3.1

Early History of Related Topics

Katta G. Murty Lecture slides

Linear Algebra = Branch of Math. dealing with Systems of linear eqs. (Note: No inequalities!).

Example: Steel company has four scrap metals (SM-1 to SM-4) with following compositions.

Type	% in type, by weight, of element			
	Al	Si	C	Fe
SM-1	5	3	4	88
SM-2	7	6	5	82
SM-3	2	1	3	94
SM-4	1	2	1	96

Need to blend into a mixture with composition by weight : Al = 4.43%, Si = 3.22%, C = 3.89%, Fe = 88.46%. Determine how to prepare this mixture.

Elimination Method for linear eqs. developed by 3rd century BC by Chinese & Indian mathematicians.

Unknown in Europe until C F Gauss discovered it (19th century) while studying linear eqs. to approximate orbit of asteroid Ceres using **Method of least squares**. W. Jordan discussed this method in a book he wrote (1888). Hence method called GJ Elimination method. (A better version called G Elimination method).

Both GJ & G methods cannot directly handle inequalities.

Linear Programming LP, Methods for handling systems of linear eqs. & ineqs. Started with:

Farka's Lemma: Fundamental theoretical result on linear ineqs. (1901)

G B Dantzig's **Simplex Method** to solve LPs (1947).

So, LP, a 20th century topic.

All efficient implementations for LP are iterative methods requiring sol. of 2 systems of linear eqs. in each iteration.

GJ Method for Solving Linear Eqs.

Consider system

(1) $$Ax = b$$

where **coefficient matrix** A is $m \times n$ matrix, & RHS vector $b \in R^m$.

Matrix $A = (a_{ij})_{m \times n}$: Started as a convenient way to represent the coefficients in a system of linear equations. Matrices and matrix algebra introduced by J J Sylvester, and Cayley in 1840s.

$A_{i.} = i$th row vector of the matrix A, $i = 1$ to m.

$A_{.j} = j$th column vector of the matrix A, $j = 1$ to n.

Unit or Identity Matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Permutation Matrix; One whose rows can be rearranged to

become the unit matrix.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$A_{m\times n}$. Products

- Ax defined only if x is a column vector in R^n.
- πA defined only if π is a row vector in R^m.

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -2 & 0 \end{pmatrix}, B = \begin{pmatrix} 5 & -3 & -4 \\ 3 & 0 & -3 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -3 & 1 \\ -2 & -3 & 4 \end{pmatrix}$$

$$d^1 = \begin{pmatrix} -4 \\ -5 \\ 2 \\ 0 \end{pmatrix}, d^2 = \begin{pmatrix} -2 \\ -11 \\ 2 \end{pmatrix}, d^3 = \begin{pmatrix} 10 \\ -3 \end{pmatrix}$$

$$e^1 = (0, 3, -1, 4), \quad e^2 = (1, -1, 3), \quad e^3 = (2, 2)$$

Main Computational tools

Row Operations:

1. Multiply a row by a nonzero scalar.

2. Add a scalar multiple of a row to another.

3. Interchange two rows.

Example:

$A_{1.}$	1	0	3	-5
$A_{2.}$	0	1	2	4
$(1/2)A_{1.}$				
$A_{2.} - 5A_{1.}$				

Gauss-Jordan Pivot Operation or Pivot Step: Specified by specifying the *pivot column* and *pivot row*.

IMPORTANT CONDITION: *Pivot Element*, element in the pivot row and the pivot column, should be nonzero.

The GJ Pivot operation consists of a series of row operations to convert pivot column into *unit column* with 1 entry in pivot row.

Example: Here PC = pivot column, PR = pivot row.

		PC				
1	0	1	−1	2	1	
6	4	$\boxed{2}$	−4	0	6	PR
3	−2	0	1	−1	4	
1	0	−1	3	4	10	

History: Developed as a tool for solving systems of linear equations by the *Elimination method*

Example:

x_1	x_2	x_3	
2	2	2	12
−1	0	1	1
1	−1	0	−2

Constraints in (1) correspond to rows of coefficient matrix A. System (1) in constraint representation is:

(1) $\qquad A_{i.}x = b_i$, $i = 1$ to m.

Original tableau				
x_1	x_2	$\ldots$	x_n	
a_{11}	a_{12}	$\ldots$	a_{1n}	b_1
a_{21}	a_{22}	$\ldots$	a_{2n}	b_2
$\vdots$	$\vdots$		$\vdots$	$\vdots$
a_{m1}	a_{m2}	$\ldots$	a_{mn}	b_m

Example:

x_1	x_2	x_3	
2	1	2	12
-1	1	3	1
1	2	5	13

Result: If, after performing some GJ pivots on the tableau, a row becomes all 0s (i.e., $0 = 0$ equation), then corresponding row

in original tableau represents a **redundant constraint**, and can be deleted without effecting the set of solutions.

