Chapter 8: GKZ decompositions and toric Mori theory

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The nef and the effective cones. In this chapter we study the nef and the effective cone of a toric variety. Let $X = X(\Delta)$ be a toric variety, where Δ is a fan in N such that $|\Delta|$ is convex and of dimension equal to $n = \mathrm{rk}(N)$ (the latter condition can always be achieved after writing $X \simeq X' \times (k^*)^r$ for a suitable X').

The nef (respectively, ample) cone in $\mathrm{Pic}(X)_{\mathbb{Q}}$ is the set of classes of nef (respectively, ample) $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisors D. They are denoted by $\mathrm{Nef}(X)$ and $\mathrm{Amp}(X)$, respectively. Note that both these sets are convex cones.

Intersecting with a complete curve $C \subseteq X$ defines a linear function on $\mathrm{Pic}(X)_{\mathbb{Q}}$. It follows from Proposition 8 in Chapter 6 that the nef cone is equal to the set of α in $\mathrm{Pic}(X)_{\mathbb{Q}}$ such that $(\alpha \cdot C) \geq 0$ for every complete invariant curve C in X. In particular, the nef cone is polyhedral. X is quasiprojective if and only if the ample cone is not empty.

Exercise 1. Use Proposition 9 in Chapter 6 to show that if X is quasiprojective, then the ample cone is the interior of the nef cone.

Remark 1. The assertion in Exercise 1 holds on any projective variety. On the other hand, in the case of a non-projective complete variety, the nef cone can exhibit a rather pathological behavior. For example, Fujino gives in [Fu1] an example of a nonprojective complete toric variety such that the nef cone has dimension equal to $\dim_{\mathbb{Q}} \mathrm{Pic}(X)_{\mathbb{Q}}$. In the opposite direction, Payne gives in [Pa] an example of a smooth complete toric threefold with no nontrivial nef line bundles.

We denote by $\mathrm{N}_1(X)_{\mathbb{Q}}$ the set of numerical classes of complete curves on X. This is the quotient of the $\mathbb{Q}$-vector space with basis given by the irreducible complete curves on X by the subspace

$$\{\sum_i \lambda_i [C_i] \mid \sum_i \lambda_i (D \cdot C_i) = 0 \,\mathrm{for\,every\,divisor}\, D \,\mathrm{on}\, X\}.$$

The **Mori cone** $\mathrm{NE}(X)_{\mathbb{Q}}$ of X is the convex cone generated in $\mathrm{N}_1(X)_{\mathbb{Q}}$ by the classes of irreducible complete curves in X.

Note that inside $\mathrm{N}_1(X)_{\mathbb{Q}}$ we have the vector subspace W generated by the classes of the irreducible complete invariant curves. The intersection product gives a bilinear pairing

$$\mathrm{Pic}(X)_{\mathbb{Q}} \times \mathrm{N}_1(X)_{\mathbb{Q}} \to \mathbb{Q}.$$

By definition, this induces an injective map $\mathrm{N}_1(X)_{\mathbb{Q}} \hookrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}^*$, so $\dim_{\mathbb{Q}} N_1(X)_{\mathbb{Q}} \leq \dim_{\mathbb{Q}} \mathrm{Pic}(X)_{\mathbb{Q}}$. On the other hand, it follows from Corollary 4 in Chapter 6 that the induced map $\mathrm{Pic}(X)_{\mathbb{Q}} \to W^*$ is injective. We conclude that $\mathrm{N}_1(X)_{\mathbb{Q}}$ is generated by the classes of the irreducible complete invariant curves, and the intersection product induces a duality between $\mathrm{Pic}(X)_{\mathbb{Q}}$ and $N_1(X)_{\mathbb{Q}}$. Moreover, we deduce from Proposition 8 in Chapter 6 that the dual of the convex cone in $N_1(X)_{\mathbb{Q}}$ generated by the invariant curves is equal to the nef cone. This implies that the Mori cone is generated by the classes of complete irreducible invariant curves, and its dual is the nef cone.

Exercise 2. Show that if $|\Delta|$ is convex and n-dimensional, then the nef cone of $X(\Delta)$ is a pointed cone. If $X(\Delta)$ is quasiprojective, then the Mori cone of $X(\Delta)$ is pointed, too.

If Δ is an arbitrary fan and $X = X(\Delta)$, the **effective cone** $\mathrm{Eff}(X)$ is the set of classes of effective $\mathbb{Q}$-divisors on X. Equivalently, $\mathrm{Eff}(X)$ is the set of classes of $\mathbb{Q}$-T-divisors D such that P_D is nonempty. Therefore $\mathrm{Eff}(X)$ is the convex cone in $\mathrm{Cl}(X)_{\mathbb{Q}}$ generated by the classes of the invariant divisors $D_1, \ldots, D_d$. In particular, it is a polyhedral cone of dimension equal to the rank of $\mathrm{Cl}(X)$.

Example 1. If $X = U_\sigma$ is an affine toric variety, then $\mathrm{Eff}(X)$ is equal to $\mathrm{Cl}(X)_{\mathbb{Q}}$.

The cone of effective line bundles is the intersection $\mathrm{Eff}(X) \cap \mathrm{Pic}(X)_{\mathbb{Q}}$ (it is clear that this cone contains the nef cone whenever the nef cone is defined). A line bundle $\mathcal{L}$ is in this cone if and only if some positive multiple has a section. Note that since $\mathrm{Eff}(X)$ is polyhedral, so is its intersection with $\mathrm{Pic}(X)_{\mathbb{Q}}$. It follows from general considerations (see [Laz]) that if X is projective, then a line bundle $\mathcal{L}$ lies in the interior of the cone of effective line bundles if and only if $\mathcal{L}$ is big.

Exercise 3. If $|\Delta|$ spans $N_{\mathbb{R}}$ as a convex cone and $X = X(\Delta)$, then the effective cone $\mathrm{Eff}(X)$ is a pointed cone.

Faces of the nef cone. We discuss now the connection between the faces of the nef cone of X and the fiber spaces with total space X and quasiprojective base. Recall that $f\colon X \to Y$ is a **fiber space** if f is proper and projective, and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Every proper morphism $g\colon X \to Z$ admits a unique factorization $g = h \circ f$, where $f\colon X \to Y$ is a fiber space and h is finite: take $Y = \mathcal{S}pec(g_*\mathcal{O}_X)$. If f is a fiber space, then f is uniquely determined by the irreducible complete curves $C \subseteq X$ which are contracted by f (note that by Zariski's Main Theorem, a fiber space has connected fibers).

Lemma 1. *A morphism of toric varieties $f = \phi_*\colon X' \to X$ is a fiber space if and only if it is proper and the corresponding group homomorphism ϕ is surjective.*

Proof. The "if" part was proved in Corollary 3 in Chapter 4. In order to show that if f is a fiber space then $\phi\colon N' \to N$ is surjective, we may restrict to the torus inside X, and hence assume that $X = T_N$. If $\widetilde{N}$ is the image of ϕ, we see that f factors through $T_{\widetilde{N}} \to T_N$. Moreover, the induced morphism $X' \to T_{\widetilde{N}}$ is a fiber space, so $\mathcal{O}(T_N) = \mathcal{O}(T_{\widetilde{N}})$. This shows that ϕ is surjective, and completes the proof.

Suppose now that $X = X(\Delta)$, where $|\Delta|$ is convex and n-dimensional. Let D be a nef $\mathbb{Q}$-T-divisor on X. After replacing D by a linearly equivalent divisor, we may assume that D is effective. It follows from Proposition 9 in Chapter 6 that Δ gives a refinement of the normal fan Δ_D to P_D, so we have a fiber space $f\colon X \to X(\Delta_D)$.

Moreover, if we put

$$\phi_D(v) := \min_{u \in P_D} \langle u, v\rangle$$

for every v in $|\Delta|$, then ϕ_D induces a strictly concave function on $|\overline{\Delta_D}|$, where $\overline{\Delta_D}$ is the nondegenerate fan corresponding to Δ_D. Therefore we get an ample divisor A on $X(\Delta_D)$ such that $f_D^* A = D$.

If C is an irreducible complete curve on X, then $f_D(C)$ is a point if and only if $(D \cdot C) = 0$. Therefore such a curve is contracted by f_D if and only if its class in $\mathrm{N}_1(X)_{\mathbb{Q}}$ belongs to the face of the Mori cone $\mathrm{NE}(X)_{\mathbb{Q}} \cap D^{\perp}$. Since f is determined by the curves it contracts, it follows that f_D depends only on the smallest face τ of the nef cone which contains D.

Moreover, this face τ is the image by the injective linear map

$$f^*\colon\ \mathrm{Pic}(X(\Delta_D))_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$$

of $\mathrm{Nef}(X(\Delta_D))$. The fact that f^* is injective follows from the fact that $P_{f^*(E)} = P_E$: if $P_{f^*(E_1)} = u + P_{f^*(E_2)}$, then u lies in the linear span of P_D and $P_{E_1} = u + P_{E_2}$. On the other hand, it is clear that $f^*(\mathrm{Nef}(X(\Delta_D)))$ is contained in $\mathrm{Nef}(X)$ and is zero on every curve contracted by f_D, so it is contained in τ. Conversely, we have seen that every element in the relative interior of τ is the pull-back of an ample line bundle on $X(\Delta_D)$, so τ is contained in $f^*(\mathrm{Nef}(X(\Delta)))$.

Suppose now that $g\colon X \to Y$ is a fiber space such that Y is quasiprojective. If H is an ample divisor on Y, then an irreducible complete curve C on X is contracted if and only if $(g^*H \cdot C) = 0$. It follows that there is a unique isomorphism $h\colon X(\Delta_{g^*H}) \to Y$ such that $g = h \circ f_{g^*H}$. We have proved the following

Proposition 1. *Let $X = X(\Delta)$ be a toric variety, with $|\Delta|$ convex and n-dimensional. There is a bijection between the faces τ of the nef cone of X and the fiber spaces $f\colon X \to Y$ with Y quasiprojective, such that if τ corresponds to f, then $\tau = f^*(\mathrm{Nef}(Y))$. Moreover, an irreducible complete curve C on X is contracted by f if and only if it lies in $\tau^\perp$. Every fiber space $X \to Y$ with Y quasiprojective is toric.*

Remark 2. If τ_1 and τ_2 are faces of $\mathrm{Nef}(X)$ corresponding to $f_1\colon X \to Y_1$ and $f_2\colon X \to Y_2$, respectively, then $\tau_1 \subseteq \tau_2$ if and only if there is $h\colon Y_2 \to Y_1$ such that $h \circ f_2 = f_1$.

Pull-back of line bundles by rational maps. We review the definition of pull-backs of Cartier divisors by rational maps. Suppose that Δ and Δ' are fans in N and N', respectively, and let $\phi\colon N \to N'$ be a group homomorphism. We assume that if $\tau_1, \ldots, \tau_d$ are the rays in Δ, then $\phi(\tau_i) \subseteq |\Delta'|$ for every i. If $U = \cup_{i=1}^d U_{\tau_i}$, then ϕ induces a morphism of toric varieties $f = \phi_*\colon U \to X(\Delta')$. Since we have $\mathrm{codim}(X(\Delta) \smallsetminus U, X(\Delta)) \geq 2$, there are canonical isomorphisms induced by restriction

$$\mathrm{Div}_T(X(\Delta)) = \mathrm{Div}_T(U), \quad \mathrm{Cl}(X(\Delta)) = \mathrm{Cl}(U).$$

It follows that by pulling-back $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisors and line bundles via f, we get $f^*\colon \mathrm{Cart}_T(X(\Delta'))_{\mathbb{Q}} \to \mathrm{Cl}(X(\Delta))_{\mathbb{Q}}$ inducing $f^*\colon \mathrm{Pic}(X(\Delta'))_{\mathbb{Q}} \to \mathrm{Cl}(X(\Delta))_{\mathbb{Q}}$. Note that in this discussion we may replace U by any open subset V on which f is defined and such that the codimension of $X(\Delta) \smallsetminus V$ is at least two.

Base loci and Zariski decompositions. In order to motivate the later study of the effective cone, we discuss now the Zariski decomposition and the Iitaka fibration of a divisor. The usual setup is on a complete variety, but for later applications we keep assuming only that $|\Delta|$ is convex and of maximal dimension.

If D is a Cartier divisor $X(\Delta)$, then we denote by $\mathfrak{b}_{|D|}$ the ideal defining the base locus of $\mathcal{O}(D)$. It is characterized by the fact that $\mathfrak{b}_{|D|} \otimes \mathcal{O}(D)$ is the image of

$$\Gamma(X, \mathcal{O}(D)) \otimes \mathcal{O}_X \to \mathcal{O}(D).$$

Proposition 2. *Let D be a T-Cartier divisor on a toric variety $X(\Delta)$, with $|\Delta|$ convex and of maximal dimension.*

(1) *There is a positive integer q such that $\mathfrak{b}_{|qmD|} = \mathfrak{b}_{|qD|}^m$ for all $m \geq 1$.*

(2) *If* $P_{qD} = Q + |\Delta|^\vee$, *where* Q *is a lattice polytope, then* $\mathfrak{b}_{|qmD|}$ *and* $\mathfrak{b}^m_{|qD|}$ *have the same integral closure for every* $m \geq 1$.

(3) *Suppose that* $P_D = Q + |\Delta|^\vee$ *for a lattice polytope* Q, *and put*

$$\psi(v) = \min_{u \in P_D} \langle u, v \rangle$$

for every v *in* $|\Delta|$. *The ideal* $\mathfrak{b}_{|D|}$ *is locally principal, defining an effective Cartier divisor* F *if and only if for every cone* σ *in* Δ *there is* u_σ *in* $P_D \cap M$ *such that* $\psi(v) = \langle u_\sigma, v \rangle$ *for* v *in* σ. *In this case we have* $\psi_F = \psi_D - \psi$, $P_D = P_{D-F}$ *and* $\mathfrak{b}_{|mD|} = \mathcal{O}(-mF)$ *for every* $m \geq 1$.

Proof. The assertion in (1) follows from the fact that the section ring R_D of $\mathcal{O}(D)$ is a finitely generated k-algebra (see Exercise 9 in Chapter 6), plus standard results on graded rings. See for example Proposition 1.2 in [Bou].

Since $P_{mqD} = mP_{qD}$, in order to prove (2) it is enough to show the following: if $u_1, \ldots, u_r$ are in M such that $P_D = \mathrm{conv}\{u_1, \ldots, u_r\} + |\Delta|^\vee$, then the ideal $\mathfrak{a}$ defining the base locus of the linear series corresponding to $\chi^{u_1}, \ldots, \chi^{u_r}$ has the same integral closure as $\mathfrak{b}_{|D|}$. Note that we clearly have $\mathfrak{a} \subseteq \mathfrak{b}_{|D|}$.

