

LECTURE 6: THE ARTIN-MUMFORD EXAMPLE

In this chapter we discuss the example of Artin and Mumford [AM72] of a complex unirational 3-fold which is not rational (in fact, it is not even stably rational). As we will see, the invariant that detects non-stable-rationality is the torsion in the 3rd singular cohomology group. In order to prove that this torsion is nontrivial in the Artin-Mumford example, we follow Beauville's argument in [Bea15], making use of the Brauer group.

1. A TOPOLOGICAL INVARIANT THAT DETECTS NON-STABLY-RATIONAL VARIETIES

In this section all varieties are defined over \mathbf{C} . Our goal is to show that if X is a smooth, complex projective variety such that $H^3(X, \mathbf{Z})$ has nonzero torsion, then X is not stably rational. More generally, we will show that two stably birational varieties have the torsion in the third singular cohomology groups isomorphic.

Definition 1.1. Two varieties X and Y are stably birational if there are non-negative integers m and n such that $X \times \mathbf{P}^m$ and $Y \times \mathbf{P}^n$ are birational. With this terminology, a variety is stably rational if and only if it is stably birational to a point.

Proposition 1.2. *If X and Y are smooth, complete, complex algebraic varieties that are stably birational, then there is an isomorphism of Abelian groups*

$$H^3(X, \mathbf{Z})_{\text{tors}} \simeq H^3(Y, \mathbf{Z})_{\text{tors}}.$$

Proof. The assertion follows from the definition of stable birationality if we prove the following two statements:

i) If X is any complex algebraic variety and n is a non-negative integer, then

$$H^3(X, \mathbf{Z})_{\text{tors}} \simeq H^3(X \times \mathbf{P}^n, \mathbf{Z})_{\text{tors}}.$$

ii) If X and Y are smooth, complete, birational complex algebraic varieties, then

$$H^3(X, \mathbf{Z})_{\text{tors}} \simeq H^3(Y, \mathbf{Z})_{\text{tors}}.$$

For both these assertions, we will make use of the following consequence of the Universal Coefficient theorem: for every topological space W , the group $H^1(W, \mathbf{Z})$ has no torsion (see Remark 1.3 in Appendix 1).

In order to prove i), note that by Proposition 4.1 in Appendix 1, we have $H^i(\mathbf{P}^n, \mathbf{Z}) \simeq \mathbf{Z}$ if i is even, with $0 \leq i \leq 2n$, and $H^i(\mathbf{P}^n, \mathbf{Z}) \simeq 0$, otherwise. In particular, all cohomology groups $H^i(\mathbf{P}^n, \mathbf{Z})$ are free and finitely generated, hence by the Künneth theorem, we have

$$H^3(X \times \mathbf{P}^n, \mathbf{Z}) \simeq \bigoplus_{i+j=3} H^i(X, \mathbf{Z}) \otimes_{\mathbf{Z}} H^j(\mathbf{P}^n, \mathbf{Z}) \simeq H^3(X, \mathbf{Z}) \oplus H^1(X, \mathbf{Z}).$$

Since $H^1(X, \mathbf{Z})_{\text{tors}} = 0$, we conclude that

$$H^3(X \times \mathbf{P}^n, \mathbf{Z})_{\text{tors}} \simeq H^3(X, \mathbf{Z})_{\text{tors}}.$$

We now prove ii). It follows from the Weak Factorization theorem that any two smooth complete varieties are joined by a sequence of blow-ups and blow-downs of smooth, complete varieties, along smooth centers. Therefore it is enough to consider the case when Y is the blow-up of X along the smooth subvariety W . In this case, it follows from Proposition 4.3 in Appendix 1 that

$$H^3(Y, \mathbf{Z}) \simeq H^3(X, \mathbf{Z}) \oplus H^1(W, \mathbf{Z}).$$

Using the fact that $H^1(W, \mathbf{Z})$ has no torsion, we deduce an isomorphism

$$H^3(Y, \mathbf{Z})_{\text{tors}} \simeq H^3(X, \mathbf{Z})_{\text{tors}}.$$

□

Corollary 1.3. *If X is a smooth, complete, complex algebraic variety that is stably rational, then $H^3(X, \mathbf{Z})_{\text{tors}} = 0$.*

Proof. The assertion follows from the proposition and the fact that if Y is a point, then $H^3(Y, \mathbf{Z}) = 0$. □

Remark 1.4. It follows from the Universal Coefficient theorem (see Remark 1.4 in Appendix 1) that

$$H^3(X, \mathbf{Z})_{\text{tors}} \simeq \text{Ext}_{\mathbf{Z}}^1(H_2(X, \mathbf{Z}), \mathbf{Z}),$$

hence $H^3(X, \mathbf{Z})_{\text{tors}} \neq 0$ if and only if $H_2(X, \mathbf{Z})_{\text{tors}} \neq 0$.

