Ball Centers of Special Polytopes

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Abstract

We discuss the concepts of ball centers of convex polytopes (bounded convex polyhedra), and computational issues related to them, and how these issues are tied up with the way the polytope is represented. Ball centers play an important role in the sphere methods for linear programs, which points out the importance of efficient methods for computing them, at least approximately. We show that if a polytope in R^n is either an n-dimensional simplex, or the convex hull of a linearly independent set of vectors in R^n, then its ball center is unique and can be computed directly from the data.

Key words: Linear Programming (LP), Interior point methods (IPMs) , ball centers of a polytope, Sphere methods-1, 2 for LPs.

1 Ball Centers of Polytopes and Problems Related to Them

Currently, Interior Point Methods (IPMs) are the most popular methods for large scale linear programming (LP), most software systems for solving large scale LPs are based on some IPM. The concept of the *center* of the feasible region is an important concept in IPMs for LP.

Sphere methods (Murty [2006-1, 2], Murty, Oskoorouchi [2008-1, 2]) are new IPMs being developed for large scale LP. The center used in sphere methods is called the **ball center**, it is the center of a largest ball inside the feasible region. Consider LPs in the form

$$\begin{aligned} \text{minimize} \quad z(x) &= cx \\ \text{subject to} \quad Ax &\geq b \end{aligned} \tag{1}$$

where A is an $m \times n$ data matrix, with a known initial interior feasible solution $x^0 \in R^n$ (i.e., satisfying $Ax^0 > b$). The constraints in (1) include all bound constraints on individual variables. We assume that the rows of A, denoted by $A_{i.}$ for $i = 1$ to m, have been normalized, so that $||A_{i.}|| = 1$ ($||.||$ denotes the Euclidean norm) for all $i = 1$ to m.

For any matrix E we will denote by $E_{i.}$, $E_{.j}$ its i-th row, j-th column respectively. The symbol e denotes a vector in which all the entries are 1; it may be a row or column of appropriate dimension, depending on the context.

Let K denote the set of feasible solutions of (1). Here we will assume that K is bounded, i.e. that K is a convex polytope of full dimension n, represented by the system of linear constraints (1). Let $K^0 = \{x : Ax > b\}$ denote the interior of K.

The orthogonal distance from a point $y \in R^n$ to the facetal hyperplane $\{x : A_{i.}x = b_i\}$ of the set K is: minimum$\{||x - y|| : A_{i.}x = b_i\} = A_{i.}y - b_i$, because $||A_{i.}|| = 1$.

For each $x \in K^0$, let $\delta(x)$ denote the radius of the largest ball inside K with x as center. For $x \in K^0$, $\delta(x)$ is the minimum among the orthogonal distances of x to the various facetal hyperplanes of K, so, $\delta(x) = \text{Min}\{A_{i.}x - b_i : i = 1 \text{ to } m\}$, since $||A_{i.}|| = 1$ for all i. And, the largest ball inside K with x as its center is $B(x, \delta(x)) = \{y : ||y - x|| \leq \delta(x)\}$.

Definitions: A **ball center** of a convex polytope, is a point in it which is the center of a largest ball inside that polytope.

Also, given a hyperplane H satisfying $H \cap K^0 \neq \emptyset$, a **ball center of K on H** is a point in $H \cap K$ that is the center of a largest radius ball inside K with its center restricted to H.

Now consider the following problems:

Problem 1: Given an interior point x in the polytope K, what is the radius $\delta(x)$ of the largest ball with x as center, that is contained inside K?

Problem 2: We want to find a largest ball inside the polytope K. Is it unique, and if so what is its center and radius? If not, what is the set of all points, each of which is the center of a largest ball inside K? Also, given an interior point of K, how can we check whether it is the center of a largest ball inside K?

Hence when the polytope K is specified through linear constraints, for each point x in the interior of K, Problem 1 mentioned above can be solved efficiently using the formulas given above.

