior feasible solution x^0 to the center of the intersection of the feasible region with the objective hyperplane through x^0, before beginning the descent moves. Using this centering strategy, that method obtains an optimum solution for an LP by a very efficient descent method that uses no matrix inversions. Here we extend that method into a descent method for solving quadratic programs (QP). The advantages of this method are: (i) all the constraints in the problem never appear together in any matrix inversion operations performed in the algorithm, (ii) each iteration in the algorithm consists of essentially three steps, one step requires no matrix inversions, a second step requires solving a system of linear equations involving a small subset of constraints, a third step involves matrix operations involving only the coefficient matrix of the objective function. So, compared to other existing methods for QP, the new method is able to handle it with minimal matrix inversion computations.

Key words: Quadratic programming (QP), interior point method.

1 Introduction

We consider the quadratic program (QP)

$$\begin{array}{lrcl} \text{Minimize} & Q(x) = cx + (1/2)x^T Dx & & \\ & \text{subject to} \quad Ax & \geq & b \end{array} \qquad (1)$$

where the objective coefficient matrix D is a symmetric matrix of order n, the constraint coefficient matrix A is of order $m \times n$, and b, c are column and row vectors of appropriate orders [2, 3, 5]. Let K denote the set of feasible solutions. For simplicity we assume that K is bounded. We also assume that an interior point x^0 of K (i.e., a point satisfying $Ax^0 > b$) is available.

In this paper we assume that D is positive definite, i.e., that $Q(x)$ is strictly convex. Strategies for relaxing this assumption are discussed briefly in Section 7.

Let $K^0 = \{x : Ax > b\}$, it is the interior of K. We assume that the row vectors of A, denoted by $A_{i.}$ for $i = 1$ to m, are normalized so that their Euclidean norm $||A_{i.}|| = 1$ for all i. For each $x \in K^0$, we define $\delta(x) = \min\{A_{i.}x - b_i : i = 1 \text{ to } m\}$, $\delta(x)$ is the radius of the largest ball that can be inscribed within K with its center at x.

In [4], in the iteration when x^0 is the current interior feasible solution, the centering step has the aim of finding an $x \in K^0$ on the objective plane through x^0, that maximizes $\delta(x)$ so as to get the largest ball inscribed in K with center at an interior feasible solution that has the same objective value as x^0. In our problem here, the set of all points with the same objective value as x^0 is a nonlinear surface and not a hyperplane; so we will not constrain the center to have the same objective value as x^0 in the centering step here, but will allow only moves that keep the objective value the same or decrease it while increasing $\delta(x)$.

2 The Centering Strategy

When x^0 is the current interior feasible solution for (1), the problem of finding the largest inscribed sphere inside K with center at a point where the objective value $Q(x)$ is $\leq Q(x^0)$ is the following constrained max-min problem:

$$\begin{array}{lrcll} \text{Maximize} & \delta & & & \\ \text{subject to} & \delta - A_{i.}x & \leq & -b_i, & i = 1, ..., m \\ & Q(x) & \leq & Q(x^0) & \end{array} \qquad (2)$$

If $(\bar{x}, \bar{\delta})$ is an optimum solution of this problem, then $\bar{\delta} = \delta(\bar{x})$, and the ball $B(\bar{x}, \bar{\delta})$ with $\bar{x}$ as center, and $\bar{\delta}$ as radius is a largest inscribed sphere required. This problem (2) is itself a quadratic program. This type of model may have to be solved several times before we get a solution for our original QP (1), and for implementing our algorithm an exact solution of (2) is not essential, so solving (2) exactly will be counterproductive. Using the special max-min structure of (2), we now develop an efficient procedure for getting an approximate solution to (2), similar to the one developed in [4] for the corresponding centering problem in the algorithm discussed there for LP.

Procedure for Getting an Approximate Solution for (2)

Since our goal is to increase the minimum distance of x from the facetal hyperplanes of K, an approximate solution of (2) can be obtained through line searches in directions perpendicular to the facetal hyperplanes of K. So, in this procedure, for finding the new center $x \in K^0 \cap \{x : Q(x) \leq Q(x^0)\}$, we only consider moves in directions among $\Gamma = \{A_{i.}^T, -A_{i.}^T : i = 1, ..., m\}$ which are descent directions for $Q(x)$ at the current point.

