LECTURE 7: STABLE RATIONALITY AND DECOMPOSITION OF THE DIAGONAL

In this lecture we discuss a criterion for non-stable-rationality based on the decomposition of the diagonal in the Chow group. This criterion goes back to the work of Bloch and Srinivas [BS]. It was only recently realized by Voisin in [Voi] that one can use this point of view in a very efficient way to prove the non-stable-rationality of very general elements of many interesting families. This led to a flurry of activity in the field. In the first section we discuss general facts about decomposition of the diagonal and in the second section we discuss, following the work of Colliot-Thélène and Pirutka [CTP16], the behavior of decomposition of the diagonal in families degenerating to singular varieties with nice resolutions.

1. Decomposition of the diagonal

Let k be an arbitrary field. All schemes over k are assumed to be separated and of finite type. Recall that if X is a complete scheme over k, then we have the degree map

$$\deg\colon\ \mathrm{CH}_0(X) \to \mathbf{Z}, \quad \sum_i n_i[p_i] \to \sum_i n_i \cdot \deg\big(k(p_i)/k\big).$$

We denote by $\mathrm{CH}_0(X)_0$ the kernel of this map.

Definition 1.1. A complete scheme X is CH_0-*trivial* if the degree map $\mathrm{CH}_0(X) \to \mathbf{Z}$ is an isomorphism. Equivalently, X has a cycle of degree 1 and $\mathrm{CH}_0(X)_0 = 0$. We say that X is *universally* CH_0-*trivial* if X_F is CH_0-trivial for every field extension F/k. Equivalently, X has a cycle of degree 1 and $\mathrm{CH}_0(X_F)_0 = 0$ for every extension F/k (note that the existence of a cycle of degree 1 is clear when $X(k) \neq \emptyset$, hence it is always satisfied if k is algebraically closed).

Example 1.2. The projective space $\mathbf{P}^n_k$ is universally CH_0-trivial. Indeed, for every field extension F/k, the degree map $\mathrm{CH}_0(\mathbf{P}^n_F) \to \mathbf{Z}$ is an isomorphism by Example 1.11 in Appendix 3.

Remark 1.3. Note that if X is CH_0-trivial, then X is connected. Indeed, if p and q are closed points on different connected components of X, then $\deg\big(k(q)\big) \cdot [p] - \deg\big(k(p)\big)[q]$ is a nonzero (even non-torsion) element of $\mathrm{CH}_0(X)_0$. We deduce that if X is universally CH_0-trivial, then X is geometrically connected[1].

Definition 1.4. Let X be a complete variety over a field k, with $\dim(X) = n$. We say that X has a decomposition of the diagonal if we can write the class of the diagonal Δ_X on $X \times X$ as

$$[\Delta_X] = \alpha \times [X] + Z \quad \text{in} \quad \mathrm{CH}_n(X \times X) \tag{1}$$

[1] Recall that a scheme X over k is geometrically connected if for every field extension F/k, the scheme X_F is connected; in fact, it is enough to have $X_{\overline{k}}$ connected, where $\overline{k}$ is the algebraic closure of k.

for some $\alpha \in \mathrm{CH}_0(X)$ with $\deg(\alpha) = 1$, and some Z, the class of a cycle supported on $X \times V$ for a proper closed subset V of X. We say that X has a *rational decomposition of the diagonal* if there is a positive integer N such that we can write

$$N \cdot [\Delta_X] = N \cdot (\alpha \times [X]) + Z \quad \text{in} \quad \mathrm{CH}_n(X \times X), \tag{2}$$

with α and Z as before.

Example 1.5. The projective space $\mathbf{P}^n$ has a decomposition of the diagonal. Indeed, it follows from Proposition 2.1 in Appendix 3 that $\mathrm{CH}_n(\mathbf{P}^n \times \mathbf{P}^n)$ is freely generated by $[L_i \times L_{n-i}]$, for $0 \le i \le n$, where the L_i are i-dimensional linear subspaces of $\mathbf{P}^n$. We can thus write

$$[\Delta_{\mathbf{P}^n}] = \sum_{i=0}^{n} a_i \cdot [L_i \times L_{n-i}],$$

for unique integers $a_0, \ldots, a_n$. If $\pi \colon \mathbf{P}^n \times \mathbf{P}^n \to \mathbf{P}^n$ is the projection onto the second component, then

$$[\mathbf{P}^n] = \pi_*[\Delta_X] = \sum_{i=0}^{n} a_i \cdot \pi_*([L_i \times L_{n-i}] = a_0[\mathbf{P}^n].$$

Therefore $a_0 = 1$, which implies that for every $p \in \mathbf{P}^n(k)$, we can write

$$[\Delta_{\mathbf{P}^n}] = [\{p\} \times \mathbf{P}^n] + Z \quad \text{in} \quad \mathrm{CH}_n(\mathbf{P}^n \times \mathbf{P}^n),$$

where Z is the class of a cycle supported on $\mathbf{P}^n \times H$, for some hyperplane H.

The following important criterion for the decomposition of the diagonal is due to Bloch and Srinivas [BS].

