Chapter 7: Cohomology of line bundles

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The function associated to a $\mathbb{Q}$-T-divisor. Let $X = X(\Delta)$ be a toric variety. We have associated to every $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor D a function ψ_D defined on the support of Δ. We generalize this now to the case of a non necessarily $\mathbb{Q}$-Cartier divisor.

Let $D = \sum_{i=1}^{d} a_i D_i$ be a $\mathbb{Q}$-T-divisor. We define $\psi_D \colon |\Delta| \to \mathbb{R}$ as follows: if the primitive generators of the rays of a cone σ in Δ are $v_1, \ldots, v_r$, and if v is in σ, then

$$\psi_D(v) := \sup \left\{ -\sum_{i=1}^{r} \lambda_i a_i \mid v = \sum_{i=1}^{r} \lambda_i v_i,\ \lambda_i \geq 0 \,\text{for all}\, i \right\}.$$

Note that if v lies in a face τ of σ and $v = \sum_{i=1}^{r} \lambda_i v_i$ with $\lambda_i \geq 0$, then $\lambda_i = 0$ whenever v_i is not in τ. This implies that ψ_D is well-defined. In particular, we see that $\psi_D(v_i) = -a_i$ for all i.

Lemma 1. *The above supremum is a maximum, so in particular it is finite. If v lies in $|\Delta| \cap N_{\mathbb{Q}}$, then $\psi_D(v)$ is rational. For every u in P_D and every v in $|\Delta|$ we have*

$$\langle u, v \rangle \geq \psi_D(v).$$

Proof. Suppose that $v_1, \ldots, v_r$ are the primitive generators of a cone σ in Δ. If u is in P_D, then $\langle u, v_i \rangle \geq -a_i$. We deduce that for $v = \sum_{i=1}^{r} \lambda_i v_i$ with $\lambda_i \geq 0$ we have

$$\langle u, v \rangle \geq -\sum_{i=1}^{r} \lambda_i a_i.$$

Therefore $\langle u, v \rangle \geq \psi_D(v)$.

In general, P_D might be empty. However, in order to show that $\psi_D(v)$ is finite it is enough to restrict to U_σ, in which case the corresponding polyhedron is nonempty. In order to show that the supremum is a maximum, fix a maximal linearly independent subset of the $v_1, \ldots, v_r$. Writing the rest of the v_i in terms of these, we see that for a given v, $\psi_D(v)$ is equal to the maximum of a suitable linear function on a convex cone. In addition, if v is in $N_{\mathbb{Q}}$ then this cone is rational, so $\psi_D(v)$ is rational.

If D is $\mathbb{Q}$-Cartier, then our present definition of ψ_D agrees with the old one. It is clear that $\psi_D(\lambda v) = \lambda \psi_D(v)$ if $\lambda \geq 0$. By definition, D is $\mathbb{Q}$-Cartier if and only if the restriction of ψ_D to every cone in Δ is linear.

Exercise 1. Show that if D and E are $\mathbb{Q}$-T-divisors on X, then

$$\psi_{D+E}(v) \leq \psi_D(v) + \psi_E(v)$$

for every v in $|\Delta|$, with equality if either D or E is $\mathbb{Q}$-Cartier.

Given a $\mathbb{Q}$-T-divisor D and u in M, consider the closed cone

$$Z_D(u) := \{v \in |\Delta| \mid \langle u, v\rangle \geq \psi_D(v)\}.$$

Note that $Z_D(u)$ is a (nonempty) cone. Since $\psi_D(v_i) = -a_i$ for every i, we deduce from Lemma 1 that

$$P_D = \{u \in M_{\mathbb{R}} \mid \langle u, v\rangle \geq \psi_D(v) \,\text{for all}\, v \,\text{in}\, |\Delta|\}.$$

Therefore u is in P_D if and only if $Z_D(u) = |\Delta|$.

Lemma 2. *For every $\mathbb{Q}$-T-divisor D and for every cone σ in Δ, the restriction of ψ_D to σ is concave. In particular, for every u in M the set $\sigma \smallsetminus Z_D(u)$ is convex.*

Proof. Suppose that $v_1, \ldots, v_r$ are the primitive generators of the rays of σ. If w and w' are in $|\Delta|$ and $0 \leq t \leq 1$, write

$$w = \sum_{i=1}^{r} \lambda_i v_i, \quad w' = \sum_{i=1}^{r} \lambda'_i v_i$$

with λ_i, $\lambda'_i \geq 0$ such that

$$\psi_D(w) = -\sum_{i=1}^{r} \lambda_i a_i, \quad \psi_D(w') = -\sum_{i=1}^{r} \lambda'_i a_i.$$

Since $tw + (1-t)w' = \sum_{i=1}^{r}(t\lambda_i + (1-t)\lambda'_i)v_i$, we deduce $\psi_D(tw + (1-t)w') \geq t\psi_D(w) + (1-t)\psi_D(w')$. The last assertion is a formal consequence of the fact that ψ_D is concave.