So, this procedure consists of a series of moves beginning with x^0, generating a sequence of points $x^r \in K^0 \cap \{x : Q(x) \leq Q(x^0)\}$, $r = 1, 2, ...$ When at x^r look for a **profitable direction to move** at x^r, which is a direction $p \in \Gamma = \{A_{i.}^T, -A_{i.}^T : i = 1, ..., m\}$ satisfying:

(i): $\nabla Q(x^r)p < 0$, and

(ii): $\delta(x^r + \alpha p)$ increases as α changes from 0 to positive values.

For any $x \in K^0$ define $T(x) = \{i : 1 \leq i \leq m$, and i ties for the minimum in $\delta(x) = \text{minimum}\{A_{i.}x - b_i : i = 1, ..., m\}\}$. $T(x)$ is known as the **index set of touching constraints** at x, because it is the index set of facetal hyperplanes of K which are tangents to the ball $B(x, \delta(x))$ if each constraint in (1) defines a facetal hyperplane for K. In [4] it has been shown that a direction p satisfies condition (ii) above at x^r iff all the entries in $\{A_{t.}p : t \in T(x^r)\}$ are of the same sign. So, for any given direction p, both (i), (ii) can be checked easily to determine if p is a profitable direction to move at x^r.

If a profitable direction $p \in \Gamma$ to move at x^r has been found, the step length α to move at x^r in the direction p to get the next point in the sequence $x^{r+1} = x^r + \alpha p$ is defined to be: $\alpha = \text{minimum}\{\beta_1, \beta_2\}$ where

> $\beta_1 =$ the value of β that minimizes $Q(x^r + \beta p)$ over $\beta \geq 0$. Finding β_1 therefore requires minimizing a quadratic function in the single variable β, which can be solved easily.

β_2 = the value of β that maximizes $\delta(x^r + \beta p)$ over $\beta \geq 0$. In [4] it has been shown that this can be found by solving the following 2-variable linear program in which the variables are θ, β.

$$\begin{array}{rrcll} & \text{Maximize} \quad \theta & & & \\ \text{subject to} & \theta - \beta A_{i.}p & \leq & A_{i.}x^r - b_i & \quad i = 1, \ldots, m \\ & \theta, \beta & \geq & 0 & \end{array}$$

which can be found with at most $O(m)$ effort. [4] discusses how to solve this efficiently.

Once β_1, β_2 are determined, let $\alpha = \text{minimum}\{\beta_1, \beta_2\}$, take the next point in the sequence to be $x^{r+1} = x^r + \alpha p$, and continue the procedure in the same way with x^{r+1}.

The procedure continues as long as profitable directions $p \in \Gamma$ to move at the current point can be found.

When there are several profitable directions to move at the current point in this procedure, efficient selection criteria to choose the best among them can be developed. In fact, additional directions can be included in Γ to improve the quality of the approximation obtained. When there are no profitable directions to move at the current point, or when improvement in the value of the radius of the inscribed ball becomes smaller than some selected tolerance, take the current point in the sequence as the center selected by this procedure.

As can be seen, the procedure used in this centering strategy does not need any matrix inversion, and only solves a series of 2-variable LPs, and single variable quadratic function minimization problems, which can be solved very efficiently. Hence this centering strategy can be expected to be efficient.

What is the Purpose of Maximizing the Radius of the Inscribed Ball in this Centering Step?

Our goal is to find an optimum solution to the original quadratic program (1). Then, why are we focussing on the seemingly unrelated problem of maximizing the radius of the inscribed ball in this centering step? The reason is the following.