We show that $\mathfrak{b}_{|D|}$ is contained in the integral closure of $\mathfrak{a}$ on each U_σ, where σ is a cone in Δ. If $u(\sigma)$ in M is such that $D|_{U_\sigma} = \mathrm{div}(\chi^{-u(\sigma)})|_{U_\sigma}$, then $\mathfrak{b}_{|D|}$ is spanned on U_σ by $\{\chi^{u-u(\sigma)} \mid u \in P_D \cap M\}$. For every such u in $P_D \cap M$, we may write $u = a_1u_1 + \ldots + a_ru_r + w$ where $a_1, \ldots, a_r$ are nonnegative rational numbers such that $\sum_j a_j = 1$, and where w is in $|\Delta|^\vee$. If ℓ is a positive integer such that all ℓa_j are integers, and ℓw is in M, then we have

$$(\chi^{u-u(\sigma)})^\ell = \chi^{\ell w} \cdot \prod_{j=1}^r (\chi^{u_i - u(\sigma)})^{\ell a_j} \in \mathfrak{a}^\ell,$$

so $\chi^{u-u(\sigma)}$ lies in the integral closure of $\mathfrak{a}$. This completes the proof of (2).

For (3), note that if $\mathfrak{b}_{|D|} = \mathcal{O}(-F)$, then $\mathcal{O}(D-F)$ is globally generated. Since F is effective we have $P_{D-F} \subseteq P_D$ and the reverse inclusion follows from the definition of F and the fact that the vertices of Q are lattice points. We deduce that $\psi = \psi_{D-F}$, so ψ has the required property.

Conversely, suppose that for each cone σ in Δ we can find u_σ in $P_D \cap M$ such that $\psi(v) = \langle u_\sigma, v \rangle$ for every v in σ. It follows that there is a Cartier T-divisor E such that $\psi = \psi_E$. By the definition of ψ we see that it is a concave function, so Proposition 3 in Chapter 6 implies that $\mathcal{O}(E)$ is globally generated. We have $\psi \geq \psi_D$, hence there is an effective Cartier divisor F such that $\psi_F = \psi_D - \psi$. For every u in $mP_D \cap M$, the divisor $\mathrm{div}(\chi^u) + mD - mF$ is effective, so $\mathfrak{b}_{|mD|}$ is contained in $\mathcal{O}(-mF)$. Since $\mathcal{O}(mD - mF)$ is globally generated, it follows that $\mathfrak{b}_{|mD|} = \mathcal{O}(-mF)$.

Remark 3. The condition in (3) is equivalent with the fact that the identity on N gives a morphism of fans from Δ to the normal fan of P_D.

We use Proposition 2 to describe the Zariski decomposition of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor D on the toric variety $X = X(\Delta)$. We assume that $D = \sum_{i=1}^{d} a_i D_i$, where all a_i are nonnegative. Note that if P_D is nonempty, then we can achieve this after replacing D with a linearly equivalent divisor.

Let q be such that qP_D is a lattice polytope and qD is a Cartier divisor. We want to construct a proper, birational morphism $\pi\colon Y \to X$ such that the base locus of $\mathcal{O}(\pi^*(qD))$ is a Cartier divisor. The minimal such π is the normalized blow-up of X along the base locus of $\mathcal{O}(qD)$. A description of this variety can be given using the local description of normalized blow-ups in Chapter 4. However, it is easier to use Proposition 2 instead.

Suppose that Δ' is a fan refinement of Δ inducing a proper, birational morphism of toric varieties $\pi\colon Y \to X$. Note that we have $P_{\pi^*D} = P_D$. It follows from Proposition 2 (see also Remark 3) that the base locus of $\mathcal{O}(\pi^*(qD))$ is an effective Cartier divisor if and only if Δ' refines also the normal fan of P_D. Therefore there is a unique minimal such fan Δ' (in the sense that every other fan with the same property refines Δ'). Its cones are the intersections of the cones in Δ with the cones in the normal fan to P_D.

Note that the morphism $\pi\colon Y \to X$ does not depend on q, but only on D. On Y we have a decomposition $\pi^*D = F + E$ with the following properties:

(1) F is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor.
(2) E is nef.
(3) For every m divisible enough, multiplication by an equation defining mF induces an isomorphism

$$\Gamma(Y, \mathcal{O}(mE)) \simeq \Gamma(Y, \mathcal{O}(mD)).$$

Note that the $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor E is characterized by

$$\psi_E(v) = \min_{u \in P_D} \langle u, v \rangle.$$

A decomposition as above is called a **Zariski decomposition** of D. We mention that existence of Zariski decompositions has important consequences for the behavior of linear series on a given variety. In general, such a decomposition exists if the section ring R_D of $\mathcal{O}(D)$ is finitely generated (in particular, one could deduce the existence of a Zariski decomposition on a toric variety from Exercise 9 in Chapter 6). On the other hand, there are varieties for which there is no Zariski decomposition decomposition, even if one allows E and F to have $\mathbb{R}$-coefficients (see [Na] for such an example). More on toric Zariski decompositions, in the relative setting, can be found in [FS].

Exercise 4. Let $\pi\colon Y \to X$ and $\pi^*D = F + E$ be the Zariski decomposition of the effective divisor D. Show that if X is complete, then $(E^{n-1} \cdot F) = 0$.

Remark 4. For general varieties there is no Zariski decomposition, but there is a notion of **asymptotic Zariski decomposition**. In this context, an asymptotic version of the orthogonality relation in the above exercise was proved in [BDPP]. This was part of their description of the effective cone as the dual of the cone of moving curves (see Proposition 6 below for a proof of this result for toric varieties).

Let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor on the toric variety X. Consider the proper birational morphism $\pi\colon Y \to X$ that we described above and the Zariski decomposition $\pi^*D = F + E$. If $q \geq 1$ is such that qD is a Cartier divisor and qP_D is the Minkowski sum of a lattice polytope and $|\Delta|^\vee$, then qP has integral coefficients and $\mathcal{O}(qE)$ is a globally generated line bundle. By Proposition 9 in Chapter 6 we get a morphism $f\colon Y \to Y'$. Moreover, we have an ample divisor A on Y' such that $f^*A = E$. Note that Y' is the toric variety defined by the normal fan to the polytope of E, which is the same as the normal fan to P_D.

The morphism f does not depend on q; this is the **Iitaka fibration** corresponding to D. See [Laz] for the general definition of the Iitaka fibration. For general varieties, this fibration is unique only up to birational equivalence; however, we see that for toric varieties there is a canonically defined morphism.

Exercise 5. Let g be the rational map $f \circ \pi^{-1}$. Show that g^*A is defined, $D' = D - g^*A$ is effective, and if q is divisible enough, then qD' is the divisorial part in the base locus of qD.

If X is complete, then f is the Stein factorization of the morphism $f_{\mathcal{O}(E)}$ defined by $\mathcal{O}(qE)$. We denote by $f_{\mathcal{O}(qD)}$ the rational map defined by $\mathcal{O}(qD)$, so we have the following commutative diagram

$$\begin{tikzcd}
& Y \arrow[dl, "\pi"'] \arrow[dr, "f"] \arrow[dd] & \\
X \arrow[dr, dashed, "f_{\mathcal{O}(qD)}"'] & & Y' \arrow[dl, "f_{\mathcal{O}(qP)}"'] \\
& \mathbb{P}(H^0(X,\mathcal{O}(qD))) = \mathbb{P}(H^0(Y,\mathcal{O}(qP))) &
\end{tikzcd}$$

Note that if X is complete, then for every q such that qD is Cartier and qP_D is a lattice polytope the dimension of the image of the rational map $f_{\mathcal{O}(qD)}$ is equal to the dimension of P_D. In particular, we recover the assertion that the Iitaka dimension of D is equal to the dimension of P_D.

Our next goal is to let D vary. We will see that if we fix the normal fan Δ_{P_D} to P_D and the support of D' consisting of, say, r components (D' is as in the above exercise), we get in the effective cone of $X(\Delta)$ an open cone isomorphic to the product between the ample cone of $X(\Delta_{P_D})$ and $(\mathbb{Q}_+^*)^r$. However, in order to account for all the ample divisors on $X(\Delta_{P_D})$, we need to allow D to be non-necessarily $\mathbb{Q}$-Cartier.

GKZ decompositions. Let X be the toric variety associated to a fan Δ in N, such that $|\Delta|$ is convex, of dimension $\mathrm{rk}(N)$. We denote by $\Delta(1)$ the set of rays in Δ, and by $D_1, \ldots, D_d$ the prime T-divisors. Consider a $\mathbb{Q}$-T-divisor $D = \sum_{i=1}^d a_i D_i$ such that $a_i \geq 0$ for all i, so that the origin lies in the corresponding polyhedron P_D. We use our considerations on Zariski decomposition to describe the effective cone. All the results can be found in [OP], but we follow the more geometric approach from [HKP].

We denote by Δ_D the normal fan to the polyhedron P_D (note that Δ_D may be degenerate). We consider also the set

$$I_D := \{i \mid \langle u, v_i\rangle > -a_i \,\text{for every}\, u \,\text{in}\, P_D\}.$$

It is clear that Δ_D and I_D do not change if we replace D by a linearly equivalent divisor or by a positive multiple. If D is $\mathbb{Q}$-Cartier, then I_D is precisely the set of prime divisors which appear in the base locus of $\mathcal{O}(mD)$ for all m divisible enough.

We see that every cone in Δ_D is generated by a subset of $\Delta(1) \smallsetminus I_D$. Indeed, this follows from the equations defining P_D and Lemma 12 in Chapter 2.

Recall that by Proposition 2 in Chapter 6, if we write P_D as the Minkowski sum of a polytope and a cone, then the cone is equal to $\cap_{\sigma\in\Delta}\sigma^\vee = |\Delta|^\vee$. By Proposition 5 in Chapter 4, this implies that the support of Δ_D is equal to the support of Δ.

We define $\phi_D \colon |\Delta| \to \mathbb{R}$ by

$$\phi_D(v) := \min_{u\in P_D} \langle u, v\rangle.$$

It follows from definition that the maximal cones in Δ_D are precisely the maximal sets on which ϕ_D is linear.

If W_D is the linear span of P_D, then Δ_D induces a nondegenerate fan in $N_{\mathbb{R}}/W_D^{\perp}$ which we denote by $\overline{\Delta_D}$. The canonical projection $g_D \colon N_{\mathbb{R}} \to N_{\mathbb{R}}/W_D^{\perp}$ induces a rational map $f_D \colon X(\Delta) \to X(\overline{\Delta_D})$. Since Δ and Δ_D have the same support, it follows that every ray in Δ is mapped by g_D inside a cone in $\overline{\Delta_D}$. In particular, we can pull-back Cartier divisors via f_D to get Weil divisors on X.

Since the origin lies in P_D, it follows that ϕ_D induces a function on $|\overline{\Delta_D}|$, which is strictly concave by the definition of Δ_D. Therefore it defines an ample divisor A on $X(\overline{\Delta_D})$. Note that we have $P_D = P_A$. Indeed, if u is in P_A, then

$$\langle u, v_i\rangle \geq \phi_D(v_i) \geq -a_i$$

for every i, so u is in P_D. Conversely, if u is in P_D, then u is in W_D and

$$\langle u, v_i\rangle \geq \min_{w\in P_D} \langle w, v_i\rangle = \phi_D(v_i)$$

for every i, so u is in P_A.

Consider $F := D - f_D^*(A)$. Since $f_D^*(A) = -\sum_{i=1}^d \phi_D(v_i) D_i$ and $\phi_D(v_i) \geq -a_i$, we see that F is effective. Moreover, the support of F is the union of those D_i with i in I_D. We call F the **fixed part** of D and the decomposition $D = f_D^*(A) + F$ the **GKZ decomposition** of D.

Suppose now that D is replaced by $D' = D + \mathrm{div}(\chi^u)$, with u in M (assuming that D' is effective, too). We have $\Delta_D = \Delta_{D'}$ and $I_D = I_{D'}$. Since u is in P_D, in particular u lies in $M \cap W_D$, so $A' = A + \mathrm{div}(\chi^u)$ is an ample divisor on $X(\overline{\Delta_D})$. We see that the corresponding decomposition for D' is $D' = f_D^*(A') + F$.

Remark 5. With the above notation, suppose that D is $\mathbb{Q}$-Cartier, and assume that q is such that qD is Cartier and qP_D is the Minkowski sum of a lattice polytope and of $|\Delta|$. It follows from definition that F is the divisorial part in the base locus of $|qD|$.

Conversely, suppose that Σ is a possibly degenerate fan in N with $|\Sigma| = |\Delta|$ and I is a subset of $\Delta(1)$ such that

(1) Every cone in Σ is generated by rays in $\Delta(1) \smallsetminus I$.
(2) $X(\Sigma)$ is quasiprojective.

We denote by $\mathcal{S}$ the set of such pairs (Σ, I). Note that since $\Delta(1)$ is finite, so is $\mathcal{S}$.

If W is the largest linear subspace contained in all cones in Σ, then Σ induces a nondegenerate fan $\overline{\Sigma}$ in $N_{\mathbb{R}}/W$. If v is in $N_{\mathbb{R}}$ we denote by $\overline{v}$ its class in $N_{\mathbb{R}}/W$. The natural projection $N_{\mathbb{R}} \to N_{\mathbb{R}}/W$ induces a rational map $f_{\Sigma}\colon X(\Delta) \dashrightarrow X(\Sigma)$. Since Σ and Δ have the same support, we see that f_{Σ} is defined on the complement of a closed subset of codimension at least two. Therefore we have a pull-back map f_{Σ}^* which takes $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisors on $X(\Sigma)$ to $\mathbb{Q}$-T-divisors on $X(\Delta)$.

In order to obtain a closed cone in $\mathrm{Eff}(X)$ we work with nef (non necessarily ample) divisors on $X(\Sigma)$. If A is a nef effective $\mathbb{Q}$-T-divisor on $X(\Sigma)$ and $\lambda = (\lambda_i)_i$ is in $\mathbb{Q}_+^{|I|}$, consider

$$D := f_{\Sigma}^*(A) + F, \quad \text{with } F = \sum_{i \in I} \lambda_i D_i. \tag{1}$$

We claim that $P_D = P_A$. Indeed, if u is in $M_{\mathbb{R}}$, then u is in P_D if and only if

$$\langle u, v_i \rangle \geq \psi_A(\overline{v_i}) - \lambda_i$$

for every i (if i is not in I, we put $\lambda_i = 0$). Note that if we put this condition only for the rays in $\Delta(1) \smallsetminus I$, then we get the condition for all rays, since each cone in Σ is generated by rays in $\Delta(1) \smallsetminus I$ and ψ_A is linear on each of these cones. This proves our claim.

Since A is nef, this implies that

$$\phi_D(v) := \min_{u \in P_D} \langle u, v \rangle = \psi_A(\overline{v})$$

for every v in $|\Delta|$. Moreover, if A is ample then Σ is the normal fan to P_A, so $\Sigma = \Delta_D$. It follows that in this case the decomposition in (1) is the GKZ decomposition of D.

Note that if A is not ample, then we have a fiber space $f_A\colon X(\Sigma) \to X(\Delta_A)$ and an ample divisor A' on $X(\Delta_A)$ such that $D = (f_A \circ f_{\Sigma})^*(A') + F$ is the GKZ decomposition of D.