Cohomology of $\mathcal{O}(D)$. Let D be a $\mathbb{Q}$-T-divisor on $X(\Delta)$. For every u in M we consider the subset $Z_D(u)$ of $|\Delta|$. Recall that we have

$$H^0(X, \mathcal{O}(\lfloor D \rfloor)) = \oplus_{u \in P_D} k\chi^u.$$

As we have seen, u is in P_D if and only if $|\Delta| \smallsetminus Z_D(u)$ is empty. This is the case if and only if the relative singular cohomology group $H^0(|\Delta|, |\Delta| \smallsetminus Z_D(u); k)$ is zero. We generalize this to higher cohomology groups. Let $H^p_{Z_D(u)}(|\Delta|; k)$ denote the relative singular cohomology group $H^p(|\Delta|, |\Delta| \smallsetminus Z_D(u); k)$.

Proposition 1. *If D is a $\mathbb{Q}$-T-divisor on the toric variety $X = X(\Delta)$, then for every p in $\mathbb{N}$ we have a decomposition*

$$H^p(X, \mathcal{O}(\lfloor D \rfloor)) = \oplus_{u \in M} H^p(X, \mathcal{O}(\lfloor D \rfloor))_u,$$

and canonical isomorphisms for every u in M

$$H^p(X, \mathcal{O}(\lfloor D \rfloor))_u = H^p_{Z_D(u)}(|\Delta|; k).$$

We start with an equivalent formulation in terms of relative simplicial cohomology. For a given D and for u in M, we consider the simplicial complex $\Sigma_D(u)$ on Δ such that the set of cones $\{\sigma_0, \ldots, \sigma_r\}$ is in $\Sigma_D(u)$ if and only if $\sigma_0 \cap \ldots \cap \sigma_r$ is not contained in $Z_D(u)$. Equivalently, $\{\sigma_0, \ldots, \sigma_r\}$ is not in $\Sigma_D(u)$ if and only if if for every v_i in $\sigma_0 \cap \ldots \cap \sigma_r$, the coefficient of D_i in $D + \mathrm{div}(\chi^u)$ is nonnegative. We denote also by Σ the full simplicial complex on Δ.

Lemma 3. *If D is a $\mathbb{Q}$-T-divisor on $X(\Delta)$, then for every p we have a decomposition of $H^p(X, \mathcal{O}(\lfloor D \rfloor))$ as in Proposition 1 and for every u in M we have a canonical isomorphism*

$$H^p(X, \mathcal{O}(\lfloor D \rfloor))_u = H^p(\Sigma, \Sigma_D(u); k).$$

Proof. We compute $H^p(X, \mathcal{O}(\lfloor D \rfloor))$ via Čech cohomology associated to the cover

$$X = \bigcup_{\sigma \in \Delta} U_\sigma.$$

For every σ we have a canonical decomposition

$$\Gamma(U_\sigma, \mathcal{O}(\lfloor D \rfloor)) = \oplus_u k \cdot \chi^u,$$

the direct sum being over those u in M such that $\langle u, v_i \rangle + a_i \geq 0$ for all v_i in σ. Note that this condition is equivalent with the fact that σ is contained in $Z_D(u)$.

The differentials in the Čech complex respect the M-grading, so we get a decomposition $H^p(X, \mathcal{O}(\lfloor D \rfloor)) = \oplus_{u \in M} H^p(X, \mathcal{O}(\lfloor D \rfloor))_u$. Moreover, we see that the component of degree u in the Čech complex can be identified to the complex of simplicial cochains for the pair $(\Sigma, \Sigma_D(u))$, so we get a canonical isomorphism

$$H^p(X, \mathcal{O}(\lfloor D \rfloor))_u = H^p(\Sigma, \Sigma_u; k).$$

Proof of Proposition 1. If σ is a cone not contained in $Z_D(u)$, then by Lemma 2 the set $\sigma \smallsetminus Z_D(u)$ is convex, hence either empty or contractible. Since Σ and Σ_u are, respectively, the nerves of the covers

$$|\Delta| = \bigcup_{\sigma \in \Delta} \sigma, \quad |\Delta| \smallsetminus Z_D(u) = \bigcup_{\sigma \in \Delta} (\sigma \smallsetminus Z_D(u)),$$

and all intersections of the elements in the cover are either empty or contractible, we deduce a canonical isomorphism

$$H^p(|\Delta|, |\Delta| \smallsetminus Z_D(u); k) = H^p(\Sigma, \Sigma_u; k),$$

and we conclude by Lemma 3.