Let $B(\bar{x}, \bar{\delta})$, the ball with center $\bar{x}$ and radius $\bar{\delta}$ be the ball constructed in this centering step. Then in this iteration the algorithm uses the direction $\hat{x} - \bar{x}$ as a descent direction for a line search step to minimize $Q(x)$ over $\{\bar{x} + \lambda(\hat{x} - \bar{x}) : \lambda \geq 0$, and λ such that $\bar{x} + \lambda(\hat{x} - \bar{x}) \in K\}\}$, where $\hat{x}$ is a point that minimizes $Q(x)$ over the ball $B(\bar{x}, \bar{\delta})$. There are efficient polynomial time algorithms for computing $\hat{x}$, but its computation is perhaps the most expensive computational operation in this algorithm. Maximizing $\bar{\delta}$, the radius of the ball found in this centering step, helps to reduce the number of times this expensive step has to be used in this algorithm.

3 Descent Step Using a Descent Direction

Let $B(\bar{x}, \bar{\delta}) = \{x : (x - \bar{x})^T(x - \bar{x}) \leq \bar{\delta}^2\}$ be the ball with center $\bar{x}$, and radius $\bar{\delta}$, obtained in the centering step. In this step we solve the problem

$$\begin{array}{rll} \text{Minimize} & Q(x) = cx + (1/2)x^T Dx & \\ \text{subject to} & (x - \bar{x})^T(x - \bar{x}) & \leq \bar{\delta}^2 \end{array} \tag{3}$$

This is the problem of minimizing a quadratic function inside a ball for which efficient polynomial time algorithms exist. Associating the Lagrange multiplier $\lambda \in R^1$ with the constraint, the KKT optimality conditions for this problem are

$$\begin{array}{rcl} c^T + Dx + 2\lambda(x - \bar{x}) & = & 0 \\ \lambda \geq 0, \quad \bar{\delta}^2 - (x - \bar{x})^T(x - \bar{x}) & \geq & 0 \\ \lambda(\bar{\delta}^2 - (x - \bar{x})^T(x - \bar{x})) & = & 0 \end{array}$$

Since $\lambda \in R^1$, this problem can be solved efficiently (in polynomial time) using the KKT conditions, see [1, 5] for complete details of this algorithm. The algorithm becomes simpler when D is positive definite or semidefinite, but even if D is not positive semidefinite, it can be solved efficiently using the KKT conditions.

Let $\hat{x}$ be the optimum solution computed for (3). If $\hat{x}$ is an interior point of $B(\bar{x}, \bar{\delta})$, or if it is a boundary point of both $B(\bar{x}, \bar{\delta})$ and K, or if $\nabla Q(\hat{x}) = 0$; then $\hat{x}$ is an optimum solution of (1), terminate.

Otherwise, using $\hat{x} - \bar{x}$ as the descent direction for $Q(x)$ at $\bar{x}$, do a line search to minimize $Q(x)$ on the line segment $\{\bar{x} + \lambda(\hat{x} - \bar{x}) : \lambda \geq 0$, and λ such that $\bar{x} + \lambda(\hat{x} - \bar{x}) \in K\}$. Let λ_1 be the optimum step length for this line search. If $\bar{x} + \lambda_1(\hat{x} - \bar{x})$ is an interior point of K; then terminate if $\nabla Q(x) = 0$ at this point, otherwise define this point as the output of this step.

If however, $\bar{x} + \lambda_1(\hat{x} - \bar{x})$ is a boundary point of K, let $I = \{i : i$-th constraint in (1) is satisfied as an equation by $\bar{x} + \lambda_1(\hat{x} - \bar{x})\}$. If the following system in Lagrange multipliers $\pi_I = (\pi_i : i \in I)$

$$\begin{array}{rcl} c + (\bar{x} + \lambda_1(\hat{x} - \bar{x}))^T D - \sum_{i \in I} \pi_i A_{i.} & = & 0 \\ \pi_i \geq 0 \quad \text{for all} & & i \in I \end{array} \tag{4}$$

has a feasible solution, then $\bar{x} + \lambda_1(\hat{x} - \bar{x})$ is an optimum solution of (1), terminate. However, it may not be productive to check if system (4) is feasible every time this step ends up at this stage. If this operation of checking the feasibility of (4) is not carried out, or if (4) turns out to be infeasible, then take the output of this step as $\bar{x} + (\lambda_1 - \epsilon)(\hat{x} - \bar{x})$ where ϵ is some preselected positive tolerance for the current point to be an interior point of K.