For our polytope K specified by the system of linear constraints (1), a ball center of K is a point $x \in K$ which maximizes $\delta(x)$, i.e., it is an optimum solution of the problem of maximizing $\delta = \text{minimum}\{A_{i.}x - b_i : i = 1 \text{ to } m\}$ subject to the constraints in (1). This is the LP

$$\begin{aligned} \text{Maximize} \quad & \delta \\ \text{subject to} \quad & \delta \leq A_{i.}x - b_i, \quad i = 1 \text{ to } m \end{aligned} \tag{2}$$

So, a ball center of K represented by the system of constraints (1), is an optimum solution

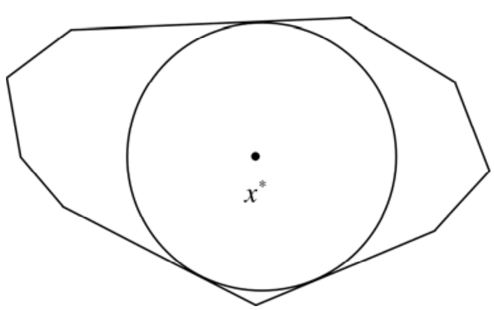

Figure 1: Polytope K and the largest ball inside it are shown. When the largest inscribed ball in K is unique as here, its center x^* is the ball center of K.

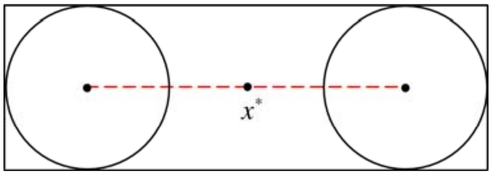

Figure 2: A 2-dimensional polytope K for which the largest inscribed ball is not unique. S, the set of centers of all such balls, the optimum face of (8.2.1) in the x-space, is the dashed line segment in this polytope, x^* is a point in it.

x of the LP (2), and the δ in that optimum solution is the radius of the largest ball inside K, it is $\delta(x)$ defined above for that point x.

An optimum solution of the LP (2) may not be unique, so a ball center for a polytope may not be unique. Figure 1 illustrates a polytope which has a unique ball center, and Figure 2 illustrates one in which the ball center is not unique.

Linear programming theory provides efficient methods to cheque whether $(x, \delta(x))$ is an optimum solution of (2) for a given interior point x of K; whether an optimum solution of (2) is unique, and if not it provides a representation of the set of all optimum solutions of (2) by a system derived from the system of constraints in (2) with some inequalities in it converted into equations. So, when the polytope K is represented by a system of linear constraints, all the questions in Problem 2 can be answered efficiently.

One of the two steps in each iteration of sphere methods for LP is the centering step, which tries to compute a good approximation to a ball center of a polytope, beginning with an initial interior feasible solution. The sphere methods in (Murty and Oskoorouchi [2008-1, 2]) solve this using a series of line searches. Here we encounter the following problem.

> **Problem 3:** Given an interior point $\bar{x}$ of K, a $y \in R^n$, $y \neq 0$ is said to be a **profitable direction** at $\bar{x}$, if $\delta(\bar{x} + \alpha y)$ increases strictly as α increases from 0. How can we check efficiently whether a given $y \neq 0$ is a profitable direction at $\bar{x}$? How can we check whether there exists a profitable direction at $\bar{x}$, and if so how can we compute one such direction efficiently?

Answers to Problem 3 are provided when the polytope is represented through a system of linear constraints in (Murty [2006-1, 2]). For instance for our polytope K represented by (1), for each interior point x of K, define $T(x)$ = set of all indices i satisfying: $A_{i.}x - b_i = \text{Minimum}\{A_{p.}x - b_p : p = 1 \text{ to } m\} = \delta(x)$. The hyperplane $\{x : A_{i.}x = b_i\}$ is a tangent plane to the ball $B(x, \delta(x))$ for each $i \in T(x)$, therefore $T(x)$ is called the **index set of touching constraints in (1)** at $x \in K^0$. See Figure 3.