Remark 1. Note that the decomposition in Proposition 1 does not depend only on $\mathcal{O}(\lfloor D \rfloor)$, but it also depends on D. More precisely, the natural isomorphism

$$H^p(X, \mathcal{O}(\lfloor D \rfloor)) \to H^p(X, \mathcal{O}(\lfloor D + \mathrm{div}(\chi^w) \rfloor))$$

takes the component of degree u to the component of degree $u - w$.

Remark 2. Note that the isomorphisms in Proposition 1 and Lemma 3 are compatible with adding an effective divisor. More precisely, suppose that $D' = D + F$, where F is an effective $\mathbb{Q}$-T-divisor. We have an inclusion $\mathcal{O}(\lfloor D \rfloor) \hookrightarrow \mathcal{O}(\lfloor D' \rfloor)$ which induces

$$\phi\colon H^p(X, \mathcal{O}(\lfloor D \rfloor))_u \to H^p(X, \mathcal{O}(\lfloor E \rfloor))_u$$

for every p in $\mathbb{N}$ and every u in M.

On the other hand, since F is effective we have $\psi_D(v) \geq \psi_E(v)$ for every v in $|\Delta|$. Therefore $|\Delta| \smallsetminus Z_{D'}(u) \subseteq |\Delta| \smallsetminus Z_D(u)$ and $\Sigma_{D'}(u) \subseteq \Sigma_D(u)$. We get induced morphisms

$$\psi_1\colon H^p_{Z_D(u)}(|\Delta|) \to H^p_{Z_{D'}(u)}(|\Delta|),$$

$$\psi_2\colon H^p(\Sigma, \Sigma_D(u); k) \to H^p(\Sigma, \Sigma_{D'}(u); k).$$

It follows from the proof of Proposition 1 and Lemma 3 that the isomorphisms in those results identify ϕ with ψ_1 and, respectively, ψ_2.

It follows from Proposition 1 that if $|\Delta|$ is convex and ψ_D is concave, then $H^p(X, \mathcal{O}(\lfloor D \rfloor)) = 0$ for $p \geq 1$. Indeed, since $|\Delta|$ is contractible, the long exact sequence of cohomology of a pair shows that $H^p(|\Delta|, |\Delta| \smallsetminus Z_D(u); k)$ is isomorphic to the reduced cohomology group $\widetilde{H}^{p-1}(|\Delta| \smallsetminus Z_D(u); k)$. This is zero for $p \geq 1$, since $|\Delta| \smallsetminus Z_D(u)$ is convex, hence either empty or contractible.

For example, we may apply this if X is affine, and we get that $H^p(X, \mathcal{O}(D)) = 0$ for all $p \geq 1$ and all T-divisors D. Of course, this is a particular case of the fact that every coherent sheaf on an affine variety has no higher cohomology.

The same argument can be applied to deduce the following corollary.

Corollary 1. *Suppose that $|\Delta|$ is convex. If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor such that some positive multiple mD is Cartier and $\mathcal{O}(mD)$ is globally generated, then $H^p(X, \mathcal{O}(\lfloor D \rfloor)) = 0$ for every $p \geq 1$.*

Proof. After writing $X = X' \times (k^*)^r$ for a suitable X', we reduce the statement to the case when $|\Delta|$ has dimension equal to $\mathrm{rk}(N)$. In this case ψ_D is concave, so the assertion follows from Proposition 1.

Corollary 2. *If $f = \phi_* \colon Y \to X$ is a proper morphism of toric varieties such that the corresponding lattice homomorphism ϕ is surjective, then*

$$f_*(\mathcal{O}_Y) = \mathcal{O}_X \quad \text{and} \quad R^p f_*(\mathcal{O}_Y) = 0 \quad \text{for all}\, p > 0.$$

In particular, every toric variety has rational singularities.

Proof. We may assume that $X = U_\sigma$, where σ is a cone of maximal dimension. Note that the first equality was proved in Chapter 4, Corollary 3. The second equality follows from Corollary 1, since the support of the fan defining Y is equal to $\phi^{-1}(\sigma)$, so it is convex. The last assertion follows from the fact that every toric variety has a toric resolution of singularities (see Chapter 5).

Corollary 3. *Let D and E be $\mathbb{Q}$-T-divisors on X and u, w in M. We write*

$$D + \mathrm{div}(\chi^u) = \sum_{i=1}^{d} \alpha_i D_i, \quad E + \mathrm{div}(\chi^w) = \sum_{i=1}^{d} \beta_i D_i.$$

If for all i, we have $\alpha_i \geq 0$ if and only if $\beta_i \geq 0$, then for every p in $\mathbb{N}$ there is a canonical isomorphism

$$H^p(X, \mathcal{O}(\lfloor D \rfloor))_u = H^p(X, \mathcal{O}(\lfloor E \rfloor))_w.$$

Proof. The assertion follows from Lemma 3 and the fact that the hypothesis implies $\Sigma_D(u) = \Sigma_E(w)$.