Theorem 1.6. *Given a smooth, geometrically connected, complete n-dimensional variety X over the field k, the following are equivalent:*

1. *X is universally CH_0-trivial.*
2. *There is $\alpha \in \mathrm{CH}_0(X)$ with $\deg(\alpha) = 1$, and we have $\mathrm{CH}_0(X_K)_0 = 0$, where $K = k(X)$.*
3. *X has a decomposition of the diagonal. Moreover, in this case for every $\alpha \in \mathrm{CH}_0(X)$ with $\deg(\alpha) = 1$, we have a decomposition as in (1).*

Proof. The implication (1)⇒(2) is trivial. We begin by proving (2)⇒(3). Consider the Cartesian diagram

$$\begin{array}{ccc} X_K & \xrightarrow{j} & X \times X \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(K) & \longrightarrow & X, \end{array}$$

where π is the projection onto the second component. Note that we have a morphism

$$j^* \colon Z_n(X \times X) \to Z_0(X_K), \quad j^*\big(\sum_i n_i[V_i]\big) = \sum_i n_i[j^{-1}(V_i)].$$

It follows from definition that

$$\deg\big(j^*([V])\big) = \begin{cases} \deg\big(k(V)/K\big), & \text{if } \pi(V) = X; \\ 0, & \text{otherwise.} \end{cases}$$

Given $\alpha \in Z_0(X)$ with $\deg(\alpha) = 1$, both $j^*(\alpha \times [X])$ and $j^*(\Delta_X)$ have degree 1, hence

$$j^*(\alpha \times [X]) \sim_{\rm rat} j^*(\Delta_X).$$

It is straightforward to see that in this case there is a nonempty open subset U of X such that

$$\big([\Delta_X] - \alpha \times [X]\big)|_{X \times U} \sim_{\rm rat} 0.$$

Using the exact sequence in Example 1.8 in Appendix 3, we deduce that

$$[\Delta_X] - \alpha \times [X] = i_*(\beta) \in \mathrm{CH}_n(X \times X)$$

for some $\beta \in \mathrm{CH}_n(X \times V)$, where $V = X \smallsetminus U$ and $i\colon X \times V \hookrightarrow X \times X$ is the inclusion. Therefore X has a decomposition of the diagonal.

We now prove (3)⇒(1). Given a decomposition of the diagonal (1) for X, we clearly obtain an induced decomposition of the diagonal for X_F, whenever F is a field extension of k. We see that after replacing X by X_F, it is enough to show that given a decomposition of the diagonal (1), we have $\mathrm{CH}_0(X)_0 = 0$. In order to show this, we make use of the action of correspondences on Chow groups (see §4 in Appendix 3).

By considering the cycle classes in (1) as correspondences $X \vdash X$, we obtain

$$[\Delta_X]^* = (\alpha \times [X])^* + Z^* \quad \text{as maps} \quad \mathrm{CH}_0(X) \to \mathrm{CH}_0(X).$$

Recall that if Γ_f is the graph of a morphism $f\colon X \to X$, then $[\Gamma_f]^* = f^*$ (see Example 4.5 in Appendix 3). Since Δ_X is the graph of the identity morphism, we see that $[\Delta_X]^*$ is the identity.

Let us compute $(\alpha \times [X])^*(\beta)$ for $\beta \in \mathrm{CH}_0(X)$. Note that if $p_1, p_2\colon X \times X \to X$ are the projections onto the two components, then it follows from definition that

$$(\alpha \times [X])^*(\beta) = (p_1)_*\big((\alpha \times [X]) \cdot p_2^*(\beta)\big) = (p_1)_*\big(p_1^*(\alpha) \cdot p_2^*(\beta)\big) = \alpha \cdot (p_1)_*\big(p_2^*(\beta)\big) = \deg(\beta)\alpha,$$

where the third equality follows from the projection formula and the fourth one follows easily from definitions (this is also a special case of the compatibility between proper push-forward and flat pull-back).

Finally, we note that $Z^*(\beta) = 0$ for every β. This follows by applying Proposition 4.8 to the correspondence Z'. We thus conclude that for every $\beta \in \mathrm{CH}_0(X)$, we have $\beta = \deg(\beta) \cdot \alpha$, hence $\mathrm{CH}_0(X)_0 = 0$. This completes the proof. □

Remark 1.7. The proof of the theorem shows that under the same assumptions, given a positive integer N, the following are equivalent:

1. There is $\alpha \in \mathrm{CH}_0(X)$ with $\deg(\alpha) = 1$ and $N \cdot \mathrm{CH}_0(X_F)_0 = 0$ for every field extension F/k.
2. There is $\alpha \in \mathrm{CH}_0(X)$ with $\deg(\alpha) = 1$ and we have $N \cdot \mathrm{CH}_0(X_K)_0 = 0$, where $K = k(X)$.

(3) X has a rational decomposition of the diagonal, with N as in (2). Moreover, in this case for every $\alpha \in \mathrm{CH}_0(X)$ with $\deg(\alpha) = 1$, we have a decomposition as in (2).

