6.1

Simplex Method for LP

Katta G. Murty Lecture slides

First: Put problem in **Standard Form** which is:

Min $z = cx$ subject to $Ax = b$ and $x \geq 0$.

All variables nonnegative variables. Only equality constraints.

Every LP can be put in standard form by following simple steps.

1. Convert obj. to min. form.

2. Convert ineq. constraints involving 2 or more variables into eqs. with appropriate **slacks** (slack vars. are always nonnegative vars.).

$-2x_1 - x_4 + x_7 \leq -13$ becomes $-2x_1 - x_4 + x_7 + s_1 = -13$, $s_1 \geq 0$.

$x_1 - 2x_5 \geq -8$ becomes $x_1 - 2x_5 - s_2 = -8$, $s_2 \geq 0$.

3. A variable with only one bound (lower or upper). Convert into eq. with slack. Use eq. to eliminate variable.

$x_1 \leq 6$ becomes $x_1 + s_3 = 6$ or $x_1 = 6 - s_3$ where $s_3 \geq 0$.

$x_2 \geq 4$ becomes $x_2 - s_4 = 4$ or $x_2 = 4 + s_4$ where $s_4 \geq 0$.

4. A Variable with both lower & upper bound restrictions.

If lower bound 0, leave it as nonegativity restriction; & treat upper bound converted into an eq. as a constraint.

If lower bound $\neq 0$, say $6 \leq x_3 \leq 10$, make $x_3 = 6 + s_5$ where $0 \leq s_5 \leq 4$. Now treat bounds on s_5 as above.

5. Put all equality constraints in detached coeff. tableau form. If there are any unrestricted variables, eliminate them by pivoting.

Example: max $z' = x_1 - 2x_2 + x_3 - x_4$

subject to $x_1 + x_2 + x_3 + x_4 \geq 6$

$x_1 - x_2 - x_3 - x_4 \leq -7$

$-2x_1 + x_2 - x_3 = 12$

$2 \leq x_1 \leq 10$, $x_2 \geq 5$, $x_3 \geq 0$, x_4 unrestricted.

Second: Transform all RHS constants in constraint rows into nonnegative numbers.

Example: min $z = -3x_4 + 2x_2 - 2x_1 - x_3 + 2$

s. to $\quad x_4 - 2x - 2 + x_1 - x_3 = -12$

$-x_2 + 2x_3 + x_5 - 2x_4 + x_1 = -2$

$3x_5 - 2x_1 + x_4 - 2x_3 = 6$

and $x_j \geq 0 \ \forall \ j$.

The resulting tableau, called **Original Tableau** is of following form:

	x_1	$\ldots$	x_j	$\ldots$	x_n	$-z$	
Original constraint rows	a_{11}	$\ldots$	a_{1j}	$\ldots$	a_{1n}	0	$b_1 \geq 0$
	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
	a_{m1}	$\ldots$	a_{mj}	$\ldots$	a_{mn}	0	$b_m \geq 0$
Original obj. row	c_1	$\ldots$	c_j	$\ldots$	c_n	1	$-z_0$

Third: Look for variables whose col. vecs. (among constraint rows only, ignoring obj. row) are unit vecs.

1. If all unit vectors found, for $i = 1$ to m, select a variable associated with ith unit vec. as **ith basic variable** or **basic variable in ith row**. Leads to a **feasible basic vector**.

Price out all basic columns: For $i = 1$ to m, convert cost coeff. of ith basic var. to 0, by subtracting suitable multiple of ith row from obj. row.

Now select $-z$ as basic variable in obj. row. The matrix consisting of basic cols. in proper order in present tableau is unit matrix, so present tableau is **canonical tableau WRT present basic vector**. Go to Phase II: Simplex algo. to solve original LP with it.

2. If one or more unit vectors are missing in original tableau, we don't have fesible basic vector to start simplex algo.

Now we construct a Phase I problem to find feasible basic vector for original problem first. For $i = 1$ to m, if

> if original tableau has ith unit vector, select a variable associated with it as ith basic variable;
>
> if original tableau does not have ith unit vector, introduce an **artificial variable** associated with ith unit vector

& 0 cost coeff. in original obj. row, and select it as ith basic var. All artificials are required to be ≥ 0.