Example:

x_1	x_2	x_3	
2	1	2	12
-1	1	3	1
1	2	5	14

Result: If, after performing some GJ pivots on the tableau, a row becomes the $0 = 1$ equation (the fundamental inconsistent equation), the original system has no solutions.

The GJ method for solving linear eqs.

Start with the original tableau (detached coefficient tableau of original system). Suppose it has m rows & n variables.

1. Select the order in which rows 1 to m in the tableau are to be chosen as pivot rows.

2. **General Step:** Suppose row r is the pivot row for the pivot step in the present tableau. Let $\bar{a}_{r1}, \ldots, \bar{a}_{rn}, \bar{b}_r$ be the

coefficients of the variables and the updated RHS constant in row r in the present tableau.

2.1. If $(\bar{a}_{r1}, \ldots, \bar{a}_{rn}) \neq 0$ select a variable x_j for a j such that $\bar{a}_{rj} \neq 0$ as the basic variable in row r, and the column of x_j in the present tableau as the pivot column, and perform the GJ pivot step. If row r is the last pivot row in the selected order, go to 3 if 2.3 given below has never occurred so far. If row r is not the last pivot row in the selected order, with the resulting tableau go back to 2 to perform a pivot step with the next pivot row in the selected order.

2.2. If $(\bar{a}_{r1}, \ldots, \bar{a}_{rn}) = 0$ and $\bar{b}_r = 0$, this row is called the "0 = 0" equation. This indicates that the constraint in the original system (1) corresponding to this row is a redundant constraint and can be eliminated without changing the set of solutions.

If row r is the last pivot row in the selected order, go to 3 if 2.3 given below has never occurred so far. If row r

is not the last pivot row in the selected order, with the present tableau go back to 2 to perform a pivot step with the next pivot row in the selected order.

2.3. If $(\bar{a}_{r1}, \ldots, \bar{a}_{rn}) = 0$ and $\bar{b}_r \neq 0$, this row is called the "$0 = \alpha$" equation for $\alpha = \bar{b}_r \neq 0$, or an inconsistent or infeasible equation. In this case the original system (1) has no solution.

3. Make all the nonbasic variables in the final tableau equal to 0, and the basic variable in each row equal to the updated RHS constant in that row in the final tableau. This is a basic solution to (1), terminate.

EXAMPLE 1:

	x_1	x_2	x_3	x_4	x_5	RHS
	-1	0	$\boxed{1}$	1	2	10
	1	-1	2	-1	1	5
	1	-2	5	-1	4	20
	2	0	0	-2	1	25
	3	-2	5	-3	5	45
	5	-2	5	-5	6	70

EXAMPLE 2:

	x_1	x_2	x_3	RHS
	1	1	$\boxed{1}$	5
	-1	3	1	6
	0	4	2	11
	2	3	1	10

EXAMPLE 3:

	x_1	x_2	x_3	x_4	x_5	RHS
	2	1	-1	-2	0	4
	4	1	5	6	4	9
	6	2	4	4	4	12
	10	3	9	10	8	22

1. In final tableau, each remaining row has a basic variable selected. The vector of basic variables in their proper order called **basic vector** selected. In final tableau, columns of basic variables in proper order form unit matrix.

In LP literature, final tableau called **canonical tableau** WRT selected basic vector. In math. books it is called **reduced row echelon form** (except for some rearrangement of columns).

2. Basic sol. WRT basic vector obtained by fixing nonbasic variables $= 0$.

Basic sol. called

> **nondegenerate** if all basic variables nonzero in the basic sol.
>
> **degenerate** if at least one basic variable zero in basic sol.

3. What is rank of coeff. matrix of original system?

4. When can we conclude that sol. obtained is the unique sol.? What is the general sol. if it is not unique?

5. Can we find a nonnegative sol. to system using GJ method?

6. Simplex method for LP is an extension of GJ method to:

> **a).** Find nonnegative sol. if one exists, or establish that there is no nonnegative sol. So, this method can solve systems of linear inequalities.

b). minimize a linear obj. func. subject to linear inequalities.

HW problems: In solving all these problems, use pivot elements of 1 or -1 as far as possible to avoid getting fractions.

3.1 (AS) We need a nonnegative solution to the following system of linear equations.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
1	-2	0	1	0	-1	6	4
1	-1	2	-2	1	8	-17	-1
0	2	5	1	-5	3	-4	3

Ignoring the nonnegativity restrictions on the variables, solve the system of eqs. by GJ method.