4 Descent Step Using the Touching Costraints

We will first provide the motivation for this step. Assume that the centering step is carried out exactly, and suppose $B(\bar{x}, \bar{\delta}) = \{x : (x - \bar{x})^T(x - \bar{x}) \leq \bar{\delta}^2\}$ is the ball with center $\bar{x}$ and radius $\bar{\delta}$ obtained in the centering step in this iteration. $T(\bar{x}) = \{i : A_{i.}\bar{x} = b_i + \bar{\delta}\}$ is the index set of **touching constraints** in this iteration, this is the index set of facetal hyperplanes of K that are touching the ball $B(\bar{x}, \bar{\delta})$ and hence are tangent hyperplanes for it. Actually $T(\bar{x})$ is the index set of linear constraints in (2) that are active at its optimum solution, all other linear constraints in (2) are inactive at its optimum solution; and the same thing is also true for the problem obtained by replacing x^0 in (2) by $\bar{x}$. So, $(\bar{x}, \bar{\delta})$ is an optimum solution for (2) when x^0 there is replaced by $\bar{x}$, i.e., for

$$\begin{array}{rrcll} \text{Maximize} & \delta & & & \\ \text{subject to} & \delta - A_{i.}x & \leq & -b_i, & i = 1, ..., m \\ & Q(x) & \leq & Q(\bar{x}) & \end{array} \tag{5}$$

It often happens the the index set of touching constraints for the ball obtained from an optimum solution of (5) with $Q(\bar{x})$ replaced by $Q(\bar{x}) - \gamma$ remains the same as $T(\bar{x})$, for a range of values of γ, say $0 \leq \gamma \leq \gamma_1$. In this range $0 \leq \gamma \leq \gamma_1$, let $\delta(\gamma)$ denote the optimum radius of the ball, and $x(\gamma)$ the center. Beginning with $\delta(0) = \bar{\delta}$, clearly, $\delta(\gamma)$ decreases as γ increases to γ_1. From these facts we see that in the range $\delta(0) \geq \delta(\gamma) \geq \delta(\gamma_1)$, $x(\gamma)$ is the optimum solution of

$$\begin{array}{rll} \text{Minimize} & Q(x) & \\ \text{subject to} & A_{i.}x & = b_i + \delta(\gamma), \quad i \in T(\bar{x}) \end{array} \tag{6}$$

Replacing the parameter $\delta(\gamma)$ by the symbol s, an optimum solution for (6) can be obtained by solving

$$
\begin{aligned}
c^T + Dx - \sum_{i \in T(\bar{x})} \pi_i A_{i.} &= 0 \qquad (7)\\
A_{i.}x &= b_i + s, \quad i \in T(\bar{x})
\end{aligned}
$$

where $\pi_{T(\bar{x})} = (\pi_i : i \in T(\bar{x}))$ is the vector of lagrange multipliers for (6). If $(x(s), \pi_{T(\bar{x}}(s))$ is a solution of (7) as a function of the parameter s, then $x(s)$ defines a straight line in R^n in terms of the parameter s. The above argument shows that by carrying out a line search step on this straight line, we can decrease the value of $Q(x)$ to reach $Q(x(\gamma_1))$; and any further decrease in the value of $Q(x)$ below this will lead to an optimal touching constraint index set for the ball different from $T(\bar{x})$.

Even when (2) is solved approximately, we may improve the objective value by carrying out this work with the ball obtained. That is what this step does.

Denoting the ball obtained in the centering step by the same symbol $B(\bar{x}, \bar{\delta}) = \{x : (x - \bar{x})^T (x - \bar{x}) \leq \bar{\delta}^2\}$, denote the touching constraint index set by the same symbol as above $T(\bar{x}) = \{i : A_{i.}\bar{x} = b_i + \bar{\delta}\}$. With this $T(\bar{x})$, get the solution $(x(s), \pi_{T(\bar{x})})$ for system (7). Then do a line search to minimize $Q(x)$ over the line segment $\{x(s) : s \text{ such that } x(s) \in K\}$. Suppose $s = s_1$ gives the optimum $x(s)$ in this line search step.