Define:

$w =$ sum of artificials introduced.

w called **infeasibility measure** or Phase I obj. func. measures how far away present Phase I sol. is from feasibility to original problem.

$w \geq 0$ always, and if w becomes 0, all artificials must be 0, so sol. feasible to original prob.

Introduce: (sum of artificials) $-w = 0$ as **Phase I obj. row** at bottom of tableau. Now original obj. row called **Phase II obj. row**.

Price out all basic cols. in both obj. rows.

Select $-z, -w$ as basic vars. in Phase II, I obj. rows. Now we have **Phase I canonical tableau** WRT present basic vec. Go to Phase I.

Examples:

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
1	1	-1	0	1	0	0	10
-1	0	3	0	0	1	0	1
1	0	-2	1	0	0	0	2
-10	2	20	-2	-1	3	1	0

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
1	-1	1	1	1	0	0	0	1
1	1	0	-3	-1	1	0	0	2
1	1	0	-2	-1	0	1	0	0
2	-3	-1	1	-5	-2	3	1	0

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
1	1	-1	0	0	1	0	0
1	-1	1	1	1	0	0	4
1	1	1	0	1	1	0	6
-2	-1	-3	-1	1	2	1	0

Phase II: Simplex algorithm to solve original problem

Begins with canonical tableau WRT feasible basic vector, of following form:

Canonical form WRT feasible basic vec. $x_B = (x_1, \ldots, x_m, -z)$

BV	x_1	$\ldots$	x_m	x_{m+1}	$\ldots$	x_n	$-z$	
x_1	1	$\ldots$	0	$\bar{a}_{1,m+1}$	$\ldots$	$\bar{a}_{1n}$	0	$\bar{b}_1 \geq 0$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
x_m	0	$\ldots$	1	$\bar{a}_{m,m+1}$	$\ldots$	$\bar{a}_{mn}$	0	$\bar{b}_m \geq 0$
$-z$	0	$\ldots$	0	$\bar{c}_{m+1}$	$\ldots$	$\bar{c}_n$	1	$\bar{z}$

$$x_j \geq 0 \forall j, \text{ min } z$$

Present BFS is:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$z = \bar{z}$$

The updated entries in obj. row, $\bar{c}_j$ are called **relative** or **reduced cost coefficients**.

Result: Optimality criterion in simplex algorithm: If relative cost coeffs. of all nonbasic variables are ≥ 0, present BFS is optimal. Remember relative cost coeffs. of basic variables are always 0.

Example:

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
x_1	1	0	0	-1	3	2	0	3
x_2	0	1	0	-2	-1	1	0	4
x_3	0	0	1	3	1	-1	0	0
$-z$	0	0	0	4	6	7	1	-10

$x_j \geq 0 \ \forall \ j$, min z.

Equation given by obj. row is:

or $z =$

Present BFS is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$$z =$$

How to get a better sol. if opt. criterion not satisfied?

Consider following example.

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
x_1	1	0	0	1	-1	2	0	3
x_2	0	1	0	2	-1	1	0	10
x_3	0	0	1	-1	2	2	0	6
$-z$	0	0	0	-6	-8	2	1	-100

$$x_j \geq 0 \;\forall\; j, \text{ min } z.$$

From obj. row we get: $z = 100 - 6x_4 - 8x_5 + 2x_6$.

The present BFS & obj. value are:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$$z =$$

Obj. value will decrease if nonbasics x_4 or x_5 with relative cost $\bar{c}_j < 0$ are increased from present values of 0. That's why nonbasics with $\bar{c}_j < 0$ are called **eligible nonbasic variables to enter basic vector**.

A step in Simplex Algo. Selects one eligible var. & tries to increase its value from 0 to λ say. This variable called **entering variable** in this step. All nonbasics other than entering var. remain fixed at 0. Updated col. of entering var. in present canonical tableau called **pivot column** in this step.

Suppose x_4 selected as entering var. When value of x_4 changed from 0 to λ, from tableau new sol. as function of λ is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 - \lambda \\ 10 - 2\lambda \\ 6 + \lambda \\ \lambda \\ 0 \\ 0 \end{pmatrix}$$

$$z = 100 - 6\lambda$$

As $\lambda \uparrow$ $z \downarrow$. So, should give λ max. possible value. But basic variables in rows 1, 2 (in which pivot col. has positive entry) keep $\downarrow$ as $\lambda \uparrow$.

x_1, 1st basic var. becomes 0 when λ reaches 3, and will be < 0 if $\lambda > 3$.

x_2, 2nd basic var. becomes 0 when λ reaches $5 = 10/2$, and will be < 0 if $\lambda > 5$.