Corollary 1.8. *Let X and Y be stably birational smooth, complete varieties over k. If Y is geometrically connected, has a zero-cycle of degree 1, and has a decomposition of the diagonal, then the same holds for X. In particular, if X is a smooth, complete, stably rational variety, then X has a decomposition of the diagonal.*

Proof. By assumption, we have open subsets $U \subseteq X \times \mathbf{P}^m$ and $V \subseteq Y \times \mathbf{P}^n$ that are isomorphic. For every field extension F/k, we have $U_F \simeq V_F$, and since U_F is dense in X_F, we conclude that X_F is connected. By applying Theorem 4.6 in Appendix 3, we obtain an isomorphism $\mathrm{CH}_0(X_F) \simeq \mathrm{CH}_0(Y_F)$ which is compatible with the degree maps. Theorem 1.6 implies that Y_F is CH_0-trivial, hence X_F is CH_0-trivial, and another application of the theorem implies that X has a decomposition of the diagonal. The last assertion in the corollary follows from the first one and from Example 1.5. □

Remark 1.9. Note that if X is a complete, geometrically integral scheme over k, such that $X_{\overline{k}}$ has a decomposition of the diagonal, where $\overline{k}$ is the algebraic closure of k, then there is a finite extension k'/k such that $X_{k'}$ has a decomposition of the diagonal. Indeed, this follows easily from the definition and the fact that for $n = \dim(X)$, we have an isomorphism

$$\mathrm{CH}_n(X_{\overline{k}} \times X_{\overline{k}}) \simeq \varinjlim_{k'} \mathrm{CH}_n(X_{k'} \times X_{k'}),$$

where the direct limit is over the subfield extensions of $\overline{k}$ that are finite over k (see Remark 1.14 in Appendix 3).

By combining this remark with Theorem 1.6, we see that if X is a complete, smooth, geometrically connected variety over k such that $X_{\overline{k}}$ is universally CH_0-trivial, then there is a finite extension k'/k such that $X_{k'}$ is universally CH_0-trivial.

Remark 1.10. Suppose that X is a smooth, complete variety over k, and that we are in a setting where resolution of singularities holds (for example, assume that $\mathrm{char}(k) = 0$ or that $\dim(X) \leq 3$). In this case, if there is a dominant rational map $\mathbf{P}^n_k \dashrightarrow X$ of degree d, then X has a rational decomposition of the diagonal as in (2) with $N = d$. Note first that $X(k) \neq \emptyset$ (see Remark 1.6 in Lecture1); in particular, we have a 0-cycle of degree 1 on X. After resolving the given rational map, we obtain a dominant morphism $f\colon Y \to X$ of degree d, with Y a smooth, complete, rational variety. We claim that in this case $d \cdot \mathrm{CH}_0(X_F)_0 = 0$ for every field extension F/k, hence our assertion follows from Remark 1.7.

In order to prove the claim, we may base-change to F and thus assume that $F = k$ (note that X is geometrically connected, being unirational). Since $\deg(f) = d$, the projection formula gives

$$f_*\big(f^*(\alpha)\big) = d\alpha \quad \text{for all} \quad \alpha \in \mathrm{CH}_0(X).$$

In particular, given any $\alpha \in \mathrm{CH}_0(X)_0$, we may write $d\alpha = f_*(\beta)$ for some $\beta \in \mathrm{CH}_0(Y)$. Since

$$\deg(\beta) = \deg\big(f_*(\beta)\big) = d \cdot \deg(\alpha) = 0,$$

we conclude that $\beta \in \mathrm{CH}_0(Y)_0$, hence $\beta = 0$ since Y is rational. This completes the proof of our claim.

Remark 1.11. An interesting consequence of the previous remark is that (still assuming that we are in a setting where resolution of singularities holds), if X is a smooth complete variety such that we have two rational dominant maps $\mathbf{P}^n_k \dashrightarrow X$ with relatively prime degrees, then X has a decomposition of the diagonal.

We now show that having a rational decomposition of the diagonal is a rather weak condition. For example, we will see that this property always holds for Fano varieties in characteristic 0.

Proposition 1.12. *Given a smooth, complete, n-dimensional variety X over the field k, such that there is $\alpha \in \mathrm{CH}_0(X)$ with $\deg(\alpha) = 1$, if $\mathrm{CH}_0(X_{\overline{K}})_0 = 0$, where $K = k(X)$ and $\overline{K}$ is the algebraic closure of K, then X has a rational decomposition of the diagonal.*

Before giving the proof of the proposition, let us recall that if W is a scheme over a field F and F'/F is a field extension, then we have a pull-back map $\mathrm{CH}_p(W) \to \mathrm{CH}_p(W_{F'})$ that maps $[V]$ to $[V_{F'}]$ for every integral closed subscheme V of W. Note that this commutes with the degree maps on CH_0. It is also easy to see that if $\alpha \in \mathrm{CH}_p(W)$ maps to 0 in $\mathrm{CH}_p(W_{F'})$, then there is a subextension $F'' \subseteq F'$ of finite type over F such that the image of α in $\mathrm{CH}_p(W_{F''})$ is 0. In particular, if F' is algebraic over F, then F'' is finite over F. Note that if $N = \deg(F''/F)$, then $N\alpha = 0$ in $\mathrm{CH}_p(W)$. Indeed, the morphism

$$f\colon W_{F''} \to W$$

is finite and flat, of degree N. Therefore

$$N\alpha = f_*\big(f^*(\alpha)\big) = 0.$$

Proof of Proposition 1.12. With the notation in the proof of Theorem 1.6, recall that we have the 0-cycles $j^*(\alpha \times [X])$ and $j^*([\Delta_X])$ on X_K, both of degree 1. By hypothesis, the pull-backs to $X_{\overline{K}}$ of the two cycles are rationally equivalent. The above discussion implies that there is a positive integer N such that

$$N\big(j^*(\alpha \times [X]) - j^*([\Delta_X])\big) \sim_{\mathrm{rat}} 0.$$

We conclude as in the proof of Theorem 1.6 that

$$N\big([\Delta_X] - \alpha \times [X]\big) \sim_{\mathrm{rat}} Z,$$

where $Z \in Z_n(X \times X)$ is a cycle supported on $X \times V$, for a proper closed subset V of X. □