If $x(s_1)$ is an interior point of K; then terminate if $\nabla Q(x) = 0$ at this point, otherwise define this point as the output of this step.

If however, $x(s_1)$ is a boundary point of K, let $I = \{i : i\text{-th constraint in (1) is satisfied as an equation by } x(s_1)\}$. If the following system in Lagrange multipliers $\pi_I = (\pi_i : i \in I)$

$$
\begin{aligned}
c + x(s_1)^T D - \sum_{i \in I} \pi_i A_{i.} &= 0 \qquad (8)\\
\pi_i \geq 0 \quad \text{for all} &\qquad i \in I
\end{aligned}
$$

has a feasible solution, then $x(s_1)$ is an optimum solution of (1), terminate. However, it may not be productive to check if system (8) is feasible every time this step ends up at this stage. If this operation of checking the feasibility of (8) is not carried out, or if (8) turns out to be infeasible, then take the output of this step as a point on the line segment $\{x(s) : s \in R^1\}$ close to $x(s_1)$ but in the interior of K.

5 The Algorithm

The algorithm consists of repititions of the following iteration beginning with an initial interior point of K. We will now describe the general iteration. In each iteration, Steps 2.1 and 2.2 are parallel steps, both of which begin with the ball obtained in the centering step in the iteration.

A General Iteration

Let x^0 be the current interior feasible solution.

1. Centering Strategy: Apply the centering strategy described in Section 2 beginning with the current interior feasble solution. Let $B(\bar{x}, \bar{\delta})$ denote the ball obtained with center $\bar{x}$ and radius $\bar{\delta}$. Let $T(\bar{x}) = \{i : A_{i.}\bar{x} = b_i + \bar{\delta}\}$ is the index set of touching constraints for this ball.

2.1. Descent Step Using a Descent Direction: Apply this strategy described in Section 3 beginning with the ball $B(\bar{x}, \bar{\delta})$. If termination did not occur in this step, let x^1 denote the interior feasible solution of (1) which is the output point in this step.

2.2. Descent Step Using the Touching Constraints: Apply this strategy described in Section 4 beginning with the ball $B(\bar{x}, \bar{\delta})$. If termination did not occur in this step, let x^2 denote the interior feasible solution of (1) which is the output point in this step.

3. Move to Next Iteration: Define the new current interior feasible solution as the point among x^1, x^2 obtained in Steps 2.1, 2.2, which gives the smallest value for $Q(x)$. With it, go to the next iteration.

6 Convergence Results

In this section we discuss convergence results on the algorithm under the assumption that the centering problem is solved to optimality in every iteration.

Theorem 1: Cosider the following version of (2) with $Q(x^0)$ replaced by a parameter t.

$$\begin{array}{rlrcll} \delta[t] = & \text{Maximum value of} & \delta & & & \\ & \text{subject to} & \delta - A_{i.}x & \leq & -b_i, & i = 1, ..., m \\ & & Q(x) & \leq & t & \end{array} \quad (9)$$

$\delta[t]$ is a concave function of t in the interval of values of t for which the above problem has a feasible solution.

Proof: Let $t_{\min}, t_{\max}$ be the minimum and maximum values of $Q(x)$ over $x \in K$. Let t_1, t_2 be any pair of values in the interval $[t_{\min}, t_{\max}]$; and suppose $(x^1, \delta^1), (x^2, \delta^2)$ are optimum solutions of (9) when $t = t_1, t_2$ respectively. Let $0 < \alpha < 1$.

Since $Q(x)$ is convex, we have $Q(\alpha x^1 + (1-\alpha)x^2) \leq \alpha Q(x^1) + (1-\alpha)Q(x^2)$ $\leq \alpha t_1 + (1-\alpha)t_2$. From this we verify that $(\alpha x^1 + (1-\alpha)x^2, \alpha\delta^1 + (1-\alpha)\delta^2)$ is feasible to (9) when $t = \alpha t_1 + (1-\alpha)t_2$, but it may not be an optimum solution of (9).