The above correspondence induces a linear map

$$\Phi_{\Sigma,I}\colon\ \mathrm{Nef}(X(\Sigma)) \times \mathbb{Q}_+^{|I|} \to \mathrm{Cl}(X)_{\mathbb{Q}}$$

which is linear and injective. For injectivity, note that if $f_\Sigma^*(A_1) + F_1$ and $f_\Sigma^*(A_2) + F_2$ are linearly equivalent, then P_{A_1} is a translate of P_{A_2}, so A_1 and A_2 are linearly equivalent. If we write $F_j = \sum_{i\in I} \lambda_{i,j} D_i$ for $j = 1, 2$, then there is u in $M_{\mathbb{Q}}$ such that $\langle u, v_i\rangle$ is equal to $\lambda_{i,1} - \lambda_{i,2}$ if i is in I, and it is equal to zero, otherwise. Since each cone in Σ is generated by rays in $\Delta(1) \smallsetminus I$, we deduce that u is zero on $|\Delta|$, so $F_1 = F_2$.

We denote the image of $\Phi_{\Sigma,I}$ by $GKZ_{\Sigma,I}$. Since the nef cone is pointed and polyhedral, we see that $GKZ_{\Sigma,I}$ is a pointed polyhedral subcone of $\mathrm{Eff}(X)$. We denote the relative interior of this cone by $GKZ^\circ_{\Sigma,I}$. This is the image by $\Phi_{\Sigma,I}$ of $\mathrm{Amp}(X(\Sigma)) \times (\mathbb{Q}_+^*)^{|I|}$.

It follows from Proposition 1 that the faces of $GKZ_{\Sigma,I}$ are of the form $GKZ_{\Sigma',I'}$, where (Σ', I') is such that Σ refines Σ', I' is a subset of I and $X(\Sigma')$ is quasiprojective. For this it is enough to note that every such (Σ', I') is in $\mathcal{S}$, since every cone σ in Σ' is generated by the elements in $\Delta(1) \smallsetminus I$ which generate the cones in Σ that are contained in σ.

If D is an effective $\mathbb{Q}$-T-divisor on X, then its class $[D]$ in $\mathrm{Pic}(X)_{\mathbb{Q}}$ lies in $GKZ^\circ_{\Sigma,I}$ if and only if $\Sigma = \Delta_D$ and $I = I_D$. Moreover, it lies in $GKZ_{\Sigma,I}$ if and only if Σ refines Δ_D and $I_D \subseteq I$. In particular, this shows that if $[D]$ lies in $GKZ_{\Sigma,I}$, then GKZ_{Δ_D,I_D} is contained in $GKZ_{\Sigma,I}$. Moreover, GKZ_{Σ_1,I_1} is a subset of GKZ_{Σ_2,I_2} if and only if it is a face of this cone.

Consider now (Σ_1, I_1) and (Σ_2, I_2) in $\mathcal{S}$. We see that $GKZ_{\Sigma_1,I_1} \cap GKZ_{\Sigma_2,I_2}$ is a union of sets of the form $GKZ_{\Sigma,I}$, so it is a union of faces of both GKZ_{Σ_1,I_1} and GKZ_{Σ_2,I_2}. The lemma below shows that $GKZ_{\Sigma_1,I_1} \cap GKZ_{\Sigma_2,I_2}$ is a face of both GKZ_{Σ_1,I_1} and GKZ_{Σ_2,I_2}.

Lemma 2. *If σ is a pointed rational polyhedral convex cone, and $\tau \subseteq \sigma$ is a polyhedral convex subcone which is a union of faces of σ, then τ is a face of σ.*

Proof. A face of σ that is contained in τ is a face of τ. The assertion in the lemma follows from the fact τ can't be the union of its proper faces.

Thus we have proved the following

Theorem 1. *The set of cones*

$$\{GKZ_{\Sigma,I} \mid (\Sigma, I) \in \mathcal{S}\}$$

forms a fan whose support is the effective cone $\mathrm{Eff}(X)$. *The cone $GKZ_{\Sigma,I}$ is a face of $GKZ_{\Sigma',I'}$ if and only if Σ' refines Σ and I is contained in I'.*

The above fan refinement of $\mathrm{Eff}(X)$ is called the **GKZ decomposition** of the effective cone.

Remark 6. If (Σ, I) is in $\mathcal{S}$ and W is the common subspace of the cones in Δ, since $X(\Sigma)$ is quasiprojective we see that

$$\dim(GKZ_{\Sigma,I}) = \mathrm{rk}(\mathrm{Pic}(X(\Sigma))) + \#I \leq \#\overline{\Sigma}(1) - n + \dim(W) + \#I,$$

with equality if and only if $X(\Sigma)$ is simplicial. Note also that if W is nonzero and ℓ is the number of rays in $\Delta(1)$ that lie in W, since W is generated by these rays as a convex cone, we have $\ell \geq \dim(W) + 1$. Moreover, our hypothesis on Σ and I gives $\#\overline{\Sigma}(1) \leq \#\Delta(1) - \#I - \ell$. We deduce that in this case $\dim(GKZ_{\Sigma,I}) \leq d - n - 1$.

On the other hand, if W is zero we get

$$\dim(GKZ_{\Sigma,I}) \leq \#\Sigma(1) - n + \#I.$$

Therefore $GKZ_{\Sigma,I}$ is a cone of maximal dimension in $\mathrm{Eff}(X)$ if and only if Σ is nondegenerate and simplicial, and $I = \Delta(1) \smallsetminus \Sigma(1)$.

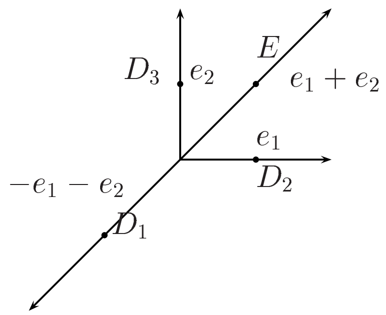
Example 2. Let Δ be the fan consisting of all faces of σ, where σ is the cone in $\mathbb{Z}^3$ generated by $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$ and $v_4 = e_1 + e_2 - e_3$, where the e_i form the standard basis of $\mathbb{Z}^3$. Since $X = X(\Delta)$ is affine, we have

$$\mathrm{Eff}(X) = \mathrm{Cl}(X)_{\mathbb{Q}} = \frac{\oplus_{i=1}^{4} \mathbb{Q} \cdot D_i}{(D_1 + D_4, D_2 + D_4, D_3 - D_4)}.$$

Let Σ_1 be the fan with maximal cones generated by $\{v_1, v_3, v_4\}$ and $\{v_2, v_3, v_4\}$, and Σ_2 the fan with maximal cones generated by $\{v_1, v_2, v_3\}$ and $\{v_1, v_2, v_4\}$. Note that D_1 is ample on $X(\Sigma_1)$. Indeed, we see that $\mathcal{O}(D_1)$ is globally generated using the fact that D_1 and D_2 are linearly equivalent; therefore D_1 is ample, since otherwise it would be principal. Similarly, $-D_1$ is ample on $X(\Sigma_2)$.

We get the decomposition of the effective cone on $X(\Delta)$ as the union $GKZ_{\Sigma_1,\emptyset} \cup GKZ_{\Sigma_2,\emptyset}$. Note that the two rational maps $X(\Delta) \dashrightarrow X(\Sigma_1)$ and $X(\Delta) \dashrightarrow X(\Sigma_2)$ are the inverses of the two resolutions of singularities π_1 and π_2 that we have seen in Example 5 in Chapter 5.

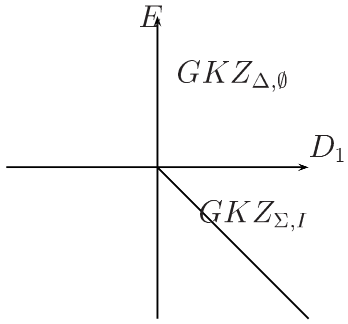
Example 3. Consider the complete fan Δ in $\mathbb{Z}^2$ with the rays below, such that $X = X(\Delta)$ is the blowing-up of $\mathbb{P}^2$ at one point



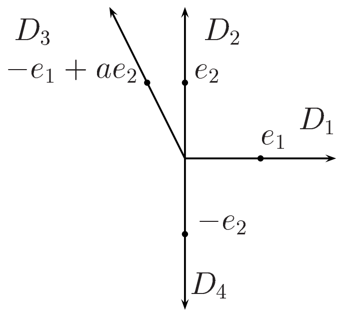
We have $\mathrm{Cl}(X)_{\mathbb{Q}} = \mathrm{Pic}(X)_{\mathbb{Q}}$ which has a basis given by D_1 and E. If Σ is the complete fan with rays through e_1, e_2 and $-e_1 - e_2$, then $X(\Sigma) = \mathbb{P}^2$, the induced map $\pi\colon X \to \mathbb{P}^2$ is the projection, and D_1 is the pull-back of a hyperplane on $\mathbb{P}^2$. Note that D_1 is linearly equivalent to $D_2 + E$ and $D_3 + E$.

The effective cone is generated by the classes of D_1, D_2, D_3 and E, so it is generated by $D_1 - E$ and E. On the other hand, a divisor $aD_1 + bE$ is nef if and only if its intersection with D_1, D_2, D_3 and E is nonnegative. One checks that this is the case if and only if $b \geq 0$ and $a \geq -b$.

We have two maximal cones in the GKZ decomposition: the nef cone $GKZ_{\Delta,\emptyset}$ and $GKZ_{\Sigma,\{E\}}$:



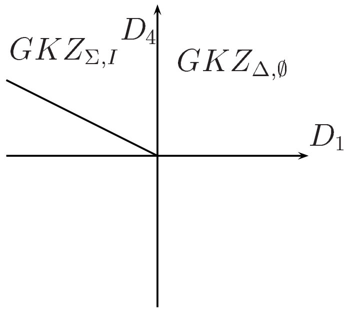
Example 4. Consider the complete fan Δ in $\mathbb{Z}^2$ with the rays below, such that $X(\Delta)$ is the Hirzebruch surface $\mathbb{F}_a$.



The Picard group $\mathrm{Pic}(X)$ has basis D_1 and D_4, and $D_1 \sim D_3$, $D_4 \sim D_2 + aD_3$. Note that the projection onto the first component gives $\pi\colon X(\Delta) \to \mathbb{P}^1$ such that D_1 and D_3 are fibers of π. Moreover, we have $(D_2^2) = -a = -(D_4^2)$.

We see that $\mathrm{Eff}(X)$ is generated by D_1 and $D_4 - aD_1$. Moreover, one can check that given a divisor $D = \alpha D_1 + \beta D_2$, the condition that $(D \cdot D_i) \geq 0$ for all i is $\alpha, \beta \geq 0$.

If Σ is the complete fan whose rays are the rays in Δ through e_1, $-e_2$ and $-e_1 + ae_2$, then we have a proper birational morphism $f\colon X(\Delta) \to X(\Sigma)$. It is easy to check that if α, β are nonnegative integers, then the base locus of the linear system $|\alpha D_2 + \beta D_4|$ contains αD_2. Therefore we get the GKZ decomposition of the effective cone, where I consists of the ray through e_2:



Remark 7. A birational map between two varieties $f\colon X' \dashrightarrow X$ is an **isomorphism in codimension one** if there are open subsets U' and U of X' and X, respectively, such that f induces an isomorphism of U' and U, and $\mathrm{codim}(X' \smallsetminus U', X')$, $\mathrm{codim}(X \smallsetminus U, X) \geq 2$.

Suppose that X' and X are toric varieties corresponding to the fans Δ' and Δ in N' and N, respectively. A toric birational map f is induced by a lattice isomorphism $\phi\colon N' \to N$. We see that f is an isomorphism in codimension one if and only if $\phi(\Delta'(1)) = \Delta(1)$. For example, both morphisms π_1 and π_2 in Example 2 are isomorphisms in codimension one.

If f is an isomorphism in codimension one, then we have an induced isomorphism between $\mathrm{Cl}(X)_{\mathbb{Q}}$ and $\mathrm{Cl}(X')_{\mathbb{Q}}$. Indeed, if $U \subseteq X$ and $U' \subseteq X'$ are isomorphic open subsets via f and their complements have codimension ≥ 2, then we have

$$\mathrm{Cl}(X)_{\mathbb{Q}} = \mathrm{Cl}(U)_{\mathbb{Q}} \simeq \mathrm{Cl}(U')_{\mathbb{Q}} = \mathrm{Cl}(X')_{\mathbb{Q}}.$$

If $|\Delta|$ and $|\Delta'|$ are convex and n-dimensional, then $|\Delta| = |\Delta'|$ and the above isomorphism preserves the GKZ decomposition of the effective cones.

Remark 8. For every fan Δ such that $|\Delta|$ is convex and n-dimensional and $X(\Delta)$ is quasiprojective, we see that $\mathrm{Nef}(X(\Delta))$ is a cone in the GKZ decomposition of the effective cone of $X(\Delta)$. This is a maximal cone if and only if Δ is simplicial. More generally, for every simplicial quasiprojective $X(\Sigma)$ which is isomorphic to $X(\Delta)$ in codimension one and with $|\Delta| = |\Sigma|$, $\mathrm{Nef}(X(\Sigma))$ corresponds to a maximal cone in the GKZ decomposition of $\mathrm{Eff}(X(\Delta))$.

The GKZ decomposition on an affine variety. We consider now the case when $X = U_\sigma$, with σ a maximal cone in N. We claim that for every (Σ, I) in $\mathcal{S}$, we have $I = \emptyset$ and $\Sigma(1) = \Delta(1)$. Indeed, since $|\Sigma| = \sigma$ it follows that σ is generated as a convex cone by the rays in Σ. On the other hand, we know that $\Sigma(1)$ is a subset of the rays in σ that are not in I. Since the rays in σ give a minimal system of generators for σ, we deduce our claim.

It follows that in this case the GKZ decomposition is given by

$$\mathrm{Cl}(U_\sigma)_{\mathbb{Q}} = \bigcup GKZ_\Sigma,$$

where we put GKZ_Σ for $GKZ_{\Sigma,\emptyset}$ The union is over all fan refinements Σ of σ such that $\Sigma(1)$ is the set of rays in σ and $X(\Sigma)$ is quasiprojective. For every such Σ we have a proper morphism

$$g_\Sigma \colon X(\Sigma) \to U_\sigma$$

which is an isomorphism in codimension one. g_Σ induces an identification of $\mathrm{Cl}(U_\sigma)_{\mathbb{Q}}$ with $\mathrm{Cl}(X(\Sigma))_{\mathbb{Q}}$ such that GKZ_Σ is identified with $\mathrm{Nef}(X(\Sigma))$. The maximal cones in this decompositions correspond to simplicial fans Σ.

Consider now the GKZ decomposition of a given divisor D on U_σ. This consists of a morphism $g_\Sigma \colon X(\Sigma) \to U_\sigma$ as above and an ample divisor A on $X(\Sigma)$ such that $(g_\Sigma^{-1})^*(A) = D$.

Proposition 3. *Let $X = X(\Delta)$ be an arbitrary toric variety and D a $\mathbb{Q}$-T-divisor on X. Then there is a unique proper morphism of toric varieties $f \colon Y \to X$ which is an isomorphism in codimension one, such that the divisor corresponding to D on Y is $\mathbb{Q}$-Cartier and f-ample.*

Proof. Because of uniqueness, the assertion is local on X, so we may assume that $X = U_\sigma$ is affine, and we may also further assume that σ has maximal dimension. In this case the assertion follows from the existence and the uniqueness of a GKZ decomposition for D.

Recall that $f \colon Y \to X$ is projective if it is proper and there is an f-ample divisor on Y.