Remark 1.5. One can avoid making use of the Weak Factorization theorem in the proof of Proposition 1.2, by using one of the classical results of Hironaka on resolution of singularities. This says that given any rational map $\varphi: X \dashrightarrow Y$ between smooth, complete varieties, there is a sequence of blow-ups with smooth centers

$$X_r \rightarrow X_{r-1} \rightarrow \dots \rightarrow X_0 = X$$

such that the induced rational map $X_r \dashrightarrow Y$ is a morphism. By applying this to a birational map $X \dashrightarrow Y$, the argument in the proof of the proposition implies that we have an isomorphism

$$H^3(X_r, \mathbf{Z})_{\text{tors}} \simeq H^3(X, \mathbf{Z})_{\text{tors}}.$$

On the other hand, since X_r and Y are smooth and complete, the birational morphism $X_r \rightarrow Y$ induces an injective map $H^3(Y, \mathbf{Z}) \hookrightarrow H^3(X_r, \mathbf{Z})$. We thus conclude that there is an injective map $\alpha: H^3(Y, \mathbf{Z})_{\text{tors}} \hookrightarrow H^3(X, \mathbf{Z})_{\text{tors}}$. By applying the same argument to the inverse rational map $Y \dashrightarrow X$, we conclude that α is an isomorphism. The advantage of this approach is that it can be applied also in positive characteristic, in dimension 3, when resolution of singularities is known, but Weak Factorization is not (of course, in this case one has to replace singular cohomology by its ℓ -adic counterpart).

Remark 1.6. The criterion in Corollary 1.3 can't be applied to prove irrationality of hypersurfaces $X \subseteq \mathbf{P}^n$, with $n \geq 4$. Indeed, in this case, it follows from the Lefschetz hyperplane theorem that the maps

$$H_i(X, \mathbf{Z}) \rightarrow H_i(\mathbf{P}^n, \mathbf{Z})$$

are injective for $i \leq n - 2$. On the other hand, the description of $H^*(\mathbf{P}^n, \mathbf{Z})$ in Proposition 4.1 in Appendix 1, together with the Universal Coefficient theorem, imply that $H_i(\mathbf{P}^n, \mathbf{Z})_{\text{tors}} = 0$ for all i . Therefore $H_2(X, \mathbf{Z})_{\text{tors}} = 0$ for $n \geq 4$, hence also $H^3(X, \mathbf{Z})_{\text{tors}} = 0$ for $n \geq 4$.

Remark 1.7. Arguing as in the proof of Proposition 1.2, we see that if X and Y are stably birational smooth, complete varieties, then

$$H^2(X, \mathbf{Z})_{\text{tors}} \simeq H^2(Y, \mathbf{Z})_{\text{tors}}.$$

However, this invariant is not useful for rationality problems. In order to explain this, let us recall that a smooth, projective variety X is *rationaly connected* if any two general points lie in the image of some map $f: \mathbf{P}^1 \rightarrow X$. This is a much better behaved notion than both rationality and unirationality and it is natural to restrict to rationally connected varieties when discussing rationality (note that it follows from definition that a unirational variety is automatically rationally connected). However, rationally connected varieties are simply connected (see for example [Deb01, Corollary 4.18]). Since $H_1(X, \mathbf{Z})$ is isomorphic to the Abelianization of $\pi_1(X)$, we conclude that if X is a smooth, projective, rationally connected variety, then $H_1(X, \mathbf{Z}) = 0$, hence $H^2(X, \mathbf{Z})_{\text{tors}} = 0$ by the Universal Coefficient theorem.

