Problem session 11

The first problem requires the use of the Tor functors. We recall first their definition and the basic properties. Suppose that A is a commutative ring and M is an A-module. The functor $M \otimes_A -$ is right exact. The category of A-modules has enough projective objects and therefore we can construct the left derived functors of the above functor. The ith derived functor is denoted by $\mathrm{Tor}_i^A(M, -)$.

It follows by definition that $\mathrm{Tor}_0^A(M, N) \simeq M \otimes_A N$ and that if we have an exact sequence of A-modules

$$0 \to N' \to N \to N'' \to 0$$

then we get a long exact sequence

$$\cdots \to \mathrm{Tor}_i^A(M, N') \to \mathrm{Tor}_i^A(M, N) \to \mathrm{Tor}_i^A(M, N'') \to \mathrm{Tor}_{i-1}^A(M, N') \to \cdots .$$

It is an easy exercise to deduce from this and from the definition of flatness that M is flat if and only if $\mathrm{Tor}_i^A(M, N) = 0$ for every N and for every $i \geq 1$. Moreover, it is enough to have this for $i = 1$ and every N.

The fact that the tensor product commutes with arbitrary direct sums implies that the same remains true for the Tor functors. A slightly trickier result is that we have $\mathrm{Tor}_i^A(M, N) \simeq \mathrm{Tor}_i^A(N, M)$ for every i, M and N. This follows using the commutativity of the tensor product and by computing Tor using free resolutions for both M and N at the same time.

The purpose of the first problem is to prove a version of the *local flatness criterion.*

Problem 1. Let A be a ring and I an ideal of A that is nilpotent, i.e. there is q such that $I^q = 0$. If M is an A-module, then the following are equivalent:

(1) M is flat over A.
(2) M/IM is flat over A/I and the canonical morphism $I \otimes M \to M$ is injective.

For the implication (2)⇒(1), suppose that the condition in (2) is satisfied.

(a) Show that $\mathrm{Tor}_1(M, A/I) = 0$.
(b) Deduce that for every A/I-module N we have $\mathrm{Tor}_1^A(M, N) = 0$ (hint: consider a free presentation of N).
(c) Prove now by induction on $m \geq 1$ that if N is an A-module annihilated by I^m, then $\mathrm{Tor}_1^A(M, N) = 0$.

By taking $m = q$, we deduce that M is flat over A.

Remark. There is another version of the local flatness criterion: if $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a local morphism of local Noetherian rings and if M is a finitely generated B-module, then the equivalence (1)⇔(2) still holds if $I \subseteq \mathfrak{m}$. Moreover, note that if we take $I = \mathfrak{m}$, then M/IM is automatically flat over A/I. For a proof, see Matsumura's book.

We will use this to describe infinitesimal deformations of a scheme. Let X be a scheme of finite type over an algebraically closed field k. An *infinitesimal deformation of* X (over A) is a Cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow g \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

with g flat and of finite type and where A is a finite local k-algebra (i.e. $\mathrm{Spec}(A)$ is supported at one point).

In fact, we will be interested in embedded deformations. Suppose that X is a closed subscheme of a scheme Y of finite type over k and that A is as above. An *infinitesimal embedded deformation of* X (over A) is a closed subscheme $\mathcal{X} \hookrightarrow Y \times \mathrm{Spec}(A)$ that is flat over $\mathrm{Spec}(A)$ and such that $\mathcal{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = X$ as closed subschemes of Y.

Suppose that $\phi\colon A' \to A$ is a morphism of local finite k-algebras inducing $f\colon \mathrm{Spec}(A) \hookrightarrow \mathrm{Spec}(A')$. If $\mathcal{X} \hookrightarrow Y \times \mathrm{Spec}(A')$ is an embedded deformation of X over A', then $f^*\mathcal{X} := \mathcal{X} \times_{\mathrm{Spec}(A')} \mathrm{Spec}(A)$ is an embedded deformation of X over $\mathrm{Spec}(A)$. For example, if $\phi\colon A' \to k$ is the canonical surjection, then $f^*\mathcal{X} = X$.

Suppose now that $\phi\colon A' \to A$ is surjective and that $\mathcal{X} \subseteq Y \times \mathrm{Spec}(A)$ is an infinitesimal embedded deformation of X. A key problem is to understand the *liftings* of $\mathcal{X}$ over $\mathrm{Spec}(A')$, i.e. the embedded deformations $\mathcal{X}'$ of X over $\mathrm{Spec}(A')$ such that $f^*\mathcal{X}' = \mathcal{X}$.

Problem 2. With the above notation, suppose that $Y = \mathrm{Spec}(R)$ is affine and that $\mathcal{X}$ is defined by the ideal $J \subseteq R \otimes_k A$. Let $\mathcal{X}'$ be a closed subscheme of $Y \times \mathrm{Spec}(A')$ defined by the ideal $J' \subseteq R \otimes_k A'$. such that $J' \cdot (R \otimes_k A) = J$. Show that $\mathcal{X}'$ is flat over $\mathrm{Spec}(A')$ if and only if the following two conditions hold for some (every) system of generators $f_1, \ldots, f_r$ of J:

(a) We have liftings f_i' of the f_i to J' that generate J' (in fact every such liftings to J' will generate J').
(b) For every relation $\sum_i g_i f_i = 0$ with the g_i in $R \otimes_k A$ there are liftings g_i' of the g_i to $R \otimes_k A'$ such that $\sum_i g_i' f_i' = 0$.

In the next problem we use the above description of liftings to study the first-order embedded deformations of X.

Problem 3. Let X be a closed subscheme of Y. Our goal is to describe the set of embedded deformations of X over $A = k[t]/(t^2)$.

(a) Suppose first that $Y = \mathrm{Spec}(R)$ is affine and let $f_1, \ldots, f_r$ be generators of the ideal I of X in Y. Show that the ideal of every embedded deformation of X over $k[t]/(t^2)$ is generated by elements of the form $f_i + t g_i$ for suitable elements $g_i \in R$ with the property that the morphism $R^r \to R$ taking e_i to g_i induces a morphism of R-modules $I \to R/I$ (note that we have also a surjection $R^r \to I$ that takes each e_i to f_i.

(b) Show that two sets $\{g_i\}_i$ and $\{g'_i\}_i$ as above define the same closed subscheme of $Y \times \mathrm{Spec}(k[t]/(t^2))$ if and only if the corresponding morphisms $I \to R/I$ are the same.

(c) Deduce that for an arbitrary Y (not necessarily affine), if $\mathcal{I}$ is the sheaf of ideals defining X, then the set of embedded deformations of X over $k[t]/(t^2)$ is isomorphic to the space of global sections of the normal sheaf of X in Y, namely to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$.