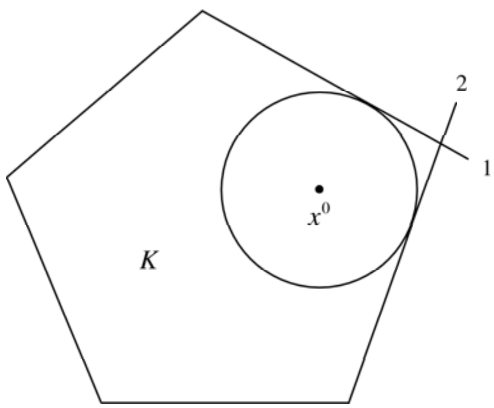

Figure 3: $x^0 \in K^0$, and the ball shown is $B(x^0, \delta(x^0))$, the largest ball inside K with x^0 as center. Facetal hyperplanes of K corresponding to indices 1, 2 are tangent planes to this ball, so $T(x^0) = \{1, 2\}$.

In (Murty [2006-1, 2], Murty and Oskoorouchi [2008-1, 2])) it has been proved that a $y \neq 0$ is R^n is a profitable direction at the interior point $\bar{x}$ of K represented by (1), iff $A_{i.}y > 0$ for all $i \in T(\bar{x})$. Also, there exists no profitable direction at $\bar{x}$ iff it is a ball center of K, which holds iff the system: $A_{i.}y > 0$ for all $i \in T(\bar{x})$, has no solution in y.

Thus answers to all the questions in Problem 3 can be derived efficiently when the polytope K is represented through a system of linear constraints.

2 Questions Related to Ball centers in Polytopes Represented as Convex Hulls of Their Extreme Points

There are two ways of representing a convex polytope. One way represents it as the set of feasible solutions of a given system of linear constraints, this is the representation we used in Section 1, and we have seen that Problems 1, 2, 3 stated in Section 1 can be solved efficiently under this representation of the convex polytope.

Another way is to give the set of all the extreme points of the polytope, and represent the polytope as the convex hull of this set. Let $\Gamma = \{x^1, ..., x^L\}$ be the set of all extreme points of a convex polytope K_1. In this way, K_1 is represented as the convex hull of Γ.

However, there are no efficient methods known to solve any of Problems 1, 2, 3 listed above on the polytope K_1 represented as the convex hull of its extreme points. All these problems on K_1 remain as open problems. Some of these problems may be hard problems, as indicated by the following theorem.

Theorem: $\Gamma \; = < \{x^1, ..., x^L\} >$, is a polytope of dimension n in R^n, which is represented as the convex hull of its extreme points; and x^0 is a given interior point of Γ. The problem of computing $\delta(x^0)$ = the radius of the largest ball inscribed in Γ with x^0 as center is NP-hard.

Proof: Transfer the origin to x^0. Then Γ becomes $< \{\bar{x}^1, ..., \bar{x}^L\} >$, where $\bar{x}^t = x^t - x^0$. In this representation, since $x^0 = 0$ is an interior point of Γ, it can be represented by a system of linear inequalities of the form $Ax \leq e$; where e is a column vector of all 1s, and A is a matrix of order $m \times n$; m being the number of facets of Γ.

Then for each for each $i = 1$ to m, $A_{i.}$ is the i-th row vector of A, and $\{x : A_{i.}x = 1\}$ is a facetal hyperplane of Γ. So the Euclidean distance between $x^0 = 0$, and the nearest point to x^0 on this facetal hyperplane is $1/||A_{i.}||$. So, $\delta(x^0) = \text{minimum}\{1/||A_{i.}||\text{: } i = 1 \text{ to } m\}$.