The following corollary strengthens the vanishing result from [Mu].

Corollary 4. *Let D be an ample $\mathbb{Q}$-T-divisor on X. If D' is a T-divisor (with integer coefficients) such that*

$$K_X + \lceil D \rceil \leq D' \leq \lceil D \rceil,$$

then $H^p(X, \mathcal{O}(D')) = 0$ for every $p \geq 1$.

Proof. We need to show that for every u in M we have $H^p(X, \mathcal{O}(D'))_u = 0$ for $p \geq 1$. Let us fix u in M.

If we write $D = \sum_{i=1}^d a_i D_i$ and $D' = \sum_{i=1}^d b_i D_i$, then $b_i = \lceil a_i \rceil - 1$ if a_i is not an integer, and $b_i \in \{a_i - 1, a_i\}$, otherwise. Let I be the set of those i such that $b_i = \lceil a_i \rceil - 1$.

Since D is ample, there is a positive integer m such that mD is Cartier and

$$H^p(X, \mathcal{O}(mD - \sum_{j \in I} D_j)) = 0$$

for every $p \geq 1$. In particular, the degree mu component of this cohomology group vanishes. By taking m large enough, we may assume in addition that for every j in I such that $a_j + \langle u, v_j \rangle > 0$, we have $a_j + \langle u, v_j \rangle \geq 1/m$.

By Corollary 3, in order to show that $H^p(X, \mathcal{O}(D'))_u = 0$ it is enough to show that for every $1 \leq i \leq d$, the coefficient α_i of D_i in $mD - \sum_{j \in I} D_j + \mathrm{div}(\chi^{mu})$ is nonnegative if and only if the coefficient β_i of D_i in $D' + \mathrm{div}(\chi^u)$ is nonnegative.

Suppose first that i is not in I. In this case a_i is an integer and $b_i = a_i$, so $\alpha_i = m\beta_i$, and our assertion follows. On the other hand, if i is in I, then $\alpha_i \geq 0$ if and only if $a_i + \langle u, v_i \rangle \geq 1/m$. By our choice of m, this is the case if and only if $\langle u, v_i \rangle > -a_i$. This is equivalent with $\langle u, v_i \rangle \geq \lfloor -a_i \rfloor + 1$, and since $\beta_i = \lceil a_i \rceil - 1 + \langle u, v_i \rangle$, this condition is further equivalent with $\beta_i \geq 0$, so the proof of the proposition is complete.

Remark 3. Suppose that D is an ample $\mathbb{Q}$-T-divisor on the projective toric variety X. If in the above corollary we take $D' = K_X + \lceil D \rceil$, then we get $H^p(X, \mathcal{O}(K_X + \lceil D \rceil)) = 0$ for every $p \geq 1$. This is the toric version of the Kawamata-Viehweg Vanishing Theorem. On the other hand, if we consider $D' = \lfloor D \rfloor$, we get $H^p(X, \mathcal{O}(\lfloor D \rfloor)) = 0$ for every $p \geq 1$, which is a particular case of Corollary 1.

Remark 4. If D is a $\mathbb{Q}$-T-divisor and if m is a positive integer, then by Corollary 3 we have a canonical isomorphism

$$H^p(X, \mathcal{O}(\lfloor D \rfloor))_u = H^p(X, \mathcal{O}(\lfloor mD \rfloor))_{mu}$$

This can be used in order to reduce the computation of the graded components in the cohomology from a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor to one which is Cartier.

Serre duality. If X is a smooth projective n-dimensional toric variety, then $\Omega_X^n = \mathcal{O}(K_X)$ is the dualizing sheaf of X (see [Ha], Chapter III.7). In particular, for every vector bundle E on X and every p we have a canonical isomorphism

$$H^p(X, E) \simeq H^{n-p}(X, E^\vee \otimes \mathcal{O}(K_X))^*.$$

We give a direct proof of this result when E is a line bundle, in a more general setup.

Proposition 2. *Let X be a complete n-dimensional toric variety. If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor, then for every p there is a canonical isomorphism of M-graded vector spaces*

$$H^p(X, \mathcal{O}(\lfloor D \rfloor)) = H^{n-p}(X, \mathcal{O}(K_X - \lfloor D \rfloor))^*.$$

Proof. We need to show that for every u in M we have a canonical isomorphism

$$H^p(X, \mathcal{O}(\lfloor D \rfloor))_u = H^{n-p}(X, \mathcal{O}(K_X - \lfloor D \rfloor))^*_{-u}. \tag{1}$$

We start by proving the proposition in the case when X is smooth and D is an integral divisor. In this case ψ_D is linear on each cone in Δ.