Were you able to obtain a nonnegative solution to the system?

Write the general solution of the system. From this can you see easily whether there exists a nonnegative solution? If so, find it.

3.2 (US): Find solutions of the following systems using GJ method. For each, check whether the sol. obtained is unique.

(a)	x_1	x_2	x_3	x_4	
	-1	-2	0	0	1
	2	5	2	2	-3
	0	1	3	0	-4
	1	0	-1	1	3
	2	-2	1	-1	2

(b)	x_1	x_2	x_3	
	3	1	4	5
	-4	-1	-6	-4
	-2	-1	0	-10

(c)	x_1	x_2	x_3	x_4	
	1	2	3	2	5
	-1	-3	-5	-1	-4
	1	0	4	1	11
	1	2	-2	-2	-13

3.3 (AS, no RC) Solve following systems of linear eqs. using GJ method. Determine following ineach case: (i) a general sol. of system in parametric form (ii) rank of coefficient matrix (iii) dimension of set of feasible solutions.

$$
\begin{aligned}
(a) \qquad & -x_2 + 3x_3 + 8x_4 - 7x_6 = -3 \\
& 7x_1 + x_2 - 2x_3 - 10x_4 + 2x_5 - 3x_6 = -4 \\
& 10x_1 + 2x_3 + 12x_4 + 5x_5 - 8x_6 = 2
\end{aligned}
$$

(a)	x_1	x_2	x_3	x_4	x_5	x_6	
	1	1	0	−1	−1	1	−3
	−1	0	1	2	−1	2	−7
	0	3	2	3	−5	1	−10
	1	1	0	2	−1	−2	8

(c)	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
	2	−1	0	1	2	−3	−2	1
	−3	1	−1	0	1	−2	−1	−4
	2	0	3	−3	−1	0	2	7
	0	2	1	−1	0	2	1	7

3.4 (AS, RC) Do the same as in 3.2 for the following problems.

(a)	x_1	x_2	x_3	x_4	x_5	x_6	
	−1	0	1	−1	−1	2	−1
	0	−1	1	0	1	1	2
	−2	−3	5	−2	1	7	4
	1	1	−3	2	0	−2	−2
	−2	−3	4	−1	1	8	3
	2	2	−4	1	2	−1	−4

(b)	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
	2	0	3	0	4	5	−1	−10
	0	3	−2	5	−3	2	1	10
	4	6	2	10	2	14	0	0
	3	−3	2	1	−1	−1	−1	−7
	9	6	5	16	2	20	−1	−7
	5	0	3	6	0	6	−1	−7

(c)	x_1	x_2	x_3	x_4	x_5	
	−2	7	5	1	4	38
	−3	2	−2	2	9	84
	−8	11	1	5	22	206
	−7	16	8	4	17	160
	−5	9	3	3	13	122

(d)	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	
	1	1	1	0	0	0	800
	0	0	0	1	1	1	300
	1	0	0	1	0	0	400
	0	1	0	0	1	0	200
	0	0	1	0	0	1	500

3.5 (I) Do the same as in 3.2 for the following set systems of eqs.

(a)	x_1	x_2	x_3	x_4	x_5	x_6	
	1	2	−1	0	3	1	3
	−1	−3	2	2	−2	−3	4
	0	−1	1	2	1	−2	7
	−1	−4	3	4	−1	−5	9
	1	1	0	2	4	−1	7

(b)	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	
	1	1	1	0	0	0	500
	0	0	0	1	1	1	1000
	1	0	0	1	0	0	600
	0	1	0	0	1	0	500
	0	0	1	0	0	1	500

(c)	x_1	x_2	x_3	x_4	x_5	
	1	1	0	0	0	3
	0	1	1	0	0	1
	0	0	1	1	0	1
	0	0	0	1	1	4
	1	0	0	0	1	5
	1	1	1	1	1	7
	1	2	1	1	1	9

$$
\begin{aligned}
(d) \qquad & -5x_1 - 2x_2 + 10x_3 + x_4 + 7x_5 + 2x_6 - 4x_7 = 2 \\
& -3x_1 + 3x_2 - 2x_3 - x_4 + 2x_5 + x_6 + 3x_7 = 3 \\
& -8x_1 + x_2 + 8x_3 + 9x_5 + 3x_6 - x_7 = 6 \\
& x_1 + 2x_3 - 3x_5 - 4x_6 + 2x_7 = 5 \\
& -7x_1 + x_2 + 10x_3 + 6x_5 - x_6 + x_7 = 8
\end{aligned}
$$