Problem session 2

Problem 1. Let k be an algebraically closed field.

(1) Let $f\colon \operatorname{Spec} k[x,y]/(x-y^2) \to \operatorname{Spec} k[x]$ be the morphism induced by $i\colon k[x] \to k[x,y]/(x-y^2)$ such that $i(x)=x$. Describe the fibers of f (both over the closed points and over the generic point). Which fibers are reduced ?

(2) Let $g\colon \operatorname{Spec} k[x,y,z]/(xy-z^2) \to \operatorname{Spec} k[z]$ be induced by $j\colon k[z] \to k[x,y,z]/(xy-z^2$, such that $j(z)=z$. Describe the fibers of g. Which fibers are reducible ?

Problem 2. Show that if X is a reduced scheme and if $U \subseteq X$ is a dense open subset, then the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is injective.

Problem 3. Give an example of a reduced scheme X over $\operatorname{Spec} k$ for a field k, and a field extension $k \subset K$ such that $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$ is not reduced. Give a similar example, where X is irreducible, but $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$ is not.

Problem 4. Let X be a scheme and x a point on X.

(1) Show that there is a canonical morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to X$. Moreover, if $f\colon X \to Y$ is a scheme morphism and $y=f(x)$, then there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{O}_{X,x} & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} \mathcal{O}_{Y,y} & \longrightarrow & Y \end{array}$$

(2) Suppose that X and Y are schemes over S, with S Noetherian and Y of finite type. Consider the set

$$F_x := \{(U,f) \mid x \in U \subseteq X, U \,\text{open}, f \in \operatorname{Hom}_S(U,Y)\}/\sim,$$

where $(U,f) \sim (V,g)$ if and only if $f=g$ in some open neighborhood of x. Show that there is a natural bijection $F_x \simeq \operatorname{Hom}_S(\operatorname{Spec} \mathcal{O}_{X,x}, Y)$.

Problem 5. Let X be a Noetherian scheme.

(1) Show that the corresponding topological space is Noetherian, i.e. there is no strictly increasing sequence of open subsets.

(2) Show that as a topological space, X admits a unique decomposition $X = X_1 \cup \ldots \cup X_r$, where X_i are maximal irreducible closed subsets (called the irreducible components of X).