Note that $Z_D(u) = \{0\}$ if and only if $\langle u, v_i \rangle + a_i < 0$ (equivalently, $\langle -u, v_i \rangle - a_i - 1 \geq 0$) for every i, which is the case if and only if $Z_{K_X - D}(-u) = N_{\mathbb{R}}$. In this case Proposition 1 shows that both sides of (1) are naturally isomorphic to k if $p = n$ and are zero, otherwise. Therefore, we may assume from now on that $Z_D(u)$ is different from $\{0\}$ and $N_{\mathbb{R}}$.

Using Proposition 1 and the long exact sequence of a pair we get canonical isomorphisms

$$H^p(X, \mathcal{O}(D))_u = H^p(N_{\mathbb{R}}, N_{\mathbb{R}} \smallsetminus Z_D(u); k) = \widetilde{H}^{p-1}(N_{\mathbb{R}} \smallsetminus Z_D(u); k),$$

$$\begin{aligned} H^{n-p}(X, \mathcal{O}(K_X - D))_{-u} &= H^{n-p}(N_{\mathbb{R}}, N_{\mathbb{R}} \smallsetminus Z_{K_X - D}(-u); k) \\ &= \widetilde{H}^{n-p-1}(N_{\mathbb{R}} \smallsetminus Z_{K_X - D}(-u); k). \end{aligned}$$

Note that both $N_{\mathbb{R}} \smallsetminus Z_D(u)$ and $Z_D(u) \smallsetminus \{0\}$ do not contain the origin and are preserved by multiplication with positive real numbers. Therefore they are homotopic equivalent with their intersections with the unit sphere S^{n-1}. Using this and Alexander duality we get canonical isomorphisms

$$\begin{aligned} H^p(X, \mathcal{O}(D))_u &= \widetilde{H}^{p-1}(S^{n-1} \smallsetminus Z_D(u); k) = \widetilde{H}_{n-p-1}(S^{n-1} \cap Z_D(u); k) \\ &= \widetilde{H}^{n-p-1}(S^{n-1} \cap Z_D(u); k)^* = \widetilde{H}^{n-p-1}(Z_D(u) \smallsetminus \{0\}; k)^*. \end{aligned}$$

In order to finish the proof in this case, it is enough to find a set which is a deformation retract of both $Z_D(u) \smallsetminus \{0\}$ and $N_{\mathbb{R}} \smallsetminus Z_{K_X - D}(-u)$. This is the content of the next lemma.

Lemma 4. *Let $X = X(\Delta)$ be a smooth toric variety, D a T-divisor on X and u in M. If Z is the union of the cones in Δ which are contained in $Z_D(u)$, then $Z \smallsetminus \{0\}$ is a deformation retract of both $Z_D(u) \smallsetminus \{0\}$ and $|\Delta| \smallsetminus Z_{K_X - D}(-u)$.*

Proof. Note that $Z_D(u) \smallsetminus \{0\}$ is contained in $N_{\mathbb{R}} \smallsetminus Z_{K_X-D}(-u)$. Indeed, if $\langle u, v\rangle \geq \psi_D(v)$ and $\langle -u, v\rangle \geq \psi_{K_X-D}(v)$, by adding these two inequalities we get $\psi_{K_X}(v) \leq 0$ which implies $v = 0$.

Let σ be a cone in Δ with the primitive generators of the rays $v_1, \ldots, v_r$. We denote by τ the face of σ generated by those v_i in $Z_D(u)$. We define a retract r_σ of the inclusion

$$j_\sigma \colon \tau \smallsetminus \{0\} \hookrightarrow \sigma \smallsetminus Z_{K_X-D}(-u).$$

If $v = \sum_{i=1}^r \lambda_i v_i$ with $\lambda_i \geq 0$, then we put

$$r_\sigma(v) = \sum_{v_i \in \tau} \lambda_i v_i.$$

Note that if v does not lie in $Z_{K_X-D}(-u)$, then $r_\sigma(v)$ is nonzero: otherwise $\langle u, v_i\rangle + a_i \leq -1 = -\psi_{K_X}(v_i)$ whenever $\lambda_i \neq 0$. This implies that $\langle u, v\rangle \leq \psi_{D-K_X}(v)$, a contradiction. It is clear that the retracts r_σ can be glued to a map $r \colon |\Delta| \smallsetminus Z_{K_X-D}(-u) \to Z \smallsetminus \{0\}$. Moreover, there are homotopies $j_\sigma \circ r_\sigma \sim \mathrm{id}$ for the various cones σ in Δ, and these homotopies can be glued together. Therefore $Z \smallsetminus \{0\}$ is a deformation retract of $N_{\mathbb{R}} \smallsetminus Z_{K_X-D}(-u)$. Similarly, if we consider the restriction of r to $Z_D(u) \smallsetminus \{0\}$ we see that $Z \smallsetminus \{0\}$ is a deformation retract of $Z_D(u) \smallsetminus \{0\}$. This completes the proof.