Corollary 1. *If $X = X(\Delta)$ is an arbitrary toric variety, then there is a projective morphism of toric varieties $f\colon Y \to X$ that is an isomorphism in codimension one and such that Y is simplicial.*

Proof. It is enough to apply the assertion in the proposition repeatedly to each prime divisor on X, and take f the composition of the resulting morphisms.

Remark 9. Given a toric variety $X = X(\Delta)$ with $|\Delta|$ convex and n-dimensional, it follows from Corollary 1 that in the GKZ decomposition of $\mathrm{Eff}(X)$ there is a maximal cone of the form $GKZ_{\Sigma,\emptyset}$, where Σ is a simplicial fan refinement of Δ such that $\Sigma(1) = \Delta(1)$ and $X(\Sigma)$ is quasiprojective.

Functions on the effective cone. We apply now the GKZ decomposition is useful to study how various asymptotic functions behave on the effective cone.

Suppose first that X is a complete n-dimensional toric variety. Recall that for every $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor divisor on X, the volume of D is given by $\mathrm{vol}(X, D) = n! \cdot \mathrm{vol}(P_D)$. We have the following

Proposition 4. *If X is a complete toric variety, then the volume function is polynomial of degree n on the intersection of $\mathrm{Pic}(X)_{\mathbb{Q}}$ with each of the cones in the GKZ decomposition of $\mathrm{Eff}(X)$.*

Proof. Since it is enough to restrict ourselves to the maximal cones in the GKZ decomposition, it is enough to consider a cone of the form $GKZ_{\Sigma,I}$, with Σ non-degenerate. If the class of D lies in this cone, and if $D = f_\Sigma^* A + F$ is the corresponding decomposition of D, we see that

$$\mathrm{vol}(X, D) = n! \cdot \mathrm{vol}_n(P_D) = n! \cdot \mathrm{vol}_n(P_A) = (A^n).$$

The last equality comes from the fact that if A is a nef divisor on a complete n-dimensional variety Y, then $\mathrm{vol}(Y, A) = (A^n)$ (see [Laz]). On a toric variety this follows from asymptotic Riemann-Roch and the fact that a nef line bundle has no higher cohomology.

Since the map $A \to (A^n)$ is polynomial of degree n on $\mathrm{Nef}(X(\Sigma))$, we deduce the assertion in the proposition.

Suppose now that $X = X(\Delta)$ is a toric variety, with $|\Delta|$ convex and of maximal dimension. We have seen that if D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor on X, then the support of the base locus of $|mD|$ is independent of m, if m is divisible enough (in fact, it is enough to have mD Cartier and mP_D the Minkowski sum of $|\Delta|^\vee$ and of a lattice polytope). This support is denoted by $B(D)$ and is called the asymptotic base locus of D.

Exercise 6. Show that if the class of the $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor D lies in $GKZ^{\circ}_{\Sigma,I}$, then $B(D)$ is the union of those D_i such that i is in I and of those $V(\tau)$ such that τ is not contained in a cone in Σ.

Recall that for a divisor D, we denote by $\mathfrak{b}_{|D|}$ the ideal defining the base locus of the complete linear system $|D|$.

Proposition 5. *If D_1 and D_2 are $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisors whose classes lie in the same cone in the GKZ decomposition of* $\mathrm{Eff}(X)$*, then $P_{D_1+D_2} = P_{D_1} + P_{D_2}$. In particular, if D_1 and D_2 are Cartier and P_{D_1} and P_{D_2} are Minkowski sums of corresponding lattice polytopes with $|\Delta|^{\vee}$, then the ideals $\mathfrak{b}_{|D_1+D_2|}$ and $\mathfrak{b}_{|D_1|} \cdot \mathfrak{b}_{|D_2|}$ have the same integral closure.*

Proof. Note that Exercise 16 in Chapter 6 gives the first assertion when both D_1 and D_2 are nef. Suppose now that D_1 and D_2 are in $GKZ_{\Sigma,I}$. If we have the corresponding decompositions

$$D_1 = f_{\Sigma}^{*}A_1 + F_1,\ D_2 = f_{\Sigma}^{*}A_2 + F_2,$$

with A_1 and A_2 in $\mathrm{Nef}(X(\Sigma))$, then we get a decomposition for D_1+D_2 $D_1 + D_2 = f_{\Sigma}^{*}(A_1 + A_2) + (F_1 + F_2)$. Therefore

$$P_{D_1+D_2} = P_{A_1+A_2} = P_{A_1} + P_{A_2} = P_{D_1} + P_{D_2}.$$

The last assertion in the proposition follows from the fact that if D is Cartier and $P_D = Q + |\Delta|^{\vee}$ for a lattice polytope Q, then the ideal defining the base locus of the linear system defined by the vertices of Q has the same integral closure as $\mathfrak{b}_{|D|}$ (see the proof of Proposition 2).

We give now an application to asymptotic orders of vanishing of line bundles. Before recalling the definition of these invariants, we introduce a notion that will be useful also later, when we will discuss singularities of pairs.

For an arbitrary variety X a **divisor over** X is a prime divisor E on a normal variety X' such that there is a birational morphism $\pi\colon X' \to X$. The **center** of this divisor on X is the closure of $\pi(E)$.

Every such divisor defines a valuation of the function field of X with discrete valuation ring $\mathcal{O}_{X',E}$. We identify two such divisors if they correspond to the same valuation. Geometrically, this means that we identify the divisor E on X' with the divisor F on X'' if there are birational morphisms $Z \to X'$ and $Z \to X''$ making the obvious diagram commute, and such that the proper transforms of E and F on Z are equal (and nonempty).

Suppose now that $X = X(\Delta)$ is a toric variety. A T-**divisor over** X is a divisor E as above such that π is a morphism of toric varieties. We see that we can identify an equivalence class of divisors as above with a primitive element v in $|\Delta| \cap N$. We denote by ord_v the corresponding valuation.

Exercise 7. Show that ord_v is given by

$$\mathrm{ord}_v(\sum_u a_u \chi^u) = \min_{a_u \neq 0} \langle u, v \rangle. \tag{2}$$

More generally, for every v in $N_{\mathbb{Q}}$ we consider the valuation of the function field of X given by (2). If $v = 0$, then ord_v is the zero valuation.

Suppose that $|\Delta|$ is convex and n-dimensional, and let us fix v in $|\Delta| \cap N_{\mathbb{Q}}$. We recall now the definition of the asymptotic invariant associated to the valuation ord_v from [ELMNP] (note, however, that we do not assume X to be complete). If b is a nonzero ideal of $\mathcal{O}_X$, we denote by $\mathrm{ord}_v(b)$ the order of the image of b in the discrete valuation ring corresponding to our valuation. If D is an effective Cartier divisor on X, then $\mathrm{ord}_v(|D|)$ is defined as $\mathrm{ord}_v(b_{|D|})$.

The asymptotic order of vanishing of D along v is

$$\mathrm{ord}_v(\| D \|) := \lim_{m\to\infty} \frac{\mathrm{ord}_v(|mD|)}{m} = \inf_m \frac{\mathrm{ord}_v(|mD|)}{m}.$$

It is clear that the definition depends only on the linear equivalence class of D, and it is easy to see that $\mathrm{ord}_v(\| qD \|) = q \cdot \mathrm{ord}_v(\| D \|)$. This shows that we can extend in a natural way the definition of the asymptotic order of vanishing to $\mathrm{Pic}(X)_{\mathbb{Q}}$.

If D is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor on X and if q is a positive integer such that qD is Cartier and $qP_D = Q + |\Delta|^{\vee}$ for a lattice polytope Q, then we have seen that $b_{|qmD|}$ and $b^m_{|qD|}$ have the same integral closure. If $\overline{b}$ is the integral closure of the ideal b, then $\mathrm{ord}_v(b) = \mathrm{ord}_v(\overline{b})$, so $\mathrm{ord}_v(\| D \|) = \frac{\mathrm{ord}_v(|qD|)}{q}$. Note that if v is nonzero, then $\mathrm{ord}_v(\| D \|) = 0$ if and only if the center of the divisor corresponding to the ray through v is not contained in $B(D)$.

Corollary 2. *The function* $\mathrm{ord}_v(\| \cdot \|)$ *is continuous on* $\mathrm{Pic}(X)_{\mathbb{Q}} \cap \mathrm{Eff}(X)$. *Moreover, it is linear on the intersection of each cone in the GKZ decomposition with* $\mathrm{Pic}(X)_{\mathbb{Q}}$.

Proof. It is enough to prove the second assertion. Note that for every sheaves of ideals a, $b \subseteq \mathcal{O}_X$ we have $\mathrm{ord}_v(a \cdot b) = \mathrm{ord}_v(a) + \mathrm{ord}_v(b)$. Given D_1 and D_2 whose classes lie in $GKZ_{\Sigma,I}$, let q be such that qD_1 and qD_2 are Cartier, and qP_{D_1}, qP_{D_2} are Minkowski sums of lattice polytopes with $|\Delta|^\vee$. We deduce using Proposition 5 that $qP_{D_1+D_2} = qP_{D_1} + qP_{D_2}$ and

$$\mathrm{ord}_v(\| D_1 + D_2 \|) = \frac{\mathrm{ord}_v(b_{|q(D_1+D_2)|})}{q} = \frac{\mathrm{ord}_v(b_{|qD_1|} \cdot b_{|qD_2|})}{q}$$

$$= \frac{\mathrm{ord}_v(b_{qD_1})}{q} + \frac{\mathrm{ord}_v(b_{qD_2})}{q} = \mathrm{ord}_v(\| D_1 \|) + \mathrm{ord}_v(\| D_2 \|).$$

Remark 10. If D_1 and D_2 are effective Cartier T-divisors, then $b_{|D_1|} \cdot b_{|D_2|} \subseteq b_{|D_1+D_2|}$. It is easy that for every v in $|\Delta|$, the asymptotic order of vanishing $-\mathrm{ord}_v(\| \cdot \|)$ is concave. It follows from Corollary 2 that if X is simplicial, then $-\mathrm{ord}_v(\| \cdot \|)$ defines a nef $\mathbb{Q}$-T-divisor on the toric variety X^{GKZ} corresponding to the GKZ decomposition of $\mathrm{Eff}(X)$.

Exercise 8. Suppose that v is in σ. If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor and $u(\sigma)$ is such that $\psi_D(w) = -\langle u(\sigma), w\rangle$ for w in σ, then

$$\mathrm{ord}_v(\| D \|) = \min_{u \in P_D} \langle u - u(\sigma), v\rangle.$$

Exercise 9. Show that if Y is a toric variety isomorphic with $X = X(\Delta)$ in codimension one and whose fan has support $|\Delta|$, and v is in $|\Delta| \cap N_{\mathbb{Q}}$, then the two divisors corresponding to the asymptotic orders of vanishing along v on X and Y, respectively, have the same class in the Picard group of $X^{GKZ} = Y^{GKZ}$.

Exercise 10. If $X = X(\Delta)$, show that the function

$$|\Delta| \cap N_{\mathbb{Q}} \to \mathrm{Nef}(X^{GKZ})$$

that takes v to the divisor class corresponding to $-\mathrm{ord}_v(\| \cdot \|)$ is piecewise linear on a suitable fan refining $|\Delta|$.

Gale duality and equations for the effective cone. Gale duality is a duality between maps from a given finite set to vector spaces. We give the definition and basic properties, and use this framework to discuss equations for the effective cone. For other applications, see [OP].

Let A be a finite set and W a finite dimensional vector space over $\mathbb{Q}$. Suppose that $i\colon A \to W$ is a map such that $i(A)$ spans W.

The **Gale dual** map $j\colon A \to V$ is the universal map such that $\sum_{a \in A} j(a) \otimes i(a) = 0$ in $V \otimes_{\mathbb{Q}} W$. We give an explicit description of j. Let $U = \oplus_{a \in A} \mathbb{Q} e_a$ be a vector space with the basis indexed by A, and consider the epimorphism $g\colon U \to W$ defined by $g(e_a) = i(a)$. Let V be the cokernel of g^*, so we have an exact sequence

$$0 \to W^* \xrightarrow{g^*} \oplus_{a\in A}\mathbb{Q}e_a^* \xrightarrow{f} V \to 0. \tag{3}$$

We define $j\colon A \to V$ by $j(a) = f(e_a^*)$. Note that

$$V = \oplus_{a\in A}\mathbb{Q}e_a^*/(\sum_{a\in A}\langle z, i(a)\rangle e_a^* \mid z \in W^*). \tag{4}$$

In $V \otimes_{\mathbb{Q}} W$ we have $\sum_{a\in A} j(a) \otimes i(a) = 0$. Indeed, it is enough to check that for every z in W^* we have $\sum_{a\in A}\langle z, i(a)\rangle j(a) = 0$ in V, which follows from (4).

Moreover, we see that j is universal: if $j'\colon A \to V'$ is such that $\sum_{a\in A} j'(a) \otimes i(a) = 0$ in $V' \otimes W$, then we have a morphism $h\colon \oplus_{a\in A} \mathbb{Q}e_a^* \to V'$ given by $h(e_a^*) = j'(a)$. If z is in W^*, then $\sum_{a\in A}\langle z, i(a)\rangle j'(a) = 0$, so h induces $\overline{h}\colon V \to V'$ such that $\overline{h} \circ j = j'$.

It follows from the universal property that j is unique up to the obvious notion of equivalence. By construction, we see that $j(A)$ spans V. If we dualize (3) we get

$$0 \to V^* \xrightarrow{f^*} \oplus_{a\in A}\mathbb{Q}e_a \xrightarrow{g} W \to 0. \tag{5}$$

This shows that i is the Gale dual of j, so we have a duality between maps from A to finite dimensional vector spaces, whose image spans the corresponding vector space. Moreover, note that (3) gives $\dim(V) + \dim(W) = |A|$.

Example 5. Our main example is the following. Suppose that $X = X(\Delta)$, where Δ is a fan in $N_{\mathbb{R}}$ that is not contained in any proper linear subspace. If $v_1, \ldots, v_d$ are the primitive generators of the rays in Δ and $A = \{1, \ldots, d\}$, take $i\colon A \to N_{\mathbb{Q}}$ given by $i(m) = v_m$. The Gale dual map is $j\colon A \to \mathrm{Cl}(X)_{\mathbb{Q}}$, such that $j(m)$ is the class $[D_m]$ of D_m in $\mathrm{Cl}(X)_{\mathbb{Q}}$. Indeed, this follows from the exact sequence

$$0 \to M_{\mathbb{Q}} \to \mathbb{Q}^d \to \mathrm{Cl}(X)_{\mathbb{Q}} \to 0.$$

We use this formalism to describe equations for the effective cone. More generally, suppose that $i\colon A \to W$ and $j\colon A \to V$ are Gale dual maps. We want to describe the convex cone $\mathrm{pos}(i(A))$ generated by $i(A)$. It follows from (3) that

$$g^*(i(A)^\vee) = \{\sum_{a\in A} \lambda_a e_a^* \mid \lambda \in (\mathbb{Q}_+)^{|A|}, \sum_{a\in A} \lambda_a j(a) = 0\}. \tag{6}$$

We deduce the following

Lemma 3. *The sum $\sum_{a\in A} \alpha_a i(a)$ lies in $\mathrm{pos}(i(A))$ if and only if for every $(\beta_a)_a$ in $\mathbb{N}^{|A|}$ such that $\sum_{a\in A} \beta_a j(a) = 0$, we have $\sum_{a\in A} \alpha_a\beta_a \geq 0$.*

In particular, we get the following description of effective divisors.