Therefore the optimum objective value in (9) when $t = \alpha t_1 + (1-\alpha)t_2$, $\delta[\alpha t_1 + (1-\alpha)t_2] \geq \alpha\delta^1 + (1-\alpha)\delta^2 = \alpha\delta[t_1] + (1-\alpha)\delta[t_2]$. This shows that $\delta[t]$ satisfies Jensen's inequality required for being concave. ∎

Let $P(t)$ denote the set of feasible solutions of (9). Clearly, for $t_1 < t_2$, we have $P(t_1) \subset P(t_2)$. So, $\delta[t]$ decreases monotonically as t decreases; and since it is concave its slope decreases as t increases.

Theorem 2: The index set of touching constraints for the ball obtained in the centering step changes after each iteration in the algorithm.

Proof: This follows since the output point in each iteration in the algorithm, is selected as the best among the outputs in Steps 2.1, 2.2 in that iteration. ∎

Theorem 3: The algorithm terminates after at most $2m$ iterations.

Proof: Select an index between 1 to m, say i_1. As t is decreasing to $t_{\min}$, suppose the index i_1 is in the touching constraint index set for the ball obtained from (9) when $t = t_1$, and drops out of this set when t decreases from t_1. This implies that the system of constraints

$$\begin{array}{rcll} \delta - A_{i_1.}x & = & -b_{i_1}, & \\ \delta - A_{i.}x & \leq & -b_i, & i \neq i_1 \\ Q(x) & \leq & t & \end{array} \tag{10}$$

is feasible when $t = t_1$, and infeasible when t is slightly smaller than t_1. From convexity of $Q(x)$ we know that the set of values of t for which (10) is feasible is an interval. These facts imply that (10) is infeasible for all $t < t_1$, i.e., as t decreases below t_1, the index i_1 can never be in the touching constraint index set. So, once an index drops out of the touchnig constraint index set in the algorithm, it can never enter it in a subsequent iteration. Since the touching constraint index set changes in every iteration, these facts prove the theorem. ∎

Even if the centering step is carried out approximately, these results indicate that if it is carried to good accuracy, the algorithm will have superior performance.

7 The Case When the Matrix D is Not Positive Definite

Relaxing the positive definiteness assumption on the matrix D leads to a vast number of applications for the model (1). For example, an important model with many applications is the following $0-1$ mixed integer programming (MIP) model:

$$\begin{array}{rrcl} \text{Minimize} & cx & & \\ \text{subject to} & Ax & \geq & b \\ & x & \geq 0 & \\ x_j = 0 \text{ or } 1 \text{ for each} & j & \in & J \end{array} \tag{11}$$

where J is the subscript set for variables which are required to be binary. Solving this problem is equivalent to finding the global minimum in the quadratic program

$$\begin{array}{rrcll} \text{Minimize} & cx + M\sum_{j \in J} x_j(1-x_j) & & & \\ \text{subject to} & Ax & \geq & b & \\ & x_j & \geq & 0 & \text{for } j \notin J \\ & 0 \leq x_j & \leq & 1 & \text{for each } j \in J \end{array} \tag{12}$$

where M is a large positive penalty coefficient; which is in the form (1) with D negative semidefinite. Unlike the model (1) when D is positive definite, (12) may have many local minima, and we need to find the global minimum for (12).

Some of the steps in this algorithm can still be carried out. The approximate centering procedure can be carried out. Also, Steps 2.1 can be carried out exactly. For Step 2.2, the system of equations (6) may typically have a unique solution. Even when (6) has many feasible solutions, a solution to (7) may not even be a local minimum for (6), in fact it may be a local maximum for (6). So, the value of including Step 2.2 in the algorithm is not clear in this case. Also, many of the proofs in Section 6 based on convexity will not be valid in this nonconvex case.

However, since the ball minimization problems in Step 2.1 can be solved exactly, there is reason to hope that by adjusting the value of the penalty cost coefficient M during the algorithm, the algorithm can be made to lead to a good local minimum, and thereby offer a good heuristic approach. For this general case, these and other issues need to be pursued.

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