Problem session 3

Problem 1. Let Y and Z be two closed subschemes of a scheme X.

i) Show that there is a unique closed subscheme of X that is contained in both Y and Z and which is maximal with this property. It is called the *intersection* of Y and Z, and it is denoted by $Y \cap Z$. Show that its support is the intersection of the supports of Y and Z.
ii) Show that if $X = \operatorname{Spec} A$ and if $Y = \operatorname{Spec} A/I$ and $Z = \operatorname{Spec} A/J$, then $Y \cap Z = \operatorname{Spec} A/I + J$.
iii) Suppose that k is an algebraically closed field, and let $C \subseteq \mathbb{A}^2_k$ be a nonsingular curve (recall that if C is defined by $f = 0$ for $f \in k[x, y]$, C is nonsingular if for every point $P \in C$ either $\frac{\partial f}{\partial x}(P) \neq 0$ or $\frac{\partial f}{\partial y}(P) \neq 0$; in this case the tangent line to C at $P = (x_0, y_0)$ is defined by $\frac{\partial f}{\partial x}(P)(x - x_0) + \frac{\partial g}{\partial y}(P)(y - y_0) = 0$). Show that if L is a line, then $L \cap C$ is reduced if and only if L is not tangent to C at any point of C.

Problem 2.

i) Give an example of a morphism of schemes that has finite fibres, but is not finite.
ii) Give an example of a morphism of schemes that gives a homeomorphism onto a closed subset, but is not a closed immersion.
iii) Give an example of a morphism that is locally of finite type, but not of finite type.
(iv) Give an example of an open immersion that is not quasicompact.

Problem 3. Let k be an algebraically closed field and let X be a scheme of finite type over k whose topological space consists of one point. The *degree* of X is $\deg(X) := \dim_k \mathcal{O}(X)$.

i) Describe X if $\deg(X) \leq 3$.
ii) If $\deg(X) = 2$, describe the closed immersions $X \hookrightarrow \mathbb{A}^2_k$.

Problem 4. A morphism of schemes $f \colon X \to Y$ is called *dominant* if the set-theoretic image of f is dense in Y. Show that if X and Y are integral schemes and f is dominant, then we have an induced field homomorphism $K(Y) \hookrightarrow K(X)$ between the function fields of Y and X.

Problem 5. An integral scheme X is *normal* if all its local rings are integrally closed in their fraction field. Let $K(X)$ be the function field of X.

(i) If $U \subseteq X$ is an affine open subset, let $\widetilde{U} = \operatorname{Spec} \widetilde{\mathcal{O}(U)}$, where $\widetilde{\mathcal{O}(U)}$ is the integral closure of $\mathcal{O}(U)$ in the function field $K(X)$. Show that the induced morphisms $\widetilde{U} \to U$ can be glued to a morphism $f \colon \widetilde{X} \to X$.
(ii) Show that $\widetilde{X}$ is normal, and the morphism f is affine, surjective and birational (i.e. the induced morphism $K(X) \to K(\widetilde{X})$ is an isomorphism).

(iii) Show that f satisfies the following universal property: if $g\colon Y \to X$ is a dominant morphism and Y is a normal scheme, then there is a unique morphism $h\colon Y \to \widetilde{X}$ such that $g = f \circ h$.

Note: if X is a scheme of finite type over a field k, then for every U as above, $\widetilde{\mathcal{O}(U)}$ is a finite $\mathcal{O}(U)$-module. Therefore the morphism f is finite.