Corollary 3. *If $|\Delta|$ is a fan in $N_{\mathbb{R}}$ that does not lie in any proper subspace, and if $v_1, \ldots, v_d$ are the primitive generators of the rays in Δ, then the class of the divisor $\sum_{i=1}^{d} a_i D_i$ in $\mathrm{Cl}(X(\Delta))_{\mathbb{Q}}$ lies in the effective cone if and only if for every $(b_i)_i$ in $\mathbb{N}^d$ such that $\sum_{i=1}^{d} b_i v_i = 0$, we have $\sum_{i=1}^{d} a_i b_i \geq 0$.*

Lemma 4. *If $i\colon A \to W$ and $j\colon A \to V$ are Gale dual maps, then $\mathrm{pos}(j(A))$ is a pointed cone if and only if $\mathrm{pos}(i(A)) = W$.*

Proof. We have $\mathrm{pos}(i(A)) = W$ if and only if $\mathrm{pos}(i(A))^{\vee} = \{0\}$. By (6), this is equivalent with the fact that if λ_a is in $\mathbb{Q}_+$ for every a in A, and $\sum_{a \in A} \lambda_a j(a) = 0$, then all λ_a are zero. It is easy to see that this is equivalent with the fact that $\mathrm{pos}(j(A))$ is zero.

Corollary 4. *Let $X = X(\Delta)$ be a toric variety such that $|\Delta|$ is convex and n-dimensional. The effective cone $\mathrm{Eff}(X)$ is a pointed cone if and only if $|\Delta| = N_{\mathbb{R}}$, and $\mathrm{Eff}(X) = \mathrm{Cl}(X)_{\mathbb{Q}}$ if and only if $|\Delta|$ is a pointed cone.*

We use Corollary 3 to prove in the toric setting the description of the effective cone of line bundles from [BDPP]. The argument we present below is due to Sam Payne.

Let $X = X(\Delta)$ with $|\Delta|$ convex and of maximal dimension. We denote by X_0 the union of the U_{τ_i}, where τ_i varies over the rays in Δ. Therefore X_0 is smooth and $\mathrm{codim}(X \smallsetminus X_0, X) \geq 2$. If $C \subseteq X$ is a complete curve that is contained in X_0, we can define a linear map $\mathrm{Cl}(X)_{\mathbb{Q}} \to \mathbb{Q}$ that maps D to $(D \cdot C)$ (note that $D|_{X_0}$ is $\mathbb{Q}$-Cartier).

Exercise 11. Let $X = X(\Delta)$ and let v_i be the primitive generator of the ray τ_i corresponding to the T-divisor D_i. We know that the one-parameter subgroup f_{v_i} corresponding to v_i admits an extension $\widetilde{f}_{v_i}\colon \mathbb{A}^1 \to X$ such that $\widetilde{f}_{v_i}(0) = x_{\tau_i}$. Show that we have $\widetilde{f}_{v_i}^{*}(D_i) = \{0\}$.

Exercise 12. Let $X = X(\Delta)$ and let $v_1, \ldots, v_d$ be the primitive ray generators of Δ, corresponding to the prime T-divisors $D_1, \ldots, D_d$. Given nonnegative integers $b_1, \ldots, b_d$ such that $\sum_{i=1}^{d} b_i v_i = 0$, choose $\lambda_{i,1}, \ldots, \lambda_{i,b_i}$ in k^* for $1 \leq i \leq d$, all distinct. Define

$$f\colon \mathbb{A}^1 \smallsetminus \{\lambda_{i,j} \mid 1 \leq i \leq d, 1 \leq j \leq b_i\} \to T_N,\ f(t) = \prod_{i=1}^{d} \prod_{j=1}^{b_i} f_{v_i}(t - \lambda_{i,j}).$$

Show that f admits an extension $\widetilde{f}\colon \mathbb{P}^1 \to X_0$ such that $\widetilde{f}(\lambda_{i,j})$ is in O_{τ_i} for every i and j, $\widetilde{f}(0)$ is in T_N and $\deg(\widetilde{f}^{*} D_i) = b_i$ for all i. Note that the condition $\sum_i b_i v_i = 0$ is necessary in order to have a map $\widetilde{f}\colon \mathbb{P}^1 \to X_0$ with $\deg(f^*D_i) = b_i$ for all i.

Proposition 6. *If α is in $\mathrm{Cl}(X)_{\mathbb{Q}}$, then α is in $\mathrm{Eff}(X)$ if and only if $(\alpha \cdot C) \geq 0$ for every complete rational curve $C \subseteq X_0$ such that $C \cap T_N$ is nonempty.*

Proof. Represent α by a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor $D = \sum_{i=1}^{d} a_i D_i$ on X. It is clear that if D is effective and C is a curve which intersects the torus, then C is not contained in the support of D, so $(D \cdot C) \geq 0$.

Conversely, suppose that $(D{\cdot}C)$ for every curve as in the proposition. It follows from Exercise 12 that given nonnegative integers $b_1, \ldots, b_d$ with $\sum_{i=1}^{d} b_i v_i = 0$ there is $f\colon \mathbb{P}^1 \to X_0$ whose image intersects the torus, such that $\deg(f^*\mathcal{O}(D_i)) = b_i$ for every i. By hypothesis we have $(D{\cdot}f(C)) \geq 0$, so $(f^*D{\cdot}C) = \sum_{i=1}^{d} a_i b_i \geq 0$. It follows from Corollary 3 that the class α of D is in $\mathrm{Eff}(X)$, which completes the proof.

Remark 11. For a complete variety X, a **movable** curve in X is an irreducible complete curve $C \subseteq X$ which is the fiber of a family of curves in X that dominates X. If $X = X(\Delta)$ is a toric variety, then the set of classes of movable curves in $N_1(X)_{\mathbb{Q}}$ is the set of classes of curves C in X which intersect the torus T_N. The main result in [BDPP] shows that for a smooth variety X, the closure of the big cone is the dual of the closed convex cone generated by the classes of movable curves. We see that Proposition 6 gives a slight generalization of this statement in the toric setting.

Toric Mori theory. In general, the goal of the Minimal Model Program (MMP, for short) is to perform birational transformations on a variety X in order to make the canonical class nef, whenever this is possible. If this can not be achieved, then the theory gives fibrations of X, reducing therefore the study of X to the study of lower dimensional varieties.

In the toric setting there are several special features. First, the MMP can be carried out with respect to an arbitrary divisor, not necessarily the canonical class (in fact, it is particularly relevant to consider general divisors, since for example if X is complete, then K_X is nef if and only if X is a point). The main results of the theory that are known in arbitrary dimension (the Cone Theorem and the Contraction Theorem) follow easily for toric varieties. Moreover, the main open problems in the general case (Existence of Flips and Termination of Flips) can be settled in the toric setup. In fact, the solution of this problems for toric varieties in [Re] provided a useful model in the general development of the theory.

For arbitrary varieties, it is essential to study singularities of pairs in order to make the MMP work. Since the Cone Theorem and the Contraction Theorem are valid on arbitrary toric varieties, in this setting it is not necessary to look at singularities of pairs. However, we will see that these singularities can also be included in the picture, and in fact, the standard types of singularities admit nice toric descriptions.

We describe now the MMP for toric varieties. Let $X = X(\Delta)$ be a quasiprojective toric variety such that $|\Delta|$ is convex and of maximal dimension n. We fix a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor D on X. Our goal is "to make D nef".

Recall that the Mori cone $\mathrm{NE}(X)_{\mathbb{Q}}$ is a rational polyhedral cone, and all its rays are generated by classes of complete invariant curves (this is a strong version, in the toric setting, of the result known in general as the Cone Theorem). Moreover, given a face F of $\mathrm{NE}(X)_{\mathbb{Q}}$, there is a unique fiber space $\mathrm{cont}_F \colon X \to Y$ such that Y is quasiprojective and a curve is contracted by cont_F if and only if its class belongs to F (this is the toric version of the Contraction Theorem).

We will consider special such fiber spaces, corresponding to a ray R of the Mori cone (note that what we call "ray" is sometimes called "extremal ray" in the literature). Recall that we want to make D nef. If D is not nef, then there is a ray R of the Mori cone such that $(D \cdot R) < 0$. We consider the corresponding contraction $f = \mathrm{cont}_R$ (note that starting with the contraction of some ray R, we may always find D that is negative on that ray: simply take $D = -A$, where A is an ample divisor). We see that $-D$ is f-ample.

There are three cases to consider, which we will investigate separately:

(i) f is birational and there is a divisor E on X that is contracted by f, i.e. $\dim(f(E)) < \dim(E)$. One says that f is a **divisorial contraction**.
(ii) f is birational and an isomorphism in codimension one. In this case f is called a **flipping contraction**.
(iii) $\dim(Y) < \dim(X)$, in which case f is called a **Mori fiber space**.

We start with a general lemma describing the Picard group of Y.

Lemma 5. *If $f \colon X \to Y$ is the contraction of a ray R in the Mori cone, and if C is a complete curve whose class lies in R, then we have* *an exact sequence*

$$0 \to \mathrm{Pic}(Y)_{\mathbb{Q}} \xrightarrow{f^*} \mathrm{Pic}(X)_{\mathbb{Q}} \xrightarrow{p} \mathbb{Q} \longrightarrow 0,$$

where $p(L) = (L \cdot C)$.

Proof. Denote by q the surjective lattice morphism corresponding to f. The pull-back $f^*(M)$ is linearly equivalent to zero if and only if $\psi_M \circ q$ is linear. This implies that ψ_M is linear, hence f^* is injective. We have seen that $\mathrm{Nef}(X) \cap R^{\perp}$ is equal to $f^*(\mathrm{Nef}(Y))$, so $R^{\perp} = f^*(\mathrm{Pic}(Y)_{\mathbb{Q}})$. The fact that p is surjective follows from the existence of a divisor D on X with $(D \cdot C) < 0$.

Exercise 13. Let $f\colon X \to Y$ be a morphism of toric varieties that is a fiber space such that Y is quasiprojective. Show that if $\mathrm{rk}(\mathrm{Pic}(Y)) = \mathrm{rk}(\mathrm{Pic}(X)) - 1$, then f is the contraction of a ray in the Mori cone.

For future reference we give a toric version of the Negativity Lemma. Recall first that if $f\colon X \to Y$ is a proper morphism, then a line bundle $\mathcal{L}$ on X is called f**-nef** if for every y in Y, the restriction of L to the fiber $f^{-1}(y)$ is nef. Similarly, $\mathcal{L}$ is f**-trivial** if for every y in Y the restriction of L to $f^{-1}(y)$ is numerically trivial.

Exercise 14. Show that if $F\colon X \to Y$ is a proper morphism of toric varieties and $\mathcal{L}$ is a line bundle on X, then the following are equivalent:

i) For every (invariant) irreducible complete curve C on X such that $f(C)$ is a point, we have $(L \cdot C) \geq 0$.
ii) For every σ in the fan of Y, the restriction of L to $f^{-1}(U_\sigma)$ is nef.
iii) L is f-nef.

Exercise 15. Under the assumptions in the previous exercise, show that the following are equivalent:

i) For every (invariant) irreducible curve C on X such that $f(C)$ is a point, we have $(L \cdot C) = 0$.
ii) For every σ in Δ, the restriction of L to $f^{-1}(U_\sigma)$ is isomorphic to $\mathcal{O}_{U_\sigma}$.
iii) L is f-trivial.

Lemma 6. *Let $f\colon X \to Y$ be a proper, birational morphism of toric varieties. If D is an f-nef $\mathbb{Q}$-T-divisor on X and $f_*D \leq 0$, then $D \leq 0$. In particular, if D is f-trivial and supported on the exceptional locus (i.e. each component of D is contracted by f), then $D = 0$.*

Proof. We may clearly assume that $Y = U_\sigma$. If $v_1, \ldots, v_d$ are the primitive generators of the rays of σ, then $f_*D \leq 0$ says that $\psi_D(v_i) \geq 0$ for every i. We have D nef, so ψ_D is concave and therefore $\psi_D \geq 0$ on $|\Delta|$.

This shows that $D \leq 0$. The last assertion follows since D exceptional implies $f_*D = 0$.

Proposition 7. *Let $f\colon X = X(\Delta) \to Y$ be a divisorial contraction of a ray R and let D be a $\mathbb{Q}$-Cartier T-divisor on X such that $(D\cdot R) < 0$.*

(i) *There is a unique prime divisor E on X that is contracted by f, $E = f^{-1}(f(E))$ and f is an isomorphism over $Y \smallsetminus f(E)$. Moreover, $f_*(D)$ is $\mathbb{Q}$-Cartier and there is $a \in \mathbb{Q}_+^*$ such that $D = f^*f_*(D) + aE$. We have $\mathrm{rk}(\mathrm{Cl}(Y)) = \mathrm{rk}(\mathrm{Cl}(X)) - 1$.*

(ii) *If X is simplicial, then so is Y. Moreover, if Σ is the fan defining Y and v is the primitive generator of the ray corresponding to E, then Δ is equal to the star division $\Sigma(v)$.*

Proof. By assumption, there is a prime T-divisor E that is contracted by f. Lemma 5 implies that we can write $E \sim_{\mathbb{Q}} f^*E_1 + \lambda D$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor E_1 on Y and some $\lambda \in \mathbb{Q}$. Since $f_*E = 0$, we see that $E_1 = -\lambda \cdot f_*D$. If $\lambda = 0$, then E is f-trivial and exceptional, which contradicts Lemma 6. Hence λ is nonzero and we deduce that f_*D is $\mathbb{Q}$-Cartier. Moreover, we can write $D = f^*f_*D + aE$ for some nonzero a.

In fact, a is positive. For this it is enough to show that $(E \cdot R) < 0$. Let A be ample on X. The previous argument shows that we can write $A = f^*f_*A + bE$. If b is nonnegative, then for every $m \geq 1$ divisible enough, we have

$$H^0(X, \mathcal{O}(mA)) = H^0(Y, \mathcal{O}(mf_*A)),$$

using the fact that E is exceptional. This contradicts the fact that A is ample. Hence $b < 0$, and therefore $(E \cdot R) < 0$.

Since $(E\cdot R) < 0$, if C is a curve contracted by f, then C is contained in E. The fibers of f are connected, so we deduce that $E = f^{-1}(f(E))$. Moreover, f is a bijection over $Y \smallsetminus f(E)$ and since f is a fiber space it follows that f is an isomorphism over this open subset.

In particular, this shows that E is the only prime divisor on X that is contracted by f. Proposition 1 in Chapter 6 implies that $\mathrm{rk}(\mathrm{Cl}(Y)) = \mathrm{rk}(\mathrm{Cl}(X)) - 1$. If X is simplicial, we deduce from this and Lemma 5 that Y is simplicial, too.

For the last assertion, it is enough to check that every cone in $\Sigma(v)$ is in Δ. This follows easily from the fact that Y is simplicial, so every cone in Δ that is not in Σ contains v.

Exercise 16. Let $X = X(\Delta)$ be a quasiprojective simplicial toric variety, with $|\Delta|$ convex and of maximal dimension. If v is a primitive lattice vector in $|\Delta|$ that does not lie on the rays of Δ, show that the star division $\Delta(v)$ is a simplicial fan such that $X(\Delta(v))$ is quasiprojective and the induced morphism $X(\Delta(v)) \to X$ is a divisorial contraction.