2. LINES ON QUADRICS IN \mathbf{P}^3

In this section we work over an algebraically closed field k , of characteristic $\neq 2$. Let $\mathbf{P} = \mathbf{P}(H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2))^*) \simeq \mathbf{P}^9$ be the projective space parametrizing quadric hypersurfaces in \mathbf{P}^3 . We denote by $G = G(2, 4)$ the Grassmann variety parametrizing lines in \mathbf{P}^3 (hence G is a 4-dimensional smooth, projective, rational variety) and by I the incidence correspondence in $\mathbf{P} \times G$. Recall that if $p: I \rightarrow \mathbf{P}$ and $q: I \rightarrow G$ are induced by the projections to the two components, then it follows from Proposition 1.1 in Lecture 4 that q is a projective bundle, with fibers isomorphic to \mathbf{P}^6 . In particular, I is a smooth, projective variety of dimension 10.

if we choose coordinates x_0, \dots, x_3 on \mathbf{P}^3 , every nonzero $f \in H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2))$ can be written uniquely as

$$f = \sum_{i,j=0}^3 a_{i,j} x_i x_j, \quad \text{with } a_{i,j} = a_{j,i} \quad \text{for all } i, j.$$

The *rank* of f (and of the corresponding quadric defined by f) is the rank of the symmetric matrix $(a_{i,j})$. This is independent of the choice of coordinates. Moreover, if $\text{rank}(f) = d$, then we can find coordinates on \mathbf{P}^3 such that $f = \sum_{i=0}^{d-1} x_i^2$. Note that a quadric is smooth if and only if it has rank 4.

For $1 \leq i \leq 4$, let $W_i \subseteq \mathbf{P}$ denote the set of quadrics of rank $\leq i$. It is clear that each W_i is a closed subset of \mathbf{P} and $W_4 = \mathbf{P}$. In the following proposition we collect some basic facts about the sets W_i .

Proposition 2.1. *With the above notation, the following hold:*

- i) *We have $\dim(W_3) = 8$, $\dim(W_2) = 6$, and $\dim(W_1) = 3$.*
- ii) *The closed subset W_2 is irreducible, with $\deg(W_2) = 10$.*
- iii) *The closed subset W_3 of \mathbf{P}^3 is an irreducible hypersurface of degree 4, which is smooth at the points in $W_3 \setminus W_2$.*
- iv) *For every $Q \in W_2 \setminus W_1$, the tangent cone to W_3 at Q is isomorphic to the hypersurface in \mathbf{A}^9 given by $x_1^2 + x_2^2 + x_3^2 = 0$.*

Proof. Let $V = H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$. A quadric of rank 1 is a double plane, hence we have a bijective morphism $\mathbf{P}(V^*) \rightarrow W_1$, which gives $\dim(W_1) = 3$. Similarly, a quadric of rank ≤ 2 is a union of 2 planes in \mathbf{P}^3 , hence we have a generically 2-to-1 map $\varphi: \mathbf{P}(V^*) \times \mathbf{P}(V^*) \rightarrow W_2$, and thus W_2 is irreducible, with $\dim(W_2) = 6$. Since φ is defined by the line bundle $\mathcal{O}(1, 1)$ on $\mathbf{P}(V^*) \times \mathbf{P}(V^*)$ and $\deg(\varphi) = 2$ (this is clear if $\text{char}(k) = 0$ and we leave it as an exercise in the general case), we have

$$\deg(W_2) = \frac{1}{2}(\mathcal{O}(1, 1)^6) = \frac{1}{2} \binom{6}{3} = 10.$$

Note also that since W_3 is the zero-locus of the degree 4 equation $\det(a_{i,j}) = 0$, it follows that $\dim(W_3) = 8$.

We temporarily denote by Z the hypersurface defined (scheme-theoretically) by $\det(a_{i,j}) = 0$. We show that Z is smooth at the points in $W_3 \setminus W_2$. Given such a point P , we choose coordinates on \mathbf{P}^3 such that P is given by the polynomial $x_0^2 + x_1^2 + x_2^2$. In this case, it follows that in the chart obtained by making $a_{0,0} = 1$, the equation defining Z at P lies in $a_{3,3} + \mathfrak{m}_P^2$, where \mathfrak{m}_P is the ideal defining P in the ambient projective space. This implies that Z is smooth at P . In particular, we see that $\dim(Z_{\text{sing}}) \leq \dim(W_2) = 6$, hence Z is reduced and irreducible. Therefore we do not need to distinguish between Z and W_3 .