Remark 1.13. We note that the condition in Proposition 1.12 is always satisfied if X is a rationally connected variety over an uncountable algebraically closed field k of characteristic 0 (in particular, it is always satisfied by a Fano variety over such a field, since such a variety is rationally connected by [Deb01, Proposition 5.16]). Recall that a variety X as above is rationally connected if every two general points lie in the image of some map $f\colon \mathbf{P}^1 \to X$. In fact, on such a variety *any* two points lie on a connected chain of such curves (see [Deb01, Corollary 4.28]). Since any two points on $\mathbf{P}^1$ are rationally equivalent, we thus conclude that any two points on X are rationally equivalent, hence $\mathrm{CH}_0(X)_0 = 0$. Moreover, it is known that X is rationally connected if and only if it contains a curve $C \simeq \mathbf{P}^1$ such that the normal bundle $N_{C/X}$ is ample (see [Deb01, Corollary 4.17]). Note that this condition is preserved after base-change to an algebraically closed extension L/k, hence X_L is rationally connected as well. We thus conclude that if X is rationally connected, then $\mathrm{CH}_0(X_{\overline{k(X)}})_0 = 0$.

The following result gives a sufficient criterion for complex varieties to not have decomposition of the diagonal.

Proposition 1.14. *Let X be a smooth, projective, n-dimensional complex algebraic variety. If X has a decomposition of the diagonal, then*

i) $H^0(X, \Omega_X^m) = 0$ *for all* $m > 0$.
ii) $H^3(X, \mathbf{Z})_{\mathrm{tors}} = 0$.

Proof. We use a similar argument to that in the proof of the implication 3)⇒1) in Theorem 1.6. By hypothesis, there is $\alpha \in \mathrm{CH}_0(X)$ with $\deg(\alpha) = 1$ and a proper closed subset V of X such that

$$\big([\Delta_X] - \alpha \times [X]\big)|_{X\times(X\smallsetminus V)} \sim_{\mathrm{rat}} 0. \tag{3}$$

Since we work over an algebraically closed field, it follows from Theorem 1.6 that we may assume that $\alpha = [q]$, for some closed point $q \in X$.

Let $f\colon \widetilde{X} \to X$ be an embedded resolution of singularities of V, that is, f is projective and birational, $\widetilde{X}$ is smooth, and the inverse image $E = f^{-1}(V)$ is a simple normal crossing divisor. We may and will assume that f is an isomorphism over $X \smallsetminus V$. Let $E_1, \ldots, E_r$ be the irreducible components of E and $i_k\colon E_k \hookrightarrow \widetilde{X}$ the inclusion maps.

Let $\Gamma_f \subseteq \widetilde{X} \times X$ be the graph of f and $\Gamma'_f \subseteq X \times \widetilde{X}$ the subset obtained via the automorphism $(x, y) \to (y, x)$. Since f is an isomorphism over $X \smallsetminus V$, it follows from (3) that

$$\big([\Gamma'_f] - [\{q\} \times \widetilde{X}]\big)|_{X\times(\widetilde{X}\smallsetminus E)} \sim_{\mathrm{rat}} 0.$$

We deduce using Example 1.9 in Appendix 3 that there are $z_k \in \mathrm{CH}_n(E_k)$ such that

$$[\Gamma'_f] = [\{q\} \times \widetilde{X}] + \sum_{k=1}^{r} (1_X \times i_k)_*(z_k) \quad \text{in} \quad \mathrm{CH}_n(X \times \widetilde{X}). \tag{4}$$

We consider the homomorphism $[\Gamma'_f]_*\colon H^m(X, \mathbf{Z}) \to H^m(\widetilde{X}, \mathbf{Z})$ (see the discussion at the end of §4 in Appendix 3 for actions of correspondences on singular cohomology). We will freely use Poincaré duality to identify homology and cohomology. Since Γ_f is the graph of f, we see that

$$[\Gamma'_f]_*(u) = f^*(u) \quad \text{for all} \quad u \in H^m(X, \mathbf{Z}).$$

On the other hand, note that if $m > 0$, then

$$[\{q\} \times \widetilde{X}]_*(u) = 0 \quad \text{for all} \quad u \in H^m(X, \mathbf{Z}) \quad \text{and all} \quad q \in X.$$

This follows from the fact that $\widetilde{X} \times \{q\}$ is the graph of the constant morphism $g\colon \widetilde{X} \to X$ mapping every point to q, hence

$$[\{q\} \times \widetilde{X}]_*(u) = g^*(u) = 0 \quad \text{when} \quad m > 0.$$