Proposition 8. *Let $f\colon X = X(\Delta) \to Y$ be a flipping contraction of a ray R and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor on X such that $(D\cdot R) < 0$.*

(i) *The divisor $f_*(D)$ is not $\mathbb{Q}$-Cartier. There is a unique morphism $f^+\colon X^+ \to Y$ that is an isomorphism in codimension one, such that there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor D^+ on X^+ with $f_*(D^+) = f_*(D)$ and such that D^+ is f^+-ample.*

(ii) *If X is simplicial, then so is X^+, and f^+ is again a flipping contraction.*

Proof. If f_*D is $\mathbb{Q}$-Cartier, since f is an isomorphism in codimension one, we have $D = f^*f_*(D)$. We deduce $(D \cdot C) = 0$ for every complete curve C contracted by f, a contradiction. The second assertion in (i) follows from Proposition 3.

It follows from Lemma 5 that $\mathrm{rk}(\mathrm{Pic}(Y)) = \mathrm{rk}(\mathrm{Pic}(X)) - 1$, while $\mathrm{Cl}(X) \simeq \mathrm{Cl}(Y)$ since f is an isomorphism in codimension one. Since f^+ is also an isomorphism in codimension one, and D^+ is a $\mathbb{Q}$-Cartier divisor such that $f^+_*(D^+)$ is not $\mathbb{Q}$-Cartier, we have

$$\mathrm{rk}(\mathrm{Pic}(X^+)) \geq \mathrm{rk}(\mathrm{Pic}(Y)) + 1,\ \mathrm{Cl}(X^+) \simeq \mathrm{Cl}(Y).$$

If X is simplicial we deduce $\mathrm{rk}(\mathrm{Pic}(X^+)) \geq \mathrm{rk}(\mathrm{Cl}(X^+))$, so X^+ is simplicial. Moreover, by Exercise 13 f^+ is the contraction of a ray, hence it is a flipping contraction.

Remark 12. With the notation in the above proposition, the birational map $\phi = (f^+)^{-1} \circ f$ is called an **elementary transformation** with respect to D, or a D-flip. Since Y is quasiprojective and D^+ is f^+-ample, X^+ is quasiprojective. Note that if X is simplicial, then ϕ^{-1} is a $(-D^+)$-flip.

Exercise 17. Show that if $f\colon X \to Y$ is a flipping contraction of a ray R, then f^+ does not depend on the divisor D such that $(D \cdot R) < 0$.

The above proposition gives “existence of flips” in the toric setting. The next result shows “termination of flips”.

Proposition 9. *If f and f^+ are as in Proposition 8, then $\psi_{D^+} \geq \psi_D$ on $|\Delta|$ and $\psi_D \neq \psi_{D^+}$. Therefore there is no infinite sequence $\phi_m\colon X_m \dashrightarrow X_{m+1}$ for $m \in \mathbb{N}$ such that we have on each X_m a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor D_m with ϕ_m is a D_m-flip and $D_{m+1} = D_m^+$.*

Proof. Let $v_1, \ldots, v_d$ denote the primitive generators of the rays in the fan of X, which are the same as the rays in the fan of X^+. Since D^+ is the proper transform of D, we have $\psi_D(v_i) = \psi_{D^+}(v_i)$ for every i. On the other hand, $-D$ is f-ample and D^+ is f^+-ample, so $-\psi_D$ and ψ_{D^+} are concave. We deduce that if $t_1, \ldots, t_d$ are in $\mathbb{Q}_+$, then

$$\psi_D(\sum_i t_i v_i) \leq \sum_i t_i \psi_D(v_i) = \sum_i t_i \psi_{D^+}(v_i) \leq \psi_{D^+}(\sum_i t_i v_i),$$

which gives $\psi_D \leq \psi_{D^+}$. If $\psi_D = \psi_{D^+}$, then $X = X^+$, as the corresponding fans are given by the maximal domains of linearity for this function. This is clearly impossible by construction.

We deduce the last assertion from the fact that there are only finitely many fans with a given set of rays.

This gives the MMP in the toric context.

Corollary 5. *Let $X = X(\Delta)$ be a quasiprojective toric variety with $|\Delta|$ convex and of maximal dimension, and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor. There is a sequence*

$$X = X_0 \overset{f_1}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \ldots \overset{f_m}{\dashrightarrow} X_m$$

and divisors D_m on X_m such that

1. $D_0 = D$.
2. *Every f_m is either a divisorial contraction of a D_{m-1}-negative ray, or a D_{m-1}-flip. In the first case, $D_m = (f_m)_*(D_{m-1})$ and in the second case D_m is the proper transform of D_{m-1} via f_m,*

and we have one of the following two cases

Case 1. *D_m is nef.*

Case 2. *Every contraction of a D_m-negative ray in the Mori cone of X_m is a Mori fiber space.*

Moreover, if X is simplicial, then so is X_m.

Proof. Each divisorial contraction lowers the rank of the class group by one, while every flip preserves this rank. Moreover, Proposition 9 shows that we can't have an infinite sequence of flips, so we deduce the assertion in the corollary.

Corollary 6. *With the notation in Corollary 5, if D is effective, then the divisor D_m on X_m is nef.*

Proof. It is clear that each divisor D_m obtained via a divisorial contraction or a flip is again effective. Therefore it is enough to show that if D is effective and R is a ray in $\mathrm{NE}(X)_{\mathbb{Q}}$ such that $(D \cdot R) < 0$, then the corresponding contraction f is not a Mori fiber space. Indeed, if $\dim(f(X)) < \dim(X)$ and x is a point not in the support of D, then an irreducible curve C contained in $f^{-1}(f(x))$ is not contained in the support of D, so $(D \cdot C) \geq 0$, a contradiction.

Exercise 18. Show that if $f\colon X \to Y$ is a birational morphism, then $f_*(K_X) = K_Y$. If X and Y are isomorphic in codimension one, then K_X and K_Y correspond by the natural isomorphism $\mathrm{Cl}(X) \simeq \mathrm{Cl}(Y)$.

Remark 13. The usual notation is to run the MMP for $K_X + D$ for a suitable divisor D (the most important case being $D = 0$). Note that if $f\colon X \to Y$ is a correponding divisorial contraction, then we have $f_*(K_X + D) = K_Y + f_*(D)$. If $\phi\colon X \dashrightarrow X^+$ is a $(K_X + D)$-flip, then the corresponding divisor on X^+ is $K_{X^+} + D^+$, where we denote by D^+ the divisor corresponding to D on X^+. As a matter of terminology, a K_X-flip is simply called flip, while a $K_X + D$-flip corresponding to the contraction of a ray R such that $(K_X \cdot R) = 0$ is called a **flop** (for this definition, we need X to be simplicial, or at least $\mathbb{Q}$-Gorenstein).

Exercise 19. Show that if X is simplicial and $f\colon X \dashrightarrow X^+$ is a flop, then so is f^{-1}.

We turn now to the third type of ray contraction. For simplicity, we restrict to the case when the total space is simplicial. Let $f\colon X \to Y$ be a Mori fiber space, where $X = X(\Delta)$ is a simplicial toric variety, with $|\Delta|$ convex and of maximal dimension. We denote by Σ the non-degenerate fan defining Y, and by L the kernel of the surjective lattice morphism corresponding to f.

Proposition 10. *With the above notation, let d be the number of rays in Δ and ℓ the number of those rays that lie in L.*

(i) *Y is simplicial,* $\mathrm{rk}(L) = \ell - 1$ *and Σ has $d - \ell$ rays.*
(ii) *If σ is a maximal cone in Σ, then $f^{-1}(U_\sigma) \to U_\sigma$ is a Mori fiber space. If Y is affine, there is a finite surjective morphism of toric varieties (étale in characteristic zero) $g\colon Z \to X$ such that Z is isomorphic over Y, as a toric variety, with $Y \times \mathbb{P}^{\ell-1}$.*
(iii) *X is determined uniquely by Y, the lattice of Δ and $\Delta(1)$.*

Proof. It follows from Lemma 12 in Chapter 2 that $L_{\mathbb{Q}}$ is generated as a convex cone by the rays in Δ that lie in L. Therefore $\ell \geq \mathrm{rk}(L) + 1$. We have the following sequence of inequalities

$$\mathrm{rk}(\mathrm{Pic}(Y)) \leq \mathrm{rk}(\mathrm{Cl}(Y)) = \#\Sigma(1) - n + \mathrm{rk}(L) \leq d - \ell - n + \mathrm{rk}(L)$$
$$\leq d - n - 1 = \mathrm{rk}(\mathrm{Cl}(X)) - 1 = \mathrm{rk}(\mathrm{Pic}(X)) - 1,$$

where we used the fact that X is simplicial. As $\mathrm{rk}(\mathrm{Pic}(Y)) = \mathrm{rk}(\mathrm{Pic}(X)) - 1$ by Lemma 5, it follows that in the above sequence all inequalities are equalities. Therefore Y is simplicial, $\#\Sigma(1) = d - \ell$ and $\mathrm{rk}(L) = \ell - 1$.

If σ is a maximal cone in Σ, in order to show that $f^{-1}(U_\sigma) \to U_\sigma$ is a Mori fiber space, it is enough to show that it is a ray contraction. To see this, note that if C_1 and C_2 are two complete irreducible curves in $f^{-1}(U_\sigma)$ that are contracted by f, then their classes in $\mathrm{N}_1(X)_{\mathbb{Q}}$ are linearly dependent. This implies that their classes in $\mathrm{N}_1(f^{-1}(U_\sigma))_{\mathbb{Q}}$ are linearly dependent since the natural map $\mathrm{N}_1(f^{-1}(U_\sigma))_{\mathbb{Q}} \to \mathrm{N}_1(X)_{\mathbb{Q}}$ is injective. This in turn is a consequence of the fact that X is simplicial, so the restriction map

$$\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{Cl}(X)_{\mathbb{Q}} \to \mathrm{Pic}(f^{-1}(U_\sigma))_{\mathbb{Q}} = \mathrm{Cl}(f^{-1}(U_\sigma))_{\mathbb{Q}}$$

is surjective.

In order to prove the remaining assertion in (ii) and (iii), we may assume that $Y = U_\sigma$. Suppose that out of the primitive generators $v_1, \ldots, v_d$ of the rays in Δ, $v_1, \ldots, v_\ell$ lie in L. It follows from (i) that the rays of σ are generated by the classes $\overline{v}_{\ell+1}, \ldots, \overline{v}_d$ modulo L. Moreover, we see that if $n = \dim(X)$, then $n = (d-\ell) + (\ell - 1) = d-1$.

Since $v_1, \ldots, v_\ell$ generate $L_{\mathbb{Q}}$ as a convex cone, we see that there are positive integers $a_1, \ldots, a_\ell$ such that $a_1 v_1 + \ldots + a_\ell v_\ell = 0$.

Note that $v_1, \ldots, v_\ell$ can't all belong to the same cone in Δ. It follows that if τ is a maximal such cone, then there is $i \leq \ell$ such that

$$\tau = \tau_i := \mathrm{pos}\{v_1, \ldots, \widehat{v_i} \ldots, v_\ell, v_{\ell+1}, \ldots, v_d\}.$$

Since $L_{\mathbb{Q}}$ is contained in $|\Delta|$, it follows that the maximal cones in Δ are given by $\tau_1, \ldots, \tau_\ell$. This shows that Δ is determined by $v_1, \ldots, v_d$ and σ, so we get (iii).

If $\widetilde{N}$ is the lattice generated by $a_1v_1, \ldots, a_\ell v_\ell, v_{\ell+1}, \ldots, v_d$, and if $\widetilde{\Delta}$ is the fan Δ considered in this lattice, then it follows from the above description of Δ that $g\colon Z = X(\widetilde{\Delta}) \to X$ satisfies the requirements in (ii).

Exercise 20. Use the toric MMP to classify smooth projective toric surfaces, as follows. Let X be such a surface.

- (i) Show that if $\mathrm{rk}(\mathrm{Pic}(X)) = 1$, then X is isomorphic to $\mathbb{P}^2$.
- (ii) Suppose now that $\mathrm{rank}(\mathrm{Pic}(X)) \geq 2$, and let R be a K_X-negative ray in the Mori cone of X. Show that if D_i is an invariant divisor whose class lies in R, then $(D_i^2) \geq -1$.
- (iii) Let $f_R\colon X \to Y$ be the corresponding contraction. Show that if f_R is a divisorial contraction, then Y is smooth and f_R is the blowing-up of a fixed point in Y, with exceptional divisor D_i.
- (iv) Show that if f_R is not a divisorial contraction, then it is a Mori fiber space. In this case $Y = \mathbb{P}^1$, and X is isomorphic over Y with one of the Hirzebruch surfaces $\mathbb{F}_a$ for $a \in \mathbb{N}$.

Exercise 21. Show that if X is a smooth complete surface with invariant divisors $D_1, \ldots, D_d$, then $\sum_{i=1}^{d}(D_i^2) = 12 - 3d$. Deduce that $(K_X^2) = 12 - d$.

The walls in the GKZ decomposition. Let $X = X(\Delta)$ be a toric variety, with $|\Delta|$ convex and of maximal dimension n. If Δ has d rays, then $\dim(\mathrm{Eff}(X)) = d - n$.

We use our analysis of ray contractions to describe the walls in the GKZ decomposition

$$\mathrm{Eff}(X) = \bigcup_{\Sigma,I} GKZ_{\Sigma,I}$$

(a **wall** is a cone in the GKZ fan of dimension $d - n - 1$). Recall that $GKZ_{\Sigma,I}$ is a maximal cone if and only if Σ is a nondegenerate simplicial fan with $|\Sigma| = |\Delta|$, $\Sigma(1) \subseteq \Delta(1)$, $X(\Sigma)$ quasiprojective, and $I = \Delta(1) \smallsetminus \Sigma(1)$.

Given such a a maximal cone, a face of $GKZ_{\Sigma,I}$ is of the form $GKZ_{\Sigma',I'}$, where $X(\Sigma) \to X(\Sigma')$ corresponds to the contraction of a face, and $I' \subseteq I$. Using Exercise 13, we see that this is a facet if and only if one of the following holds:

(i) $\Sigma' = \Sigma$ and $|I \smallsetminus I'| = 1$.
(ii) $I = I'$ and $X(\Sigma) \to X(\Sigma')$ is the contraction of a ray.

The following proposition describes the walls in the GKZ decomposition in terms of the different types of ray contractions. We call a wall **exterior** if there is only one maximal cone in the GKZ decomposition having this wall as a facet; otherwise we call it **interior**.

Proposition 11. *Let $X = X(\Delta)$ be a toric variety with $|\Delta|$ convex and of maximal dimension. Suppose that $GKZ_{\Sigma,I}$ is a maximal cone in the GKZ decomposition of* $\mathrm{Eff}(X)$.