Hence max value for λ is $3 = \min\{3/1, 10/2\}$, called **minimum ratio** in this step, denoted by θ.

New sol. obtained by fixing $\lambda = \theta = 3$ in above formula. In it, 1st basic var. x_1 becomes 0, and is replaced from basic vec. by entering var. x_4. Hence x_1 called **dropping basic var.** in this step. Canonical tableau WRT new basic vec. obtained by performing GJ pivot step, with pivot col. & row 1 (row of dropping basic var. x_1) as pivot row. This pivot step transforms RHS vec. into vector of basic values in next sol.

Here is computation of min ratio, & pivot step.

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	RHS	Ratio*
x_1	1	0	0	$\boxed{1}$	-1	2	0	3	$3/1 = \min$, PR
x_2	0	1	0	2	-1	1	0	10	$10/2$
x_3	0	0	1	-1	2	2	0	6	
$-z$	0	0	0	-6	-8	2	1	-100	$\min = \theta = 3$.
				PC					

* (RHS)/(PC entry), only in rows with (PC entry) > 0.

General step in Simplex algo.

Eligible var. = any nonbasic var. with relative cost $\bar{c}_j < 0$ in present tableau.

Entering var. = an eligible var. selected to enter basic vec., x_s say.

Pivot col. (PC) = col. of entering var. in present tableau.

Pivot row (PR) = row in which min ratio attained (see below). Break ties arbitrarily.

Dropping var. = present basic var. in pivot row, which will be replaced by entering var.

PC	RHS	Ratio $\bar{b}_i/\bar{a}_{is}$
x_s		computed only if $\bar{a}_{is} > 0$
$\bar{a}_{1s}$	$\bar{b}_1$	
$\bar{a}_{2s}$	$\bar{b}_2$	
$\vdots$	$\vdots$	
$\bar{a}_{ms}$	$\bar{b}_m$	
$\bar{c}_s$	$-\bar{z}$	Min ratio = θ.

Unboundedness of Obj. func. Consider

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	RHS
x_1	1	0	0	1	-3	2	0	6
x_2	0	1	0	1	-7	1	0	7
x_3	0	0	1	1	0	1	0	8
$-z$	0	0	0	-2	-5	-10	1	-10

$$x_j \geq 0 \; \forall \; j, \text{ min } z.$$

Eligible vars. are:

Select x_5 as entering var. Making $x_5 = \lambda$, leaving x_4, x_6 at 0 leads to:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} =$$

$$z = 100 - 6\lambda$$

As $\lambda \uparrow$, all vars. remain ≥ 0, & $z \downarrow$, $\rightarrow -\infty$ as $\lambda \rightarrow +\infty$.

Why? Because PC has no positive entry.

So, in this case z unbounded below on feasible region, above formula gives a half-line in feasible region along which $z \to -\infty$.

Summary of Simplex Algorithm:

1: Initiate: with a canonical tableau WRT a feasible basic vector.

2:Opt. crit.: If rel. costs are all ≥ 0, present BFS opt., terminate.

3:If opt. violated: Identify nonbasics eligible to enter basic vec., select one as entering var. Its col. in present tableau is PC.

4:Check unboundedness: If PC ≤ 0, obj. unbounded below, terminate.

5:Pivot step: If unboundedness not satisfied, perform min ratio test, determine PR, perform pivot step and get canonical tableau WRT new basic vector. Go back to 2.

Phase I: Application of Simplex algo. to Phase I problem to find a feasible basic vec. for original problem

Consider original problem discussed earlier:

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
1	1	-1	0	0	1	0	0
1	-1	1	1	1	0	0	4
1	1	1	0	1	1	0	6
-2	-1	-3	-1	1	2	1	0

Introducing artificials t_1, t_3, the Phase I original tableau is:

x_1	x_2	x_3	x_4	x_5	x_6	t_1	t_3	$-z$	$-w$	
1	1	-1	0	0	1	1	0	0	0	0
1	-1	1	1	1	0	0	0	0	0	4
1	1	1	0	1	1	0	1	0	0	6
-2	-1	-3	-1	1	2	0	0	1	0	0
0	0	0	0	0	0	1	1	0	1	0

All vars. ≥ 0, min w

Selecting $(t_1, x_4, t_3, -z, -w)$ as the initial basic vector, we get following canonical tableau after pricing out the cols. of t_1, x_4, t_3 in both obj. rows.