Let $p_1\colon X \times \widetilde{X} \to X$ and $p_2\colon X \times \widetilde{X} \to \widetilde{X}$ be the projections onto the two components and similarly, for every $k \leq r$, let $\varphi_k\colon X \times E_k \to X$ and $\psi_k\colon X \times E_k \to E_k$ be the corresponding projections. Using the projection formula and the fact that cycle class maps commute with push-forwards and pull-backs, we obtain

$$[(1_X \times i_k)_*(z_k)]_*(u) = p_{2*}\big(p_1^*(u) \cap \mathrm{cl}((1_X \times i_k)_*(z_k))\big)$$

$$= p_{2*}(1_X \times i_k)_*\big(\varphi_k^*(u) \cap \mathrm{cl}(z_k)\big) = i_{k*}(\beta_k),$$

where $\beta_k = \psi_{k*}\big(\varphi_k^*(u) \cap \mathrm{cl}(z_k)\big)$. We thus conclude using (4) that if $m > 0$, then $f^*(u) = \sum_{k=1}^r i_{k*}(\beta_k)$, with β_k as above. Of course, by tensoring with $\mathbf{C}$, we obtain a similar result for all $u \in H^m(X, \mathbf{C})$, with $m > 0$.

On the other hand, recall that $i_{k*}\big(H^{p,q}(E_k)\big) \subseteq H^{p+1,q+1}(X)$ (see Remark 3.2 in Appendix 1). This implies that

$$H^{m,0}(\widetilde{X}) \cap \sum_{k=1}^r \mathrm{Im}\big(i_{k*}\colon H^m(E_k, \mathbf{C}) \to H^{m+2}(X, \mathbf{C})\big) = 0.$$

We thus conclude from the previous discussion that the image of

$$f^*\colon H^0(X, \Omega_X^m) \to H^0(\widetilde{X}, \Omega_{\widetilde{X}}^m)$$

is 0 whenever $m > 0$. Since this is an injective map, we obtain the assertion in i).

By Poincaré duality, in order to prove the assertion in ii), it is enough to show that $H_{2n-3}(X, \mathbf{Z})_{\mathrm{tor}} = 0$. We have seen that for every $u \in H_{2n-3}(X, \mathbf{Z})$, we can write $f^*(u) = \sum_{k=1}^r i_{k*}(\beta_k)$, with

$$\beta_k = g_{k*}\big(p_1^*(u) \cap \mathrm{cl}(z_k)\big) \in H_{2n-3}(E_k, \mathbf{Z}) \simeq H^1(E_k, \mathbf{Z}).$$

This formula implies that if u is torsion, then each β_k is torsion. On the other hand, $H^1(E_k, \mathbf{Z})$ has no torsion (see Remark 1.3 in Appendix 1), hence $\beta_k = 0$ for all k. This implies that $f^*(u) = 0$. On the other hand, since f is birational, we have $u = f_*\big(f^*(u)\big)$, hence $u = 0$. Therefore $H_{2n-3}(E_k, \mathbf{Z})_{\mathrm{tors}} = 0$, completing the proof of the proposition. □

Remark 1.15. Note that the vanishings in Proposition 1.14i) are always satisfied if X is a smooth, projective, rationally connected variety over an algebraically closed field of characteristic 0 (see [Deb01, Corollary 4.18]. Therefore, the interesting criterion for not having a decomposition of the diagonal is given by the vanishing in ii).

Remark 1.16. Under the assumptions in Proposition 1.14, since $h^2(X, \mathcal{O}_X) = h^0(X, \Omega_X^2) = 0$, we deduce from the exponential sequence and GAGA that the Chern class map $\mathrm{Pic}(X) \to H^2(X, \mathbf{Z})$ is surjective. It then follows from Proposition 3.10 in Appendix 2 that

$$\mathrm{Br}(X) \simeq H^3(X, \mathbf{Z})_{\mathrm{tors}} = 0.$$

Remark 1.17. It would be nice to know whether given a smooth, projective variety over an algebraically closed field k, and a field extension K/k, with K algebraically closed, if X_K has a decomposition of the diagonal, then the same holds for X. However, this does not seem straightforward.

2. Behavior of decomposition of diagonal in families

The key ingredient in the recent breakthrough on non-stable-rationality is a result due to Voisin [Voi] concerning the behavior of the existence of decomposition of the diagonal in families. Very roughly, this says that if a family whose general member is smooth contains a singular element that admits a "nice" resolution that does not have decomposition of the diagonal, then the general fiber also does not have such a decomposition; in particular, it is not stably birational. Our goal in this section is to discuss a stronger version of this result (and with a simpler proof than in [Voi]), due to Colliot-Thélène and Pirutka [CTP16].

We begin with the following relative version of CH_0-triviality. Let k be a fixed field.

Definition 2.1. A proper morphism $f\colon X \to Y$ of schemes (separated and of finite type) over k is CH_0-*trivial* if the induced morphism $f_*\colon \mathrm{CH}_0(X) \to \mathrm{CH}_0(Y)$ is an isomorphism. We say that f is *universally* CH_0-*trivial* if $f_K\colon X_K \to Y_K$ is CH_0-trivial for every field extension K/k. Note that when $Y = \mathrm{Spec}(k)$, then f is CH_0-trivial (resp., universally CH_0-trivial) if and only if X has this property, in the sense of Definition 1.1.

The following is a useful criterion for the CH_0-triviality of a morphism. Given a morphism $f\colon X \to Y$, for every point $y \in Y$, not necessarily closed, we denote by X_y the fiber $X \times_Y \mathrm{Spec}\big(k(y)\big)$ over y, considered as a scheme over $k(y)$.