(1) *If $f\colon X(\Sigma) \to X(\Sigma')$ is a divisorial contraction with exceptional divisor corresponding to the ray ρ of Σ, then $GKZ_{\Sigma',I\cup\{\rho\}}$ is again a maximal cone whose intersection with $GKZ_{\Sigma,I}$ is the wall $GKZ_{\Sigma',I}$.*
(1') *Given a ray ρ in I with primitive generator v, the star division $\Sigma(v)$ gives a maximal cone $GKZ_{\Sigma(v),I\smallsetminus\{\rho\}}$ in the GKZ decomposition, and its intersection with the cone $GKZ_{\Sigma,I}$ is the wall $GKZ_{\Sigma,I\smallsetminus\{\rho\}}$.*
(2) *If $f\colon X(\Sigma) \to X(\Sigma')$ is a flipping contraction and $f^+\colon X(\Sigma^+) \to X(\Sigma)$ induces the corresponding elementary transformation, then $GKZ_{\Sigma^+,I}$ is a maximal cone in the GKZ decomposition whose intersection with $GKZ_{\Sigma,I}$ is the wall $GKZ_{\Sigma',I}$.*
(3) *If $f\colon X(\Sigma) \to X(\Sigma')$ is a Mori fiber space, then $GKZ_{\Sigma',I}$ is an exterior wall.*

Moreover, every interior wall in the GKZ decomposition arises as in (1) (*equivalently, as in* (1')) *or* (2)*, and every exterior wall appears as in* (3).

Proof. If f is a divisorial contraction, we have seen that $X(\Sigma')$ is a simplicial quasiprojective variety and ρ is the only ray in Σ and not in Σ', so $GKZ_{\Sigma',I\cup\{\rho\}}$ is a maximal cone. We clearly have

$$GKZ_{\Sigma',I} \subseteq GKZ_{\Sigma,I} \cap GKZ_{\Sigma',I\cup\{\rho\}},$$

and since the GKZ decomposition gives a fan, we see that this inclusion has to be an equality by dimension considerations. This proves (1).

In the setting of (1'), it follows from Exercise 16 that $X(\Sigma(v)) \to X(\Sigma)$ is a divisorial contraction and $GKZ_{\Sigma(v),I\smallsetminus\{\rho\}}$ is a maximal cone in the GKZ decomposition. The last assertion in (1') now follows from (1).

If f is a flipping contraction, we have seen that $X(\Sigma^+)$ is a quasiprojective simplicial variety, isomorphic to $X(\Sigma)$ in codimension one. This implies that $GKZ_{\Sigma,I}$ is a maximal cone in the GKZ decomposition. Moreover, we clearly have the inclusion

$$GKZ_{\Sigma',I} \subseteq GKZ_{\Sigma,I} \cap GKZ_{\Sigma^+,I},$$

and this has to be an equality by dimension considerations.

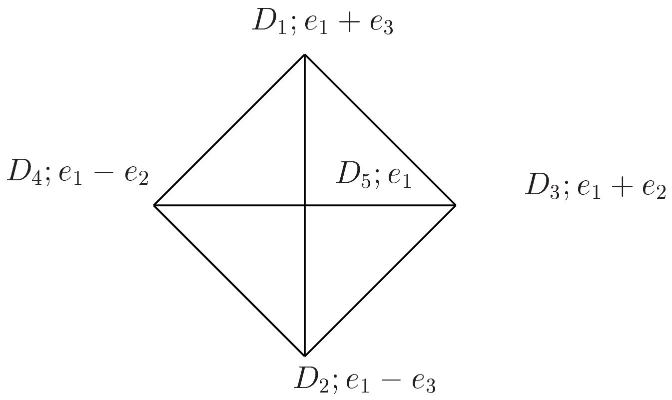
Suppose now that f is a Mori fiber space, and let us assume that there is a maximal cone $GKZ_{\Sigma'',I''}$ whose intersection with $GKZ_{\Sigma,I}$ is $GKZ_{\Sigma',I}$. We see that $I'' = I$ and $X(\Sigma'') \to X(\Sigma')$ is again a Mori fiber space. Indeed, otherwise it follows from (1), (1') and (2) that the adjacent maximal cone to $GKZ_{\Sigma'',I''}$ does not correspond to a Mori fiber space. Since $I = I''$, we see that $X(\Sigma)$ and $X(\Sigma'')$ are two Mori fiber spaces over Y, isomorphic in codimension one. It follows from Proposition 10(iii) that $\Sigma = \Sigma''$, a contradiction. This shows that $GKZ_{\Sigma',I}$ is an exterior wall.

The last assertion in the proposition follows from the fact that we have already analyzed all the possible walls in the GKZ decomposition.

Exercise 22. Let $N = \mathbb{Z}^3$ and consider the complete fan Δ in $N_{\mathbb{R}}$ with eight maximal cones generated by

$$\{\pm e_1, e_1+e_3, e_1+e_2\}, \{\pm e_1, e_1+e_2, e_1-e_3\}, \{\pm e_1, e_1-e_3, e_1-e_2\},$$
$$\{\pm e_1, e_1-e_2, e_1+e_3\}.$$

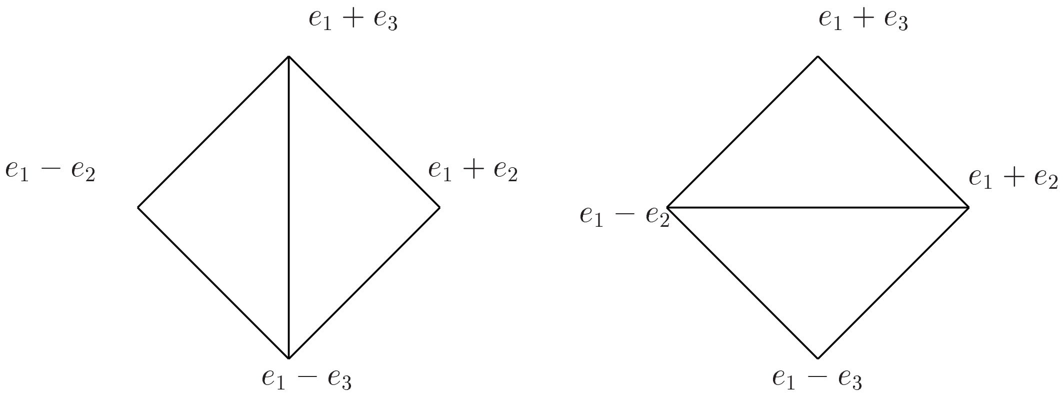
We denote the prime divisors as in the picture



and D_6 is the divisor corresponding to $-e_1$.

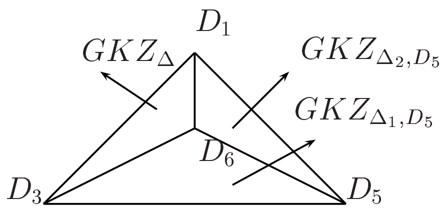
(i) Show that $X = X(\Delta)$ is smooth, projective and the effective cone is generated by D_1, D_3 and D_5.

(ii) Let Δ_1 and Δ_2 be the fans with maximal cones generated by $\{e_1+e_3, e_1-e_2, e_1-e_3\}$, $\{e_1+e_3, e_1+e_2, e_1-e_3\}$ (respectively $\{e_1+e_3, e_1-e_2, e_1+e_2\}$, $\{e_1-e_3, e_1-e_2, e_1+e_2\}$) and the other maximal cones in Δ involving $-e_1$.



Δ_1 Δ_2

Show that the GKZ decomposition of the effective cone is given as in the picture below



The interior walls generated by D_1, D_6 (respectively D_3, D_6) correspond to divisorial contractions $X \to X(\Delta_2)$ and, respectively, $X \to X(\Delta_1)$, while the one generated by D_5 and D_6 corresponds to an elementary transformation between $X(\Delta_1)$ and $X(\Delta_2)$. The three exterior walls correspond to structures of Mori fiber spaces for $X(\Delta)$, $X(\Delta_1)$ and $X(\Delta_2)$ with basis $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that $X(\Delta_1)$ and $X(\Delta_2)$ are not smooth.

Toric singularities of pairs. We first define the discrepancy divisor. Suppose that $X = X(\Delta)$ is a $\mathbb{Q}$-Gorenstein toric variety. If $\pi\colon X' \to X$ is a birational morphism of toric varieties, then the **discrepancy divisor** is $K_{X'/X} := K_{X'} - \pi^*(K_X)$ (the pull-back $\pi^*(K_X)$ is well-defined since K_X is $\mathbb{Q}$-Cartier). Note that $K_{X'/X}$ is supported on the union of the exceptional divisors of π.

More generally, consider a pair (X, D) where X is a toric variety and D is a $\mathbb{Q}$-T-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $\pi\colon X' \to X$ is a birational morphism of toric varieties, we consider $K_{X'} - \pi^*(K_X + D)$. Note that if $r \geq 1$ is such that $r(K_X + D)$ is Cartier, then so is $r(K'_X - \pi^*(K_X + D))$.

Given a toric divisor E over X, we define the **log discrepancy** $a_E(X, D)$ as follows. Consider a birational morphism π as above such that E is a prime divisor on X'. Then $a_E(X, D) - 1$ is the coefficient of E in $K_{X'} - \pi^*(K_X + D)$. If E corresponds to the primitive lattice element v in $|\Delta|$, we see that

$$a_E(X, D) = \psi_{K_X+D}(v).$$

In particular, it does not depend on the particular model X'. Note that if $r(K_X + D)$ is Cartier, then $r \cdot a_E(X, D)$ is an integer.

The main classes of singularities that are relevant in Mori Theory are described in terms of log discrepancies. Note that a divisor E over X is exceptional if and only if the corresponding ray is not a ray in Δ. A pair (X, D) as above has

1) **log canonical** singularities if $a_E(X,D) \geq 0$ for every toric divisor E over X,
2) **log terminal** singularities if $a_E(X,D) > 0$ for every toric divisor E over X,
3) **canonical** singularities if it is log canonical and $a_E(X,D) \geq 1$ for every exceptional toric divisor E over X.
4) **terminal** singularities if it is log terminal and $a_E(X,D) > 1$ for every exceptional toric divisor E over X.

When $D = 0$, we will simply say that X has terminal (or canonical, log terminal, log canonical) singularities.

Since log discrepancies can be computed in terms of the function ψ_{K_X+D}, the above classes of singularities can be characterized using this function.

Proposition 12. *Let (X,D) be such that X is a toric variety and $D = \sum_{i=1}^d a_i D_i$ is a $\mathbb{Q}$-T-divisor with $K_X + D$ being $\mathbb{Q}$-Cartier.*

1) *(X,D) has log canonical (log terminal) singularities if and only if $\psi_{K_X+D} \geq 0$ (respectively > 0) on $|\Delta| \smallsetminus \{0\}$. Furthermore, this is equivalent with $a_i \leq 1$ (respectively < 1) for all i.*
2) *(X,D) has canonical (terminal) singularities if and only if it has log canonical (respectively log terminal) singularities and if every lattice point w in $\{v \in |\Delta| \mid \psi_{K_X+D} < (\leq) 1\}$, lies on a ray of Δ.*

Proof. The assertions follow directly from the fact that if the divisor E corresponds to v, then $a_E(X,D) = \psi_{K_X+D}(v)$. For 1), note that ψ_{K_X+D} is positive on $|\Delta| \smallsetminus \{0\}$ if and only if it is positive on the generators of the rays.

Remark 14. It follows from the above proposition that if X is a toric variety with prime T-divisors $D_1, \ldots, D_d$, then X has log terminal singularities and $(X, \sum_i D_i)$ has log canonical singularities.

Remark 15. If X is a Gorenstein toric variety, then the log discrepancies are integers. Since X has log terminal singularities, it follows that in fact it has canonical singularities.

Exercise 23. Show that if X is smooth, then it has terminal singularities.

Exercise 24. Let $\pi\colon X' \to X$ be a proper birational morphism of toric varieties. Show that if D is a $\mathbb{Q}$-T-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier and if $D' = \pi^*(K_X + D) - K_{X'}$ then (X,D) has terminal singularities if and only if so does (X', D') and for every T-divisor E on X' contracted by π, the coefficient of E in D' is negative.

Exercise 25. Let $f\colon X' \to X$ be a finite surjective morphism of toric varieties. If D is a $\mathbb{Q}$-T-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier and $D' = f^*(K_X + D) - K_{X'}$, then (X, D) has terminal (canonical, log terminal, log canonical) singularities if and only if so does (X', D').

We defined singularities of pairs using all toric divisors over X. However, they all can be described using one resolution of singularities.

Proposition 13. *Let X be a toric variety and D a $\mathbb{Q}$-T-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $\pi\colon X' \to X$ is a toric resolution of singularities, and $E_1, \ldots, E_m$ are the prime T-divisors on X', write*

$$K_{X'} - \pi^*(K_X + D) = \sum_{i=1}^{m} b_i E_i.$$

1) *(X, D) is log canonical (log terminal) if and only if $b_i \geq (>) - 1$ for every i.*
2) *(X, D) is canonical (terminal) if and only if*
   i) *$b_i \geq -1 (> -1)$ for every i.*
   ii) *$b_i \geq 0 (> 0)$ for every i such that E_i is exceptional.*
   iii) *$b_i + b_j \geq -1 (> -1)$ for every i and j such that $E_i \cap E_j$ is nonempty.*

Proof. If we write $D' = \pi^*(K_X + D) - K_{X'}$, then $\psi_{K_{X'}+D'} = \psi_{K_X+D}$. The assertions in 1) follow immediately from the description of log canonical and log terminal singularities in Proposition 12.

We prove now the characterization of terminal singularities. It is clear that if (X, D) has terminal singularities, then conditions i) and ii) follow from definition. For iii), let v_i and v_j be the primitive lattice elements corresponding to E_i and E_j, respectively. Since v_i and v_j belong to the same cone in the fan of X', it follows that $v := v_i + v_j$ does not lie on a ray of Δ. Since $\psi_{K_X+D}(v) = b_i + b_j + 2 > 1$, we deduce iii).

In order to show the converse, note that i) implies that (X, D) has log canonical singularities. Suppose now that v is a lattice point in $|\Delta|$ such that $\psi_{K_X+D}(v) \leq 1$. Let σ be the cone in the fan of X' containing v in its relative interior. We write $v = a_1 v_1 + \ldots + a_r v_r$, where the v_i are the primitive generators of the rays of σ. Since σ is nonsingular, all a_i are integers. If $r = 1$, then ii) implies that v lies on a ray of Δ. If $r \geq 2$, then i) and iii) give

$$\psi_{K_X+D}(v) \geq \psi_{K_X+D}(v_i + v_j) = b_i + b_j + 2 \geq 1,$$

a contradiction. The proof of the characterization of canonical singularities is similar.

Remark 16. As we will see also in Proposition 14 below, canonical and terminal singularities behave particularly well in the case when $D = 0$. Note that in this case Proposition 13 implies that if $X' \to X$ is a resolution of singularities, and if we write

$$K_{X'/X} = \sum_{i=1}^{m} b_i E_i,$$

where the E_i are the prime T-divisors on X', then X has canonical (terminal) singularities if and only if $b_i \geq 0 (> 0)$ for all i such that E_i is exceptional. Indeed, note that $K_{X'/X}$ is supported on the union of exceptional divisors in X, so i) and iii) in Proposition 13 follow from ii).