The vectors $A_{i.}$, $i = 1$ to m are all the extreme points of the system:

$$ax^t \leq 1 \qquad \text{for all } t = 1 \text{ to } L$$

in variables $a = (a_1, \ldots, a_n)$. Hence $\delta(x^0)$ is the inverse of the optimum objective value in the problem of maximizing $||a||$ subject to the system of inequalities given above. This is the problem of maximizing the Euclidean norm, a convex function, on a convex polytope specified by a system of linear inequalities, a well known NP-hard problem. ∎

In the next section we will discuss how to find the ball center of a special polytope, a simplex; under either representation efficiently.

3 Ball Center of an n-Dimensional Simplex in R^n

3.1 Ball Center of a Simplex Represented by Constraints

Let S be an n-dimensional simplex in R^n, i.e., its representation using linear constraints is of the form

$$S = \{x : D_{i.} \geq d_i \quad \text{for } i = 1 \text{ to } n+1\}$$

where $d = (d_i) \in R^{n+1}$, and the coefficient matrix D of order $(n+1) \times n$ with rows $D_{i.}$ $i = 1$ to $n+1$ satisfies the properties that all the $(n+1)$ submatrices of it of order $n \times n$ are nonsingular, and that each row vector of D is a linear combination of the other rows with strictly negative coefficients. Without any loss of generality we will assume that all the rows of D have been normalized so that $||D_{i.}|| = 1$ for all i.

We will now show that S has a unique ball center which is the unique solution of the system of linear equations:

$$Dx - \delta e = d$$

where x will be the ball center of S in the solution and δ is the radius the largest ball inside S with center at that x.

It can be verified that the coefficient matrix of this system is nonsingular, hence this system has a unique solution, $(\bar{x}, \bar{\delta})$. The ball $B(\bar{x}, \bar{\delta})$ with $\bar{x}$ as the center and $\bar{\delta}$ as radius is inside S and touches all the facets of S, so it is the largest ball with $\bar{x}$ as center inside S.

Also, the system $Dy \geq 0$ has 0 as its unique solution, because $D_{n+1.} = \alpha_1 D_{1.} + ... + \alpha_n D_{n.}$ with all $\alpha_1, ..., \alpha_n < 0$.

Applying the conditions mentioned under Property 3, we conclude that $\bar{x}$ is the ball center of K and it is unique. So, when the simplex S is represented using linear constraints, its ball center can be computed efficiently as described above.

3.2 Ball Center of a Simplex Represented as the Convex Hull of its Extreme Points

Now suppose that S is represented as the convex hull of its set of extreme points $\Gamma = \{x^1, ..., x^{n+1}\}$. So, an $x \in S$ is represented as:

$$x = \beta_1(x^1 - x^{n+1}) + ... + \beta_n(x^n - x^{n+1}) + x^{n+1} \qquad (3)$$

where $\beta_1, ..., \beta_n \geq 0$, and $\beta_1 + ... + \beta_n \leq 1$.

Let B denote the $n \times n$ matrix with its j-th column $B_{.j} = x^j - x^{n+1}$, for $j = 1$ to n; and $\beta = (\beta_1, ..., \beta_n)^T$. Since S is a simplex we know that B is nonsingular. So, from (3), we have

$$B^{-1}x = \beta + B^{-1}x^{n+1}.$$

Using this, from the bounds on β we have

$$\begin{aligned} B^{-1}x &\geq B^{-1}x^{n+1} \\ -eB^{-1}x &\geq -1 - eB^{-1}x^{n+1} \end{aligned} \qquad (4)$$