Proposition 2.2. *A proper morphism $f\colon X \to Y$ of schemes over k is CH_0-trivial if the following two conditions hold:*

i) *For every closed point $y \in Y$, the fiber X_y is CH_0-trivial.*
ii) *For every point $\eta \in Y$ with $\dim \overline{\{\eta\}} = 1$, the degree map*
$$\mathrm{CH}(X_\eta) \to \mathbf{Z}$$
is surjective.

Proof. The surjectivity of $f_*\colon \mathrm{CH}_0(X) \to \mathrm{CH}_0(Y)$ is easy: given any closed point $y \in Y$, we have by i) a 0-cycle α_y supported on the fiber X_y such that $f_*(\alpha_y) = [y]$. Since $\mathrm{CH}_0(Y)$ is generated by $\{[y] \mid y \in Y \text{ closed}\}$, it follows that f_* is surjective.

In order to prove the injectivity of f_*, consider a 0-cycle α on X such that $f_*(\alpha) \sim_{\mathrm{rat}} 0$. Therefore we can write
$$f_*(\alpha) = \sum_{i=1}^{r} n_i \cdot \mathrm{div}_{Y_i}(\varphi_i),$$
for some 1-dimensional integral subschemes Y_i of Y and some nonzero rational functions φ_i on Y_i. If η_i is the generic point of Y_i, by applying ii) to the fibers X_{η_i}, we deduce that we

have a 0-cycle on X_{η_i} of degree 1. We thus have 1-dimensional integral subschemes $X_{i,j}$ of X that dominate Y_i, and integers $a_{i,j}$, such that if $d_{i,j} = \deg(X_{i,j}/Y_i)$, then $\sum_j a_{i,j}d_{i,j} = 1$. If $\psi_{i,j}$ is the pull-back of φ_i to $X_{i,j}$, then

$$f_*\big(\mathrm{div}_{X_{i,j}}(\psi_{i,j})\big) = \mathrm{div}_{Y_i}\big(\mathrm{Norm}(\psi_{i,j})\big) = d_{i,j}\cdot \mathrm{div}_{Y_i}(\varphi_i).$$

We deduce that if

$$\beta = \alpha - \sum_{i,j} n_i a_{i,j}\cdot \mathrm{div}_{X_{i,j}}(\psi_{i,j}),$$

then $f_*(\beta) = 0$ in $Z_0(Y)$. This implies that there are closed points $y_1,\ldots,y_s \in Y$ and 0-cycles $\beta_1,\ldots,\beta_s$, with β_i supported on X_{y_i} such that $\beta = \sum_{i=1}^s \beta_i$ and $f_*(\beta_i) = 0$ as a 0-cycle for all $i \leq s$. We deduce using i) that $\beta_i \sim_{\mathrm{rat}} 0$ in X_{y_i}, hence in X, for all i. This shows that $\alpha \sim_{\mathrm{rat}} 0$ and thus shows that $f_*\colon \mathrm{CH}_0(X) \to \mathrm{CH}_0(Y)$ is injective. □

Corollary 2.3. *If $f\colon X \to Y$ is a proper morphism of schemes over k such that for every point $\eta \in Y$ (not necessarily closed) the fiber X_η is universally CH_0-trivial, then f is universally CH_0-trivial.*

Proof. It is enough to show that for every field extension K/k, the induced morphism $f_K\colon X_K \to Y_K$ satisfies the hypothesis in the proposition. If $w \in Y_K$ is an arbitrary point and y is its image in Y, then the fiber of f_K over w is isomorphic to $(X_y)_{k(w)}$; therefore it is CH_0-trivial. Note that, in general, we have no control on the dimension of $\overline{\{y\}}$ in terms of the dimension of $\overline{\{w\}}$, which is why we need to impose the universal CH_0-triviality condition for all fibers of f. □

Example 2.4. Suppose that k is an algebraically closed field of characteristic different from 2, and X is a variety over k of dimension $n \geq 2$, having only finitely many singular points, all of them nodes (recall that this means that the projectivized tangent cone is a smooth quadric). If $f\colon \widetilde{X} \to X$ is the blow-up of the singular points, then f is universally CH_0-trivial. Indeed, the fiber over a point $x \in X$ is either $\mathrm{Spec}\,\big(k(x)\big)$ or a smooth $(n-1)$-dimensional quadric over k, which is rational by Example 3.8 in Lecture 1; therefore all fibers are universally CH_0-trivial, and we may apply Corollary 2.3.

Before stating and proving the main result of this section, we include the following technical lemma. For the definition of the specialization map that appears in the statement, see §5 in Appendix 3.

Lemma 2.5. *Let $(A, \mathfrak{m}, k)$ be a complete DVR and consider a commutative diagram with Cartesian squares*

$$\begin{array}{ccccc} W_0 & \longrightarrow & W & \longleftarrow & W_\eta \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(K) \end{array}$$

where K is the fraction field of A, and where the morphism f is smooth. For every closed point $x_0 \in W_0$, there is a closed point $x \in W_\eta$ such that $\sigma([x]) = [x_0]$, where $\sigma\colon Z_0(W_\eta) \to Z_0(W_0)$ is the specialization map.