BV	x_1	x_2	x_3	x_4	x_5	x_6	t_1	t_3	$-z$	$-w$	
t_1	1	1	-1	0	0	1	1	0	0	0	0
x_4	1	-1	1	1	1	0	0	0	0	0	4
t_3	1	1	1	0	1	1	0	1	0	0	6
$-z$	-1	-2	-2	0	2	2	0	0	1	0	4
$-w$	-2	-2	0	0	-1	-2	0	0	0	1	-6

Starting with this, apply simplex algo. to min w, ignoring original obj. z for moment.

During Phase I, use Phase I rel. costs (those in Phase I obj. row) denoted by $\bar{d}_j$.

Phase I termination cond. All Phase I rel. costs $\bar{d}_j \geq 0$.

Eligible vars. in Phase I: Original vars. x_j with $\bar{d}_j < 0$.

Other things to remember in Phase I: Any artificial is erased from tableau if it drops from basic vec., because it is no longer needed.

Conclusion at Phase I termination: If $\bar{w} =$ min value of $w > 0$, original problem **infeasible**, terminate.

If $\bar{w} = 0$, sol. from final Phase I tableau is BFS of original. Go to Phase II.

How to begin Phase II at end of Phase I if $\bar{w} = 0$

Case 1: All artificials dropped from basic vector: So, no artificial in final Phase I tableau. Erase $-w$ col. and bottom row (Phase I obj. row) from it & begin Phase II.

Case 2: Some artificials still in basic vector: Their values in sol. must be 0.

Identifying Original vars. which are 0 in every feasible sol. of original problem: Identify original variables x_j for which $\bar{d}_j > 0$ in the final Phase I tableau. Each of those variables must be $=$ 0 in every feasible sol. of original problem (because making such a variable > 0 makes $w > \bar{w} = 0$, i.e., introduces infeasibility). Fix all such variables at 0, & erase their cols. from final tableau.

Now erase $-w$ col., and last row (Phase I obj. row) from final tableau & begin Phase II. Any artificial still in basic vector will remain $= 0$, so can be left there during Phase II steps until it leaves basic vector some time.

Examples:

Min $z = x_1 + 2x_2 + 3x_3$

s. to $2x_1 + 3x_2 + x_3 - x_4 = 9$

$x_1 + 2x_2 - x_3 + x_5 = 5$

$x_1 + x_2 + 2x_3 = 4$

$x_j \geq 0 \ \forall \ j.$

$$\min\ z = x_1 - 2x_5 - 2x_6 + 5x_7 + 100$$

$$\text{s. to } x_1 - x_4 + x_7 = 2$$

$$x_1 + x_2 + x_5 + 2x_7 = 3$$

$$x_3 + x_6 + x_7 = 5$$

$$x_4 - x_7 = 0$$

$$x_j \geq 0\ \forall\ j.$$

Homework problems: Put these in standard form

7.1. max $2x_2 + x_3 + x_4$
subject to $2x_1 - x_2 - x_3 + x_4 \leq -8$
$2x_2 + x_3 - x_4 \geq 4$
$x_1 - x_2 + x_4 = 13$
$-3 \leq x_1 \leq 4$, $x_2 \geq 2$, x_3 unrestricted, $x_4 \leq 0$.

7.2. min $3x_1 - x_2 + x_3 - 2x_4$
s. to $x_1 + x_2 + 2x_3 + x_4 = 12$
$x_2 - x_3 + x_4 \geq 6$
$2x_1 + x_3 - x_4 \leq 10$
$1 \leq x_1 \leq 5$, $x_2 \leq 10$, $x_3 \geq 0$, x_4 unrestricted.

Homeworks: Solve these LPs:

7.3 (O, II) **(a)**: min $z = -x_1 - 8x_2$, s. to $-x_1 + x_2 \leq 2$
$x_1 + x_2 \leq 1$, $2x_1 + x_2 \leq 5$, $x_1, x_2 \geq 0$.