So, (4) is the representation of S = convex hull of Γ here through linear constraints. To derive the ball center of S using this representation (4), we need to normalize each constraint in (4) so that the Euclidean norm of its coefficient vector is 1. For this we need $\gamma_i = ||(B^{-1})_{i.}||$ for $i = 1$ to n, and $\gamma_{n+1} = ||eB^{-1}||$. Then from the results in Section 3.1, we know that the ball center x of S, and the radius δ of the largest ball inside S, are the solution of the system

$$\begin{pmatrix} B^{-1} & -\gamma \\ -eB^{-1} & -\gamma_{n+1} \end{pmatrix} \begin{pmatrix} x \\ \delta \end{pmatrix} = \begin{pmatrix} B^{-1}x^{n+1} \\ -1 - eB^{-1}x^{n+1} \end{pmatrix}$$

where $\gamma = (\gamma_1, ..., \gamma_n)^T$. By adding the sum of the first n equations in this system to the last, we find that in the solution of this system

$$\delta = 1/(\gamma_1 + ... + \gamma_{n+1})$$

and from the first n equations we see that the ball center of the simplex S here is

$$x = x^{n+1} + (1/(\gamma_1 + ... + \gamma_{n+1}))B\gamma$$

4 Ball Center of the Convex Hull of a Linearly Independent Set of Vectors in R^n

Let $P = \{x^1, ..., x^r\}$ be a linearly independent set of column vectors in R^n, and S_2 = convex hull of P, where $r \leq n$. Then S_2 is an $(r-1)$-dimensional simplex in its affine hull. In this section we discuss how to compute the ball center of S_2 directly.

In this case the $n \times (r-1)$ matrix

$$B_2 = (x^1 - x^r \vdots \cdots \vdots x^{r-1} - x^r)$$

is of full column rank. Find a row partition of it into $(B_{21}, B_{22})^T$ such that B_{21}, B_{22} are of orders $(r-1) \times (r-1)$ and $(n-r+1) \times (n-r+1)$ respectively, and B_{21} is nonsingular.

Let $(\underline{x}_1^j, \underline{x}_2^j)^T$, $(\underline{x}_1, \underline{x}_2)^T$ be the corresponding row partitions of the column vectors x^j for each $j = 1$ to r, and each $x \in S_2$ respectively.

Then for each $x = (\underline{x}_1, \underline{x}_2)^T \in S_2$, it can be verified that

$$\underline{x}_2 \quad = \quad B_{22}[B_{21}^{-1}(\underline{x}_1 - \underline{x}_1^r)] + \underline{x}_2^r \qquad (5)$$

(5) is the system of linear equations that defines the affine hull of S_2.

Now, the convex hull of $\{\underline{x}_1^1, \ldots, \underline{x}_1^r\} \subset R^{r-1}$ is a full dimensional simplex in R^{r-1} and its ball center $\underline{x}_1$ can be found by applying the formula derived in Section 3.2 to it. Then in the original space R^n, the ball center of S_2 is $(\underline{x}_1, \underline{x}_2)^T$ where $\underline{x}_2$ is obtained from $\underline{x}_1$ using (5).

5 Applications in Sphere Methods for Solving Large Scale LPs

Sphere methods (Murth [2006-1, 2], Murty and Oskoorouchi [2008-1, 2]) consider linear programs (LPs) in the form

$$\begin{aligned} \text{Minimize} \quad & z = cx \\ \text{subject to} \quad & Ax \geq b \end{aligned}$$

where A is an $m \times n$ matrix. Sphere methods use ball centers (i.e., centers x of largest radius balls contained inside K) subject to constraints on centers x. If the constraint on the center x is $cx = t$ for some given t, the center obtained is called a ball center of K on the objective plane $\{y : cy = t\}$.

One of the descent steps used in sphere methods is called D5.1, it consists of a sequence of at most m descent steps. Let $\{\hat{x}^1, ..., \hat{x}^s\}$ be the output points obtained in an application of D5.1. Then these s are all close to the boundary of K, scattered all around K in different directions, and typically form a linearly independent set. Also, these properties ensure that the ball center of their convex hull $< \{\hat{x}^1, ..., \hat{x}^s\} >$ is very close to a ball center of K on the objective plane through it; and hence it can be used for the next iteration in sphere methods. This application is more fully discussed in (Murty [2009]), and computational results using it will be reported in (Murty and Oskoorouchi [2008-2]).

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