Proof. Given the extension $k \hookrightarrow k' = k(x_0)$, there is an injective local homomorphism $A \hookrightarrow A'$, where $(A', \mathfrak{m}A')$ is a DVR such that $A'/\mathfrak{m}A' \simeq k'$. This is easy to see when A contains a field: indeed, in this case we have $A \simeq k[\![t]\!]$ and we may take $A' = k'[\![t]\!]$. For a proof in the mixed characteristic case, see [Mat89, Theorem 29.1]. Of course, after replacing A' by its completion, we may assume that it is complete. It is easy to see that since A is complete and $A'/\mathfrak{m}A'$ is finitely generated over k, the A-module A' is finitely generated (when A contains a field, this follows from the explicit description of A' that we have given).

Claim. There is a morphism $w\colon \operatorname{Spec}(A') \to W$ over $\operatorname{Spec}(A)$ such that the diagram

$$\begin{array}{ccc} \operatorname{Spec}(k') & \longrightarrow & \operatorname{Spec}(A') \\ \downarrow & & \downarrow \\ W_0 & \longrightarrow & W \end{array}$$

is commutative. In order to prove this, we may replace W by an affine open neighborhood of x_0 and thus assume that $W = \operatorname{Spec}(R)$ is affine. Consider now the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & R \\ \downarrow & & \downarrow \\ A' & \longrightarrow & k'. \end{array}$$

Since R is smooth over A, it is formally smooth, and since A' is complete, it follows that there is a ring homomorphism $R \to A'$ that makes the two triangles commutative. This proves our claim.

If K' is the fraction field of A', we obtain a morphism $u\colon \operatorname{Spec}(K') \to W_\eta$ over $\operatorname{Spec} K$ such that the diagram

$$\begin{array}{ccc} \operatorname{Spec}(A') & \longleftarrow & \operatorname{Spec}(K') \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle u} \\ W & \longleftarrow & W_\eta \end{array}$$

is commutative. Note that since A' finitely generated over A, the field extension K'/K is finite. It is straightforward to check that if x is the image of u, then

$$\sigma([x]) = [\overline{\{x\}} \cap W_0] = [x_0].$$

$\square$

Remark 2.6. In the above lemma, instead of assuming that A is a complete DVR, it is enough to assume that A is a Henselian local ring. However, we will not need this more general version.

The following is the main result of this section.

Theorem 2.7. *Let $(A, \mathfrak{m}, k)$ be a DVR, with fraction field K, and consider the following commutative diagram with Cartesian squares*

$$\begin{array}{ccccc} X_0 & \longrightarrow & X & \longleftarrow & X_\eta \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(K) \end{array}$$

with f proper and flat. If the following conditions hold:

i) *X_0 is geometrically integral and we have a proper, birational morphism $g\colon \widetilde{X_0} \to X_0$, with $\widetilde{X_0}$ smooth.*
ii) *The morphism g is universally CH_0-trivial.*
iii) *There is a 0-cycle of degree 1 on $\widetilde{X_0}$.*
iv) *$\widetilde{X_0}$ is not universally CH_0-trivial,*

then X_η is not universally CH_0-trivial.

Proof. The last assertion follows from the first one via Corollary 1.8. We begin the proof of the first assertion with the following reduction.

Claim. We may assume that A is complete and that the morphism

$$\deg\colon \mathrm{CH}_0(\widetilde{X_0}) \to \mathbf{Z} \tag{5}$$

is not injective. In order to see this, note that by definition, conditions iii)-iv) imply that the degree map

$$\deg\colon \mathrm{CH}_0\left((\widetilde{X_0})_L\right) \to \mathbf{Z}$$

is not injective, for some field extension L/k. As in the proof of Lemma 2.5, we can find an injective local homomorphism $A \hookrightarrow A'$, with A' a complete DVR with maximal ideal $\mathfrak{m}A'$, such that the corresponding residue field extension is $k \hookrightarrow L$. We now replace f by

$$f'\colon X' = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A') \to \mathrm{Spec}(A').$$

Note that the special fiber of f' is $(X_0)_L$ and we have the morphism

$$g' = (\widetilde{X_0})_L \to (X_0)_L$$

which again satisfies conditions i)-iv) in the theorem. Moreover, the generic fiber of f' is $(X_\eta)_{K'}$, where K' is the fraction field of A'; If it is not universally CH_0-trivial, then clearly the same holds for X_η. Therefore it is enough to prove the theorem for f', hence we may and will assume that A is complete and that (5) is not injective.

We now prove the assertion in the theorem, by considering the two group homomorphisms

$$\mathrm{CH}_0(\widetilde{X_0}) \xrightarrow{g_*} \mathrm{CH}_0(X_0) \xleftarrow{\sigma} \mathrm{CH}_0(X_\eta),$$

where σ is the specialization map. Recall that by ii), the first map is an isomorphism. Note also that both maps are compatible with the degree maps. Both maps are defined, in fact, at the level of cycles (for the specialization map, this follows from the explicit description at the end of §5 in Appendix 3).

Let $V \subseteq X_0$ be an open subset such that the induced morphism $g^{-1}(V) \to V$ is an isomorphism. In particular, V is smooth. Let U be an open subset of X such that $U \cap X_0 = V$. In this case U is smooth around V and after possibly replacing it with a smaller subset, we may assume that U is contained in the smooth locus of X.