In order to deduce the general case in the proposition, we will need the following lemma.

Lemma 5. *If $f \colon Y \to X$ is a proper, birational morphism of toric varieties, then $R^p f_* \mathcal{O}(K_Y) = \mathcal{O}(K_X)$ if $p = 0$, and it is zero, otherwise.*

Proof. We prove first the assertion in the case when Y is smooth. We may assume that $X = U_\sigma$, so Y is defined by a fan Δ with $|\Delta| = \sigma$, and we need to compute $H^p(Y, \mathcal{O}(K_Y))$. If $p = 0$, then $H^0(Y, \mathcal{O}(K_Y)) = \oplus_u k\chi^u$, the sum being over those u in M such that $\langle u, v_i\rangle \geq 1$ for every primitive generator v_i of a ray in Δ. This is equivalent with $\langle u, w_j\rangle > 0$ for every nonzero primitive generator w_j of a ray of σ, as both conditions are equivalent with $\langle u, v\rangle > 0$ for every nonzero v in σ. This proves $H^0(Y, \mathcal{O}(K_Y)) = H^0(U_\sigma, K_{U_\sigma})$.

If $p \geq 1$, then for every u in M, Proposition 1 gives $H^p(Y, \mathcal{O}(K_Y))_u = \widetilde{H}^{p-1}(\sigma \smallsetminus Z_{K_Y}(u))$. It follows from Lemma 4 that $\sigma \smallsetminus Z_{K_Y}(u)$ is homotopic equivalent to $\{v \in \sigma \smallsetminus \{0\} \mid \langle u, v\rangle \leq 0\}$. This is the complement of the origin in a pointed cone, so it is contractible, and therefore $H^p(Y, \mathcal{O}(K_Y))_u$ is zero for every u in M.

In order to deduce the general case, let $g \colon \widetilde{Y} \to Y$ be a toric resolution of singularities for Y. Applying what we have proved for g and fg we get $R^p g_* \mathcal{O}(K_{\widetilde{Y}}) = 0$ and $R^p(fg)_* \mathcal{O}(K_{\widetilde{Y}}) = 0$ for $p \geq 1$, so the

Leray spectral sequence for the composition $f \circ g$ degenerates and gives canonical isomorphisms

$$R^p f_*(g_*\mathcal{O}(K_{\widetilde{Y}})) = R^p(fg)_*\mathcal{O}(K_{\widetilde{Y}}).$$

Moreover, since $g_*\mathcal{O}(K_{\widetilde{Y}}) = \mathcal{O}(K_Y)$ and $(fg)_*\mathcal{O}(K_{\widetilde{Y}}) = \mathcal{O}(K_X)$, we get our assertion for f.

We return to the proof of Proposition 2. Suppose first that X is non necessarily smooth, but D is Cartier, and let $f\colon Y \to X$ be a toric resolution of singularities of X. Using Corollary 2, Lemma 5 and the projection formula we see that the Leray spectral sequences computing the cohomology of $\mathcal{O}(f^*D)$ and $\mathcal{O}(K_Y - f^*D)$ degenerate. Therefore Serre duality on the smooth variety Y gives the following sequence of canonical isomorphisms

$$H^p(X, \mathcal{O}(D)) = H^p(Y, \mathcal{O}(f^*D)) = H^{n-p}(Y, \mathcal{O}(K_Y - f^*D))^*$$
$$= H^{n-p}(X, f_*\mathcal{O}(K_Y) \otimes \mathcal{O}(-D))^* = H^{n-p}(X, \mathcal{O}(K_X - D))^*.$$

Suppose now that D is an arbitrary $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor on X, and fix u in M. We choose a positive integer m, which we may assume large enough, such that mD is Cartier. Using Remark 4 and what we have already proved, we have canonical isomorphisms

$$H^p(X, \mathcal{O}(\lfloor D \rfloor))_u = H^p(X, \mathcal{O}(mD))_{mu} = H^{n-p}(X, \mathcal{O}(K_X - mD))^*_{-mu}.$$

