Problem session 4

The following problems deal with the notion of constructible sets.

Problem 1. Recall that if T is a topological space, then a subset $S \subseteq T$ is *locally closed* if and only if it can be written as an intersection $U \cap F$, with U open and F closed in X. A subset $S \subseteq T$ is *constructible* if it can be written as a finite union of locally closed subsets.

(i) Show that $S \subseteq T$ is locally closed if and only if S is open in its closure $\overline{S}$.
(ii) Show that the class of constructible subsets of T is the smallest class of subsets of T that contains the closed subsets and that is closed under taking complements, finite unions and finite intersections.
(iii) Show that if $f \colon T' \to T$ is a continuous map and if $S \subseteq T$ is a constructible subset, then $f^{-1}(S)$ is constructible.

Problem 2. Let X be a Noetherian scheme and S a subset of X.

(i) Show that if S is constructible and irreducible, then S contains a nonempty open subset of $\overline{S}$.
(ii) Show that S is closed if and only if it is constructible and for every $x \in S$ and $y \in \overline{\{x\}}$ we have $y \in S$. Similarly, S is open if and only if it is constructible and for every $y \in S$ and for every x such that $y \in \overline{\{x\}}$, we have $x \in S$.

The importance of constructible subsets comes from the fact that this class is closed under taking (set-theoretic) images.

Problem 3. Give an example of a morphism $f \colon X \to Y$ of varieties over an algebraically closed field such that the set-theoretic image $f(X)$ is not locally closed in Y.

Problem 4. Prove the following theorem of Chevalley: if $f \colon X \to Y$ is a morphism of finite type of Noetherian schemes, then for every constructible subset S of X, its image $f(S)$ is constructible.

(a) Reduce the above assertion to the case when X and Y are affine integral schemes, f is dominant and $S = X$.
(b) Show that if Y is a Noetherian scheme, then any set of closed subsets of Y has a minimal element. Use this to reduce Chevalley's theorem to the following assertion: if f is a morphism as in (a), then $f(X)$ contains an open subset of Y.
(c) Let $A \hookrightarrow B$ be an A-algebra of finite type, where A and B are domains. Use the following steps to show that if $g \in B$ is a nonzero element, then there is a nonzero $f \in A$ such that every prime P in A that does not contain f can be written as $Q \cap A$ for a prime Q in B that does not contain g (note that by taking $g = 1$ we get the assertion in (b)).
(c1) Argue by induction on the number of generators of B as an A-algebra, to reduce the assertion to the case when B is generated over A by one element t.

(c2) Show that if $A[t]$ is isomorphic to the polynomial ring $A[T]$ such that $g = \sum_{i=0}^{n} \beta_i T^i$, then one can take $f = \beta_i$ for any i such that $\beta_i \neq 0$.

(c3) If this is not the case, show that we may assume that B is finite as an $A[g]$-module. Moreover, using the fact that finite morphisms are closed, show that we may assume that $B = A[g]$.

(c4) If $B = A[g]$ and $g^m + \sum_{i=1}^{m} \alpha_i g^{m-i} = 0$, with $\alpha_i \in A$, show that we may take $f = \alpha_i$ for any i such that α_i is nonzero.