Since the homomorphism (5) is not injective, we can find a nonzero $\alpha \in \mathrm{CH}_0(\widetilde{X_0})$ such that $\deg(\alpha) = 0$. Since $\widetilde{X_0}$ is smooth, it follows from Lemma 4.7 in Appendix 3 that we can write α as the class of a 0-cycle β with support inside $g^{-1}(V)$. Note that in this case $g_*(\beta)$ is a degree 0 cycle on X_0, whose support is contained in V. Since U is smooth over A and A is a complete DVR, we may apply Lemma 2.5 to deduce the existence of a 0-cycle γ on X_η, with support inside V, such that $\sigma(\gamma) = g_*(\beta)$. Since the morphism $\mathrm{CH}_0(\widetilde{X_0}) \to \mathrm{CH}_0(X_0)$ is injective, it follows that $g_*(\beta)$ is not rationally equivalent to 0, and therefore γ is not rationally equivalent to 0. On the other hand, we have

$$\deg(\gamma) = \deg\big(g_*(\beta)\big) = \deg(\beta) = 0$$

and we thus conclude that X_η is not CH_0-trivial. $\square$

The following variant of Theorem 2.7 is the one that is used in practice.

Theorem 2.8. *Let $(A, \mathfrak{m}, k)$ be a DVR, with fraction field K, where k is algebraically closed, and consider the following commutative diagram with Cartesian squares*

$$\begin{array}{ccccc} X_0 & \longrightarrow & X & \longleftarrow & X_\eta \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(K) \end{array}$$

with f proper and flat. If the following conditions hold:

i) *X_η is smooth and geometrically connected.*
ii) *X_0 is geometrically integral and we have a proper, birational morphism $g\colon \widetilde{X_0} \to X_0$, with $\widetilde{X_0}$ smooth.*
iii) *The morphism g is universally CH_0-trivial.*
iv) *$\widetilde{X_0}$ is not universally CH_0-trivial,*

then $(X_\eta)_{\overline{K}}$ is not universally CH_0-trivial, where $\overline{K}$ is an algebraic closure of K. In particular, $(X_\eta)_{\overline{K}}$ is not stably rational.

Proof. Note that since k is algebraically closed, it is automatic that there is a 0-cycle of degree 1 on $\widetilde{X_0}$. By replacing $X \to \mathrm{Spec}(A)$ with

$$X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\widehat{A}) \to \mathrm{Spec}(\widehat{A}),$$

we may assume that A is complete. If $(X_\eta)_{\overline{K}}$ is universally CH_0-trivial, then by Remark 1.9, there is a finite extension K'/K such that $(X_\eta)_{K'}$ is universally CH_0-trivial (recall that by the assumption i) and Theorem 1.6, being universally CH_0-trivial is equivalent with having a decomposition of the diagonal). The integral closure of A in K is a Dedekind domain, finite over A (we use here that A is complete, hence excellent) and let A' be its localization at a maximal ideal. The inclusion $A \hookrightarrow A'$ is a local homomorphism of DVRs, inducing an isomorphism between residue fields (here is where we use the hypothesis that k is algebraically closed). By applying Theorem 2.7 for the morphism

$$X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A') \to \mathrm{Spec}(A'),$$

we deduce that $(X_\eta)_{K'}$ is not universally CH_0-trivial, a contradiction. The last assertion in the theorem follows from Corollary 1.8. □

Remark 2.9. Given a smooth, proper morphism $f\colon X \to T$ of complex varieties, with T connected, the fibers $X_t = f^{-1}(t)$ are diffeomorphic by Ehresman's theorem In particular, the group $H^3(X_t, \mathbf{Z})$ is independent of t. In particular, if $H^3(X_{t_0}, \mathbf{Z})$ has no torsion for some $t_0 \in T$, then $H^3(X_t, \mathbf{Z})$ has no torsion for all $t \in T$.

On the other hand, we will see in the next lecture that there are examples of proper morphisms $f\colon X \to T$ whose fibers are smooth quartic hypersurfaces such that the general fiber X_t is smooth (in particular, $H^3(X_t, \mathbf{Z})$ has no torsion by Remark 1.6 in Lecture 6), but for some t_0, the fiber X_{t_0} is singular and has a resolution of singularities $\widetilde{X_{t_0}} \to X_{t_0}$ with universally CH_0-trivial fibers such that $H^3(\widetilde{X_{t_0}}, \mathbf{Z})_{\mathrm{tors}} \neq 0$. In this case, it follows from Proposition 1.14 that $\widetilde{X_{t_0}}$ is not universally CH_0-trivial, and one can apply Theorem 2.7. The existence of such behavior was one of the key insights of [Voi] and it led to many other applications.

Remark 2.10. There is another result concerning the behavior of the decomposition of the diagonal in families. Suppose that $f\colon X \to T$ is a projective morphism over an algebraically closed field k, with integral fibers, and with T smooth. If f has a section $s\colon T \to X$, then there is a countable family of subschemes $(V_i)_{i \in I}$ of closed subschemes of T such that the set of k-valued points $t \in T$ with $f^{-1}(t)$ admitting a decomposition of the diagonal is equal to $\bigcup_{i \in I} V_i(k)$. The key idea is that cycles are parametrized by Chow schemes and these have countably many components. However, the proof is rather technical, so we omit it; for a detailed proof, we refer to [CTP16, Appendice B]. This result can be combined with Theorem 2.7 in order to get a more precise statement concerning the failure of stable rationality for the fibers of suitable families.

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