(b) Min $z = -2x_1 + x_2 - 2x_3 + x_4$, s. to $x_1 - x_2 + x_4 \leq 2$
$x_2 + x_3 + 2x_4 \leq 3$, $x_1 + 2x_2 + 4x_3 - 2x_4 \leq 12$, $x_j \geq 0$.

(c) Solve using Dantzig's rule (most negative $\bar{c}_j$) for selecting the entering variable in each pivot step.

min $z = 3x_1 - 8x_2 + 2x_3 - 7x_4 - 5x_5 + 8x_6$, s. to $-x_2 + x_3 + x_4 + x_6 = 3$
$x_1 + x_2 - x_4 + x_6 = 6$, $x_2 + x_4 + x_5 - x_6 = 0$, $x_j \geq 0\ \forall\ j$.

(d) min $z = -8x_1 + 8x_2 + 14x_3 + 4x_4 + 6x_5 - 3x_6 + 3x_7$
s. to $x_1 - x_6 + x_7 = 3$, $-2x_1 - 3x_3 + x_4 + 3x_6 = 2$, $4x_3 + x_5 - x_6 = 1$
$x_2 - x_6 = 4$, $x_j \geq 0\ \forall\ j$.

7.4: (U, II) **(a)** min $z = -3x_1 + 4x_2 + x_3$, s. to $x_1 - 2x_2 + 2x_3 \leq 3$
$x_1 - x_2 - 3x_3 \leq 5$, $-x_1 + x_2 - x_3 \leq 7$, $x_j \geq 0\ \forall\ j$.

(b) : min $z = -2x_1 - x_2$, s. to $-x_1 + x_2 \leq 2$
$x_1 - 2x_2 \geq -5$, $x_1 - 3x_2 \leq 2$, $x_1, x_2 \geq 0$.

(c) : min $z = -2x_1 + 2x_2 + x_3$, s. to $x_1 - x_2 - 2x_3 \leq 3$
$x_1 - x_2 - x_3 \leq 4$, $x_1 - 2x_2 \leq 0$, $x_1, x_2, x_3 \geq 0$.

(d): min $z = -3x_1 + 2x_2 - 2x_3$, s. to $x_1 - 2x_2 + 2x_3 \leq 0$

$x_1 - x_2 - 2x_3 \leq 10, \quad -x_1 + 3x_2 - 4x_3 \leq 2, \quad -x_1 + 2x_2 - 2x_3 \leq 3$
$x_j \geq 0 \ \forall \ j.$

7.5: (U, I) **(a):** min $z = -x_1 - 2x_2$, s. to $x_1 + x_2 \geq 1$, $x_1 - x_2 \leq 2$
$-x_1 + x_2 \leq 2$, $x_1, x_2 \geq 0$.
(b):

$$\begin{array}{lrrrrrrl}
\text{Minimize} & -2x_1 & +2x_2 & +x_3 & & & & \\
\text{subject to} & & x_2 & +x_3 & -x_4 & +x_5 & +2x_6 & \leq 6 \\
 & x_1 & & +x_3 & -x_4 & +x_5 & & = 5 \\
 & -x_1 & +x_2 & -x_3 & +x_4 & & +x_6 & = 3 \\
 & & & \multicolumn{4}{l}{x_j \geq 0 \text{ for all } j} &
\end{array}$$

(c):

$$\begin{array}{lrrrrrrl}
\text{Minimize} & -2x_1 & +2x_2 & +x_3 & & & & \\
\text{subject to} & & x_2 & +x_3 & -x_4 & +x_5 & +2x_6 & \leq 6 \\
 & x_1 & & +x_3 & -x_4 & +x_5 & & = 5 \\
 & -x_1 & +x_2 & -x_3 & +x_4 & & +x_6 & = -3 \\
 & & & \multicolumn{4}{l}{x_j \geq 0 \text{ for all } j} &
\end{array}$$

If possible, determine a feasible solution where the objective function has value $= -200$.

7.6: (O, I, transition)

(a) : (O, I, transition) In solving this problem, if there is a tie for the min ratio, always select the bottommost among the rows tied as the pivot row.

min $z = -2x_3 - 10x_4$**,** **s. to** $x_1 + x_3 + 2x_4 \geq 2$
$x_2 + x_3 + x_4 \geq 4$, $x_1 + x$