By Corollary 3, in order to finish the proof it is enough to show that the coefficient of a divisor D_i in $K_X - mD + \mathrm{div}(\chi^{-mu})$ is nonnegative if and only if the coefficient of D_i in $K_X - \lfloor D \rfloor + \mathrm{div}(\chi^{-u})$ is nonnegative. Let a_i be the coefficient of D_i in D. The former condition says that $-1 - ma_i - m\langle u, v_i \rangle \geq 0$. Dividing by m and using the fact that m can be taken large enough, we see that this is equivalent with $\langle u, v_i \rangle < -a_i$. Since $\langle u, v_i \rangle$ is an integer, this is further equivalent with $\langle u, v_i \rangle \leq \lceil -a_i \rceil - 1$, which is precisely the condition that the coefficient of D_i in $K_X - \lfloor D \rfloor + \mathrm{div}(\chi^{-u})$ is nonnegative. This concludes the proof.

We deduce the following version of the Kawamata-Viehweg Vanishing Theorem.

Corollary 5. *If D is a $\mathbb{Q}$-Cartier big $\mathbb{Q}$-T-divisor on a projective toric variety X, and if D is big and nef, then $H^p(X, \mathcal{O}(-\lceil D \rceil)) = 0$ for $p < \dim(X)$.*

Proof. We show that $H^p(X, \mathcal{O}(\lfloor -D \rfloor))_u = 0$ for every u in M and every $p \geq 1$. Let m be a positive integer such that mD is Cartier. Since

$$H^p(X, \mathcal{O}(\lfloor -D \rfloor))_u = H^p(X, \mathcal{O}(-mD))_{-mu},$$

it follows that we may assume that D is a Cartier divisor. We have seen in Chapter 6 that there is a proper birational morphism of toric varieties $f\colon X \to Y$ and an ample divisor A such that f^*A is linearly equivalent with D. Moreover, it follows from Corollary 2 that $R^i f_*\mathcal{O}_X = 0$ for $i \geq 1$. We deduce from the projection formula that the spectral sequence computing the cohomology of $\mathcal{O}(-D)$ degenerates and we have

$$H^p(X, \mathcal{O}(-D)) = H^p(Y, \mathcal{O}(-A)),$$

so we have reduced the assertion to the case when D is ample, when it follows by Serre duality from Remark 3.

Interior points and the genus of a hypersurface. We give some applications of Serre duality and of the vanishing results we have on toric varieties.

Proposition 3. *If D is an ample $\mathbb{Q}$-T-divisor on a projective toric variety $X = X(\Delta)$, then*

$$\mathrm{Card}(\mathrm{Int}(P_D) \cap M) = h^0(X, \mathcal{O}(K_X + \lceil D \rceil)) = \chi(X, \mathcal{O}(K_X + \lceil D \rceil)).$$

Proof. If $D = \sum_{i=1}^d a_i D_i$, then

$$H^0(X, \mathcal{O}(K_X + \lceil D \rceil)) = \oplus_u k \cdot \chi^u,$$

the sum being over those u in M such that $\langle u, v_i \rangle + \lceil a_i \rceil - 1 \geq 0$ for all i. This condition is equivalent with $\langle u, v_i \rangle + a_i > 0$ for all i, so with the fact that u lies in the interior of P_D. The second condition follows by Corollary 4.

Example 1. Let P be a reflexive polytope in M, and let $X = X_P$ be the corresponding Fano variety. Since $D_P = -K_X$, we see that $\mathcal{O}_X(K_X + D_P) = \mathcal{O}_X$, so $h^0(X, \mathcal{O}(K_X + D_P)) = 1$. By the above proposition, we recover the fact that every reflexive polytope has only one interior lattice point (namely the origin).

We consider now an effective ample Cartier divisor D on a complete Gorenstein toric variety X. Note that we do not assume that D is a T-divisor. By adjunction, the dualizing sheaf on D is $\mathcal{O}(K_X + D)|_D$ (see [Ha], Chapter III.7). For example, if X is a Fano variety and D is in $|\mathcal{O}(-K_X)|$, then $\omega_D \simeq \mathcal{O}_D$.

Proposition 4. *Let X be a Gorenstein projective toric variety of dimension $n \geq 2$, and let D be an effective ample Cartier divisor on X. We have $h^i(D, \mathcal{O}_D) = 0$ for $1 \leq i \leq n-2$, and the arithmetic genus of D is*

$$p_a(D) = \mathrm{Card}(\mathrm{Int}(P_D) \cap M) = h^0(D, \omega_D).$$

Proof. We use the long exact sequence of cohomology for

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0. \tag{2}$$

Since $h^i(X, \mathcal{O}(-D)) = 0$ for $i < n$ by Corollary 5 and $h^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$ by Corollary 1, we deduce $h^0(D, \mathcal{O}_D) = 1$, $h^i(D, \mathcal{O}_D) = 0$ for $1 \leq i \leq n-2$ and $h^{n-1}(D, \mathcal{O}_D) = h^n(X, \mathcal{O}(-D)) = h^0(X, \mathcal{O}(K_X + D))$, by Serre duality. Hence $p_a(D) = (-1)^{n-1}(\chi(D, \mathcal{O}_D) - 1) = h^0(X, \mathcal{O}(K_X + D))$ and this is equal to $\mathrm{Card}(\mathrm{Int}(P_D) \cap M)$ by Proposition 3.