Example 6. If X is a toric surface, then X has terminal singularities if and only if it is smooth, and X has canonical singularities if and only if all its singular points are A_m-singularities. Indeed, suppose that $X = U_\sigma$ is affine and singular. We may assume that we have a basis e_1, e_2 for the lattice such that σ is generated by e_2 and $me_1 - ke_2$, where $0 < k < m$ and k and m are relatively prime. If $v = e_1$, then $\psi_{K_X}(v) = (k+1)/m \leq 1$. Therefore X doesn't have terminal singularities, and it has canonical singularities if and only if $k = m-1$. This is the case if and only if X has an A_{m-1}-singularity. Conversely, if this is the case, then adding the ray through e_1 gives a resolution of singularities X' of X such that $K_{X'/X} = 0$. Therefore X has canonical singularities by Remark 16.

We consider now how singularities of pairs behave with respect to the Minimal Model Program.

Proposition 14. *Let $X = X(\Delta)$ be a quasiprojective toric variety such that $|\Delta|$ is convex and of maximal dimension. For a $\mathbb{Q}$-T-divisor D on X such that $K_X + D$ is $\mathbb{Q}$-Cartier consider the contraction $f\colon X \to Y$ of a $(K_X + D)$-negative ray in the Mori cone of X.*

i) *If f is a flipping contraction and $f^+\colon X^+ \to Y$ induces the corresponding elementary transformation, for every toric divisor E over X we have $a_E(X^+, D^+) \geq a_E(X, D)$. In particular, if (X, D) has log canonical (or log terminal, canonical, terminal) singularities, then so does (X, D).*
ii) *If f is a divisorial contraction and (X, D) has log canonical or log terminal singularities, then so does $(Y, f_*(D))$. Moreover, if D is zero and X has canonical or terminal singularities, then so does Y.*

iii) *If $D = 0$, f is a Mori fiber space and X is simplicial and has canonical or terminal singularities, then so does Y.*

Proof. For i) note that if E corresponds to $v \in |\Delta| \cap N$, then Proposition 8 gives

$$a_E(X^+, D^+) = \psi_{K_{X^+}+D^+} \geq \psi_{K_X+D} = a_E(X, D).$$

Since X and X^+ are isomorphic in codimension one, E is exceptional over X if and only if it is exceptional over X^+. This implies the assertions about singularities of (X, D) and (X^+, D^+).

Suppose now that f is a divisorial contraction. The statement about log canonical or log terminal singularities follows from Proposition 12 and the fact that the coefficient of a prime T-divisor F in D is equal to the coefficient of the proper transform of F in $f_*(D)$. Suppose now, for example, that X has terminal singularities. It follows from Proposition 7 that if E is the divisor contracted by f, then $K_{X/Y} = aE$ for some $a \in \mathbb{Q}_+^*$. Since X has terminal singularities, so does the pair $(X, -aE)$, and therefore Y has terminal singularities by Exercise 24.

The assertion in iii) follows from the local description of Mori fiber spaces in Proposition 10 and from Exercise 25.

K-equivalent varieties. Two $\mathbb{Q}$-Gorenstein toric varieties X and Y are **K-equivalent** if there are proper birational morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ such that $K_{Z/X} = K_{Z/Y}$. Note that the existence of such f and g implies that $|\Delta_X| = |\Delta_Y|$. If this condition is satisfied, then we can always find proper birational morphisms $f\colon Z \to X$ and $g\colon Z \to X$. Given such f and g, the condition $K_{Z/X} = K_{Z/Y}$ is equivalent with $\psi_{K_X} = \psi_{K_Y}$ on $|\Delta_X| = |\Delta_Y|$. In particular, the condition does not depend on Z.

Proposition 15. *Let X and Y be two $\mathbb{Q}$-Gorenstein toric varieties.*

i) *If X and Y have canonical singularities, then X and Y are K-equivalent if and only if there are proper birational morphisms of toric varieties $f\colon Z \to X$ and $g\colon Z \to Y$ such that $K_{Z/X}$ and $K_{Z/Y}$ are linearly equivalent.*

ii) *If X and Y have terminal singularities and X and Y are K-equivalent, then they are isomorphic in codimension one.*

Proof. Suppose that $f\colon Z \to X$ and $g\colon Z \to Y$ are proper birational morphisms. If X and Y have canonical singularities, then $K_{Z/X}$ and $K_{Z/Y}$ are effective divisors. If $K_{Z/X} - K_{Z/Y}$ is linearly equivalent to zero, since $g_*(K_{Z/X} - K_{Z/Y}) = g_*(K_{Z/X})$ is effective Lemma 6 implies that $K_{Z/X} - K_{Z/Y}$ is effective. Similarly, using f we see that $K_{Z/Y} - K_{Z/X}$ is effective so $K_{Z/Y} = K_{Z/X}$.

Suppose now that X and Y are K-equivalent and have terminal singularities. If v is a primitive generator of a ray in Δ_X, then $\psi_{K_Y}(v) = \psi_{K_X}(v) = 1$, so v lies on a ray in Δ_Y since Y has terminal singularities. Therefore every ray in Δ_X is in Δ_Y, and by symmetry we see that Δ_X and Δ_Y have the same rays.

Remark 17. Roughly speaking, two varieties are K-equivalent if they are at the same level from the point of view of the canonical class. Many invariants are the same for K-equivalent smooth projective varieties. For example, this was proved for Hodge numbers by Kontsevich using motivic integration (for a toric version of the proof see Chapter 11). An important conjecture says that two K-equivalent smooth projective varieties have equivalent derived categories (see [Ka] for a proof in the case of toric varieties).

Remark 18. A proper birational morphism of toric varieties $f\colon X \to Y$ with Y $\mathbb{Q}$-Gorenstein is **crepant** if $K_{X/Y} = 0$. In this case X and Y are trivially K-equivalent. Since KX/Y is supported on the divisors contracted by f, we see that if f is an isomorphism in codimension one, then it is crepant. If Y has terminal singularities, then the converse holds: if f is crepant, then it is an isomorphism in codimension one.

Relative Minimal Model Program. We give now a sketch of the MMP in the relative context. Our approach is to reduce it to the case we have considered so far. For more details, see [Fu2] and [FS].

Let $f\colon X \to Y$ be a proper morphism of toric varieties, i.e. f is proper and there is an f-ample divisor on X. Note that if σ is a cone in the fan of Y, then the support of the fan defining $f^{-1}(U_\sigma)$ is convex, so our previous consideration apply for this open subset of X.

We start by defining relative versions of the nef cone and of the Mori cone. The relative Picard group $\mathrm{Pic}(X/Y)$ is the quotient of $\mathrm{Pic}(X)$ by the group of f-trivial line bundles and $\mathrm{Pic}(X)_{\mathbb{Q}}$ is the corresponding vector space. The relative vector space of numerical classes of curves $\mathrm{N}_1(X/Y)_{\mathbb{Q}}$ is the quotient of the vector space with basis given by irreducible complete curves $C \subseteq X$ such that $f(C)$ is a point by the subspace consisting of those $\sum_i m_i C_i$ such that for every $\mathcal{L}$ in $\mathrm{Pic}(X)$ we have $\sum_i m_i(\mathcal{L} \cdot C_i) = 0$.

The intersection pairing induces a pairing

$$\mathrm{Pic}(X/Y)_{\mathbb{Q}} \times \mathrm{N}_1(X/Y)_{\mathbb{Q}} \to \mathbb{Q}.$$

It follows by definition that the induced map $\mathrm{N}_1(X/Y)_{\mathbb{Q}} \to \mathrm{Pic}(X/Y)^*_{\mathbb{Q}}$ is injective, so $\mathrm{N}_1(X/Y)_{\mathbb{Q}}$ has finite dimension. Moreover, by Exercise 15 the map $\mathrm{Pic}(X/Y)_{\mathbb{Q}} \to \mathrm{N}_1(X/Y)_{\mathbb{Q}}$ is also injective, so the intersection pairing gives a duality between $\mathrm{Pic}(X/Y)_{\mathbb{Q}}$ and $\mathrm{N}_1(X/Y)_{\mathbb{Q}}$. Moreover, it follows from Exercise 15 that $\mathrm{N}_1(X/Y)_{\mathbb{Q}}$ is generated by classes of irreducible complete invariant curves contracted by f.

Remark 19. If Y is affine, then every complete curve on X is contracted by f, so $\mathrm{Pic}(X/Y)_{\mathbb{Q}} = \mathrm{Pic}(X)_{\mathbb{Q}}$ and $\mathrm{N}_1(X/Y)_{\mathbb{Q}} = \mathrm{N}_1(X)_{\mathbb{Q}}$.

The definition of f-nef line bundles extends in the obvious way to $\mathbb{Q}$-divisors. We denote by $\mathrm{Nef}(X/Y)$ the subset of $\mathrm{Pic}(X/Y)_{\mathbb{Q}}$ consisting of classes of f-nef $\mathbb{Q}$-divisors.

The relative Mori cone $\mathrm{NE}(X/Y)_{\mathbb{Q}}$ is the convex cone in $\mathrm{N}_1(X/Y)_{\mathbb{Q}}$ generated by classes of irreducible complete curves C on X that are contracted by f. It follows from Exercise 14 that $\mathrm{NE}(X/Y)_{\mathbb{Q}}$ is generated by irreducible invariant complete curves C on X that are contracted by f, so it is a polyhedral cone. Moreover, $\mathrm{Nef}(X/Y)$ is the dual of the relative Mori cone.

Lemma 7. *Let $f\colon X \to Y$ be a proper morphism of toric varieties, with X simplicial. If σ is a cone in the fan of Y, then we have a canonical embedding* $\mathrm{N}_1(f^{-1}(U_\sigma))_{\mathbb{Q}} \hookrightarrow \mathrm{N}_1(X/Y)_{\mathbb{Q}}$.

Proof. The map takes the class of a curve C in $f^{-1}(U_\sigma)$ to the class of the same curve in X. The fact that this is injective follows from definition and the fact that the dual restriction map $\mathrm{Pic}(X/Y)_{\mathbb{Q}} \to \mathrm{Pic}(f^{-1}(U_\sigma))_{\mathbb{Q}}$ is surjective. This in turn is a consequence of the fact that the corresponding restriction morphism between class groups is surjective and X is simplicial.

We consider now contractions of faces of the relative Mori cone. Recall that a morphism $h\colon Z \to Y$ is projective if it is proper and there is an f-ample line bundle on Z.

Proposition 16. *Let $f\colon X \to Y$ be a proper morphism of toric varieties. Given a face F of $\mathrm{NE}(X/Y)_{\mathbb{Q}}$, then f factors as*

$$X \overset{\mathrm{cont}_F}{\longrightarrow} Z \overset{h}{\longrightarrow} Y,$$

with h projective, and if C is a complete irreducible curve on X contracted by f, then C is contracted by cont_F if and only if its class in $\mathrm{N}_1(X/Y)_{\mathbb{Q}}$ lies in F.

Proof. We do the construction locally on Y. If σ is a cone in the fan of Y, then we have a map $\phi_\sigma\colon \mathrm{NE}_1(f^{-1}(U_\sigma)) \to \mathrm{NE}_1(X/Y)$ and $F_\sigma := (\phi_\sigma)^{-1}(F)$ is a face of $\mathrm{NE}_1(f^{-1}(U_\sigma))$. Since the support of the fan of $f^{-1}(U_\sigma)$ is convex, it follows from Proposition 1 that there is a unique morphism

$$\mathrm{cont}_{F_\sigma}\colon f^{-1}(U_\sigma) \to W_\sigma$$

that contracts precisely the curves whose class lies in F_σ (note that both the existence and the uniqueness assertions can be reduced easily to the case when the above mentioned support has maximal dimension). Moreover, we deduce from the construction of cont_{F_σ} (see Proposition 9 in Chapter 6) that we have an induced morphism of toric varieties $W_\sigma \to U_\sigma$ such that the obvious diagram commutes. Uniqueness of F_σ allows us to glue the morphisms cont_{F_σ} to get cont_F, as required.

From now on we assume that $f\colon X \to Y$ is a projective morphism, and for simplicity, also that X is simplicial. In this case, if R is a ray of $\mathrm{NE}(X/Y)$, using the notation in the proof of Proposition 16 we see that R_σ is either zero or a ray of $\mathrm{NE}(f^{-1}(U_\sigma))$. Therefore over each $U_\sigma \subset Y$, the contraction cont_F is either an isomorphism or the contraction of a ray in the Mori cone of $f^{-1}(U_\sigma)$.

As in the absolute case, cont_R is called a divisorial contraction if it is birational, but not an isomorphism in codimension one, a flipping contraction if it is an isomorphism in codimension one, and a Mori fiber space if $\dim(Z) < \dim(X)$. It is clear that

i) cont_R is a Mori fiber space if and only if its restriction over each $U_\sigma \subseteq Y$ is a Mori fiber space.
ii) cont_R is a flipping contraction if and only if the restriction over each U_σ is either an isomorphism or a flipping contraction.
ii) cont_R is a divisorial contraction if and only if its restriction over each U_σ is either an isomorphism or a divisorial contraction.

Using existence and uniqueness of flips in the case when the base Y is affine we deduce the corresponding statement in the case of an arbitrary flipping contraction. Furthermore, termination of flips also follows from the case of an affine base. We deduce that the analogue of Proposition 5 holds in the relative setting. Moreover, the analysis of singularities follows directly from that in the absolute case. We leave for the reader the task of stating the results in the general case.

As an application of the relative MMP we prove the following

Proposition 17. *If $X = X(\Delta)$ is a toric variety with canonical singularities, then there is a projective birational crepant morphism $f\colon Y \to X$ such that Y is simplicial and has terminal singularities.*

Proof. Let $g\colon Z \to X$ be a projective toric resolution of singularities (show that there is such a resolution!). Since g is birational, it follows that if we run the MMP for g, we can't obtain a Mori fiber space. Therefore we obtain $f\colon Y \to X$ such that Y is simplicial and has terminal singularities, and $K_{Y/X}$ is f-nef. Since $f_*(K_{Y/X}) = 0$, it follows from Lemma 6 that $-K_{Y/X}$ is effective. On the other hand, X has canonical singularities, so also $K_{Y/X}$ is effective, and therefore f is crepant.

Solutions and hints for selected exercises

Exercise 4. After replacing D by a multiple, we may assume that E and F are integral divisors and $\mathcal{O}(E)$ is globally generated. Write $\pi^*(D) = \sum_j b_j F_j$. If $f\colon Y \to Z$ is the fiber space defined by $\mathcal{O}(E)$, then it is enough to show that $\dim(f(F_j)) < \dim(X) - 1$ for every component F_j of F. If $\dim(Z) \leq \dim(X) - 2$, this is trivial. If $\dim(Z) = \dim(X)$ and $f(F_j)$ is a divisor on Z, then Lemma 12 in Chapter 2 implies that

$$P_D \cap \{u \mid \langle u, w_j \rangle = -b_j\}$$

(where w_j corresponds to F_j) gives a facet of P_D, a contradiction with the definition of F. If $\dim(Z) = \dim(X) - 1$ and F_j dominates Z, then w_j lies in the one-dimensional kernel of the lattice morphism corresponding to f. The same lemma applies to give a contradiction.

Exercise 17. If D' is another such divisor, then by Lemma 5 we have $D' \sim_{\mathbb{Q}} f^*M + aD$ for some $a \in \mathbb{Q}_+^*$. In this case $f_*(D') = M + af_*(D)$, so the proper transform of $f_*(D')$ on X^+ is f^+-ample.

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