Problem session 5

Problem 1. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Show that if $\mathcal{M}$ is an $\mathcal{O}_X$-module and $\mathcal{N}$ is an $\mathcal{O}_Y$-module, then there is a canonical morphism

$$f_*(\mathcal{M}) \otimes_{\mathcal{O}_Y} \mathcal{N} \to f_*(\mathcal{M} \otimes_{\mathcal{O}_X} f^*\mathcal{N}).$$

Show that if $\mathcal{N}$ is locally free of finite rank, then this is an isomorphism (this isomorphism is called the *projection formula*).

Problem 2. Let $i\colon Y \hookrightarrow X$ be a closed immersion defined by the ideal sheaf $\mathcal{I}$. Show that $\mathcal{F} \to i_*(\mathcal{F})$ gives an equivalence of categories between (quasi)coherent sheaves on Y and (quasi)coherent sheaves on X that are annihilated by $\mathcal{I}$. Its inverse is given by $\mathcal{G} \to i^*\mathcal{G}$.

Problem 3. Let X be an integral scheme. If $K(X)$ is the function field of X, is the constant sheaf $\underline{K(X)}$ coherent ? Is it quasicoherent ?

Problem 4. Let X be a scheme. A sheaf of $\mathcal{O}_X$-algebras is a sheaf of rings $\mathcal{A}$ on X with a morphism of sheaves of rings $\rho\colon \mathcal{O}_X \to \mathcal{A}$ (equivalently, for every U open, $\mathcal{A}(U)$ is an $\mathcal{O}_X(U)$-algebra such that the restriction maps are algebra homomorphisms). A sheaf of $\mathcal{O}_X$-algebras is called quasicoherent (or coherent) if it is so with respect to the induced $\mathcal{O}_X$-module structure.

(i) Show that if $\mathcal{A}$ is a quasicoherent sheaf of $\mathcal{O}_X$-algebras, then there is a scheme over X

$$\pi\colon \mathcal{S}pec(\mathcal{A}) \to X$$

such that for every $U \subseteq X$ affine open subset we have

$$\pi^{-1}(U) \simeq \mathrm{Spec}(\mathcal{A}(U) \to U = \mathrm{Spec}(\mathcal{O}_X(U)),$$

the morphism corresponding to $\rho(U)$. Moreover, if $V \subseteq U$ are affine open subsets, then the inclusion $\pi^{-1}(V) \subseteq \pi^{-1}(U)$ corresponds to the restriction homomorphism $\mathcal{A}(U) \to \mathcal{A}(V)$. Note that π is affine.

(ii) Show that if Y is a scheme over X, then there is a canonical bijection

$$\mathrm{Hom}_X(Y, \mathcal{S}pec(A)) \simeq \mathrm{Hom}_{\mathcal{O}_X-\mathrm{alg}}(\mathcal{A}, f_*(\mathcal{O}_Y)).$$

(iii) Show that if $f\colon Y \to X$ is an affine morphism, then we get by (ii) a morphism over X

$$Y \to \mathcal{S}pec(f_*(\mathcal{O}_Y))$$

that is an isomorphism. Deduce that there is an equivalence of categories between affine schemes over X and quasicoherent sheaves of $\mathcal{O}_X$-algebras.