Problem session 7

Problem 1. Let X be a Noetherian scheme. All locally free sheaves will be assumed of finite rank. The *Grothendieck group* $K^0(X)$ of vector bundles on X is the quotient of the free abelian group on the set of isomorphism classes of locally free sheaves on X, by the subgroup generated by relations of the form $[\mathcal{E}] - [\mathcal{E}'] - [\mathcal{E}'']$, where

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is an exact sequence of locally free sheaves on X.

1) Show that $K^0(X)$ becomes a ring with respect to $[\mathcal{E}] \cdot [\mathcal{E}'] = [\mathcal{E} \otimes \mathcal{E}']$.
2) Show that if $f\colon Y \to X$ is a morphism of Noetherian schemes, then we get an induced ring homomorphism $f^*\colon K^0(X) \to K^0(Y)$ given by $[\mathcal{E}] \to [f^*(\mathcal{E})]$. Moreover, if $g\colon Z \to Y$ is another morphism, then $(f \circ g)^* = g^* \circ f^*$.
3) The *determinant* of a locally free sheaf $\mathcal{E}$ of rank r is the sheaf $\det(\mathcal{E})$ associated to the presheaf taking the rth exterior power over the structure sheaf
$$U \to \wedge^r \mathcal{E}(U)$$
(with the obvious modification if $\mathcal{E}$ does not have constant rank). Show that $\mathcal{E} \to \det(\mathcal{E})$ induces a group homomorphism $K^0(X) \to \mathrm{Pic}(X)$.
4) Show that if X is connected, then $[\mathcal{E}] \to \mathrm{rk}(\mathcal{E})$ gives a surjective ring homomorphism $K^0(X) \to \mathbb{Z}$.

Problem 2. Let X be a Noetherian scheme. The Grothendieck group $K_0(X)$ of coherent sheaves on X is the quotient of the free abelian group on isomorphism classes of coherent sheaves on X by the subgroup generated by relations of the form $[\mathcal{F}]-[\mathcal{F}']-[\mathcal{F}'']$, where

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence of coherent sheaves on X.

i) Show that if k is a field and $X = \mathrm{Spec}(k)$, then $K^0(X) = K_0(X) \simeq \mathbb{Z}$.
ii) Show that if k is a field and $X = \mathbf{A}^1_k$, then $K^0(X) = K_0(X) \simeq \mathbb{Z}$.
iii) Show that $K_0(X)$ is a module over $K^0(X)$ with respect to the action $[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes \mathcal{F}]$.
iv) Suppose that X is projective (or more generally, proper) over a field k. Show that we get a morphisms of groups $K_0(X) \to \mathbb{Z}$ given by the *Euler-Poincaré characteristic* map
$$\mathcal{F} \to \chi(\mathcal{F}) = \sum_{p \geq 0} (-1)^p \dim_k H^p(X, \mathcal{F}).$$
v) Let $f\colon X \to Y$ be a projective morphism of Noetherian schemes. Show that we can define a group morphism $f_*\colon K_0(X) \to K_0(Y)$ by
$$f_*([\mathcal{F}]) = \sum_{p \geq 0} (-1)^p [R^p f_*(\mathcal{F})]$$
(the same assertion holds for a proper morphism). If $Y = \mathrm{Spec}(k)$, then via the isomorphism in i), this is identified with the morphism defined in iv).

v) It can be shown (using the Leray spectral sequence associated to the composition of two morphisms) that if $f\colon X \to Y$ and $g\colon Y \to Z$ are projective (more generally, proper) morphisms, then $(g \circ f)_* = g_* \circ f_*$. Prove this when either f or g is finite.

vi) Show that if $f\colon X \to Y$ is a finite morphism, then we have the *projection formula*

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$$

for every $\alpha \in K^0(Y)$ and $\beta \in K_0(X)$.

vii)* For every X we have a natural group morphism $K^0(X) \to K_0(X)$. Show that if X is nonsingular of finite dimension (this means that there is n such that all local rings of X are regular of dimension $\leq n$) and if there is an ample invertible sheaf on X, then the above morphism is an isomorphism.

Remark. Note that $K^0(X)$ behaves like a cohomology ring of X, while $K_0(X)$ behaves like an homology group. Indeed, we have a "cap-product" of $K^0(X)$ on $K_0(X)$ and both groups satisfy the usual variance properties for homology and cohomology. Moreover, note that in $K_0(X)$ we have a "fundamental class" $[\mathcal{O}_X]$ such that the isomorphism in 5) above can be interpreted as giving a version of "Poincaré duality". These K-groups play an important role in the statement of the Grothendieck-Riemann-Roch theorem.