Problem session 10

Problem 1. Let $X = \mathring{A}_k^n = \operatorname{Spec} k[x_1, \dots, x_n]$, where k is an algebraically closed field. Let Y denote the linear subspace defined by $I = (x_1, \dots, x_r)$ for some r with $1 \le r \le n$. Compute the blowing-up $\widetilde{X}$ of X along Y as follows.

1) Show that there is a surjective homomorphism of graded rings
$$R := k[x_1, \dots, x_n][y_1, \dots, y_r] \to \oplus_{m \in \mathbb{N}} I^m$$
that induces a closed embedding $\widetilde{X} \hookrightarrow X \times \mathbb{P}^{r-1}$.
2) Let J be the ideal of R generated by the degree one elements $x_i y_j - x_j y_i$ for all $1 \le i < j \le r$. Show that $\widetilde{X}$ is isomorphic with $\operatorname{Proj}(R)$.
3) Describe the exceptional divisor in the affine charts of $X \times \mathbb{P}^{r-1}$.

Remark 1. It can be shown that in fact J is the kernel of the homomorphism in 1).

Remark 2. A similar computation holds if $X = \operatorname{Spec}(R)$ is a nonsingular variety and I is an ideal of codimension r generated by r elements (the key fact is that these r elements form a regular sequence). If I defines a nonsingular subvariety of X, then locally one can always generate I by r elements.

Problem 2. Describe the proper transforms of the following affine plane curves under the blowing-up of the origin:

1) $y^2 - x^2(x+1) = 0$.
2) $y^2 - x^3 = 0$.
3) $y^2 - x^4(x+1) = 0$.

Which of these proper transforms are nonsingular ?

Problem 3. Let $X \subseteq \mathbb{P}_k^n$ be a hypersurface and $Y \subseteq \mathring{A}_k^{n+1}$ the affine cone over X. Show that the exceptional divisor of the blowing-up of Y at the origin is isomorphic to X.

Problem 4. Let A be a Noetherian ring and X a Noetherian scheme over A. Suppose that L is an invertible sheaf and that $s_0, \dots, s_n$ are global sections of L.

(1) The *base locus* of the linear span of the s_i is the subscheme B of X defined by the image of
$$\mathcal{O}_X^{\oplus(n+1)} \to \mathcal{O}_X$$
that maps each vector e_i of the standard basis to s_i. Show that B is the scheme-theoretic intersection of the closed subschemes $Z(s_i)$. In particular, the sections s_i define a morphism
$$f \colon X \smallsetminus B \to \mathbb{P}_k^n.$$
(2) Suppose that B is not supported on the whole X and let $\pi \colon \widetilde{X} \to X$ be the blowing-up of X along B. Show that there is a natural extension of $f \circ \pi$ from $\pi^{-1}(X \smallsetminus B)$ to $\widetilde{X}$.