Using the long exact sequence of cohomology for

$$0 \to \mathcal{O}(K_X) \to \mathcal{O}(K_X + D) \to \omega_D \to 0$$

and the fact that $h^i(X, \mathcal{O}(K_X)) = h^{n-i}(X, \mathcal{O}_X) = 0$ for $i < n$, we deduce $h^0(D, \omega_D) = h^0(X, \mathcal{O}(K_X + D))$.

Example 2. Let d be a positive integer and P the polytope in $\mathbb{Z}^2$ with vertices at $(0,0)$, $(d,0)$ and $(0,d)$, so $X_P = \mathbb{P}^2$. If D is in $|\mathcal{O}(D_P)|$, then D is a curve of degree d. We see that the arithmetic genus of D is $(d-1)(d-2)/2$.

Let d and e be positive integers and P the rectangle in $\mathbb{Z}^2$ with vertices at $(0,0)$, $(d,0)$, $(0,e)$ and (d,e), so $X_P = \mathbb{P}^1 \times \mathbb{P}^1$. If D is in $|\mathcal{O}(D_P)|$, then D is a curve of bidegree (d,e). We see that the arithmetic genus of D is $(d-1)(e-1)$.

Problem. (I. Dolgachev) *Let X be a Gorenstein projective toric variety of dimension $n \geq 2$, $\mathcal{L}$ an ample line bundle on X and D a general element in $|\mathcal{L}|$. Give a combinatorial description of $h^0(D, \omega_D^{\otimes m})$ for all m. Do this at least when X is a smooth threefold and $m = 2$.*

We will see later that the Riemann-Roch Theorem can be used to count the number of lattice points inside a lattice polytope. We give now the two-dimensional case, when the resulting formula is very explicit.

Proposition 5. (Pick's formula) *If P is a two-dimensional lattice polytope, we have*

$$\mathrm{Card}(P \cap M) = \mathrm{Area}(P) + \frac{1}{2} \cdot \mathrm{Perimeter}(P) + 1.$$

Proof. Note that the area is the two-dimensional volume, while the perimeter is the sum of the one-dimensional volumes of the edges. Therefore the perimeter of P is equal to $\mathrm{Card}(\partial(P) \cap M)$. We have a complete two-dimensional toric variety X and and ample Cartier divisor D on X such that P_D is a translate of D. It follows that $\mathrm{Area}(P) =$

$(D^2)/2$, and $\mathrm{Card}(P \cap M)$ is equal to $h^0(X, \mathcal{O}(D)) = \chi(X, \mathcal{O}(D))$. Since

$$\mathrm{Card}(\partial(P) \cap M) = h^0(X, \mathcal{O}(D)) - h^0(X, \mathcal{O}(K_X + D)) =$$
$$\chi(X, \mathcal{O}(D)) - \chi(X, \mathcal{O}(K_X - D)),$$

it follows that Pick's formula is equivalent with

$$\chi(X, \mathcal{O}(D)) + \chi(X, \mathcal{O}(-D)) = (D^2) + 2. \tag{3}$$

After taking a toric resolution of singularities $Y \to X$, we see using Proposition 2 that it is enough to prove (3) on Y, so we may assume that X is smooth. In this case the Riemann-Roch formula for $\mathcal{O}(D)$ and $\mathcal{O}(-D)$ gives

$$\chi(X, \mathcal{O}(D)) = \chi(X, \mathcal{O}_X) + \frac{1}{2}(D \cdot (D - K_X)) = 1 + \frac{1}{2}(D \cdot (D - K_X)),$$
$$\chi(X, \mathcal{O}(-D)) = 1 + \frac{1}{2}(D \cdot (D + K_X)).$$

Adding the two formulas gives (3) and completes the proof.

Remark 5. With the notation in the above proof, it follows from Exercise 24 in Chapter 6 that if D_i is the divisor on X corresponding to the edge Q of P, then the length of Q is given by $(D_i \cdot D)$. Therefore the perimeter of P is equal to $-(K_X \cdot D)$. If X is smooth, the Riemann-Roch formula for $\mathcal{O}(D)$ gives directly Pick's formula.

References

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