Given a point $Q \in W_2 \setminus W_1$, we choose coordinates on \mathbf{P}^3 such that Q is given by the polynomial $x_0^2 + x_1^2$. We see that if \mathfrak{m}_Q is the ideal defining Q in the ambient projective space, then in the chart obtained by making $a_{0,0} = 1$, the equation defining Z at Q lies in $(a_{2,2}a_{3,3} - a_{2,3}^2) + \mathfrak{m}_Q^3$, which implies the assertion in iv). This completes the proof of the proposition. \square

We now turn to the map $p: I \rightarrow \mathbf{P}$. Note that for every smooth quadric $H \subseteq \mathbf{P}^3$, we have precisely 2 families of lines contained in H , each of them parametrized by \mathbf{P}^1 . Indeed, if we choose coordinates on \mathbf{P}^3 such that H is the image of the Segre embedding

$$\iota: \mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3, \quad ((a_0, a_1), (b_0, b_1)) \rightarrow (a_0b_0, a_0b_1, a_1b_0, a_1b_1),$$

then the two families of lines are

$$L'_{(a_0, a_1)} = \iota(\{(a_0, a_1)\} \times \mathbf{P}^1) = (a_1x_0 = a_0x_2, a_1x_1 = a_0x_3) \quad \text{and}$$

$$L''_{(b_0, b_1)} = \iota(\mathbf{P}^1 \times \{(b_0, b_1)\}) = (b_1x_0 = b_0x_1, b_1x_2 = b_0x_3).$$

For future reference, we note that if L_1 is any member of the first family and L_2 is a member of the second family, then $L_1 \cap L_2 \neq \emptyset$. Indeed, we have

$$L'_{(a_0, a_1)} \cap L''_{(b_0, b_1)} = \{(a_0b_0, a_0b_1, a_1b_0, a_1b_1)\}.$$

We have seen that at least set-theoretically, the fiber of p over each point in $W_4 \setminus W_3$ consists of two irreducible curves, both of them isomorphic to \mathbf{P}^1 . The first assertion in the next proposition shows that this also holds scheme-theoretically.

Proposition 2.2. *The morphism $p: I \rightarrow \mathbf{P}$ is étale over $W_4 \setminus W_3$. Moreover, there is a prime divisor R on I such that $p^*(W_3) = 2R$.*

Proof. Consider a point $(H, L) \in I$ such that H is a smooth quadric. We may assume that $L = (x_0 = x_1 = 0)$ and we write $f = x_0f_0 + x_1f_1$, for some linear forms f_0 and f_1 . If $g_i = f_i|_L$, then it follows from Theorem 1.2 in Lecture 4 that in order to prove the assertion in i), it is enough to show that the map

$$H^0(L, \mathcal{O}_L(1)) \oplus H^0(L, \mathcal{O}_L(1)) \rightarrow H^0(L, \mathcal{O}_L(2)), \quad (u, v) \rightarrow ug_0 + vg_1$$

is surjective. However, if this is not surjective, then there is a point $Q \in L$ such that both g_0 and g_1 vanish at Q . In this case $\text{mult}_Q(H) \geq 2$, contradicting the fact that H is smooth.

We now prove the second assertion. Let D be the effective Cartier divisor on I given by $p^*(W_3)$. We compute the scheme-theoretic fiber $D_L \subseteq I_L$ of D over some line $L \in G$. After a suitable choice of coordinates, we have $L = (x_0 = x_1 = 0)$. We see that $I_L \simeq \mathbf{P}^6$ is the projective space of the quadrics defined by a symmetric matrix

$$\begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ a_{0,1} & a_{1,1} & a_{1,2} & a_{1,3} \\ a_{0,2} & a_{1,2} & 0 & 0 \\ a_{0,3} & a_{1,3} & 0 & 0 \end{pmatrix}$$

and D_L is defined by $(a_{0,2}a_{1,3} - a_{0,3}a_{1,2})^2$. In particular, we see that $(D_L)_{\text{red}}$ is an irreducible subset of \mathbf{P}^6 of dimension 5. Note that this is true for all $L \in G$. First, we deduce using Proposition 1.4 in Lecture 2 that $R := D_{\text{red}}$ is a prime divisor in I . Therefore we have $D = dR$ for some $d \geq 1$. Second, we have two possibilities: either $d = 2$, or $d = 1$ and the fibers of $R \rightarrow G$ are everywhere non-reduced. If $\text{char}(k) = 0$, then the second possibility contradicts generic smoothness, and we are done. In positive characteristic, we also compute the fiber of $D \rightarrow G$ over the generic point in G . We see, as above, that this is non-reduced, hence $D \neq R$, and thus $d = 2$. \square

Remark 2.3. If H is a quadric in $W_3 \setminus W_2$, then a straightforward computation shows that $p^{-1}(H)$ is nonreduced and $p^{-1}(H)_{\text{red}} \simeq \mathbf{P}^1$. In fact, H is the projective cone over a smooth conic C in a plane, and by intersecting a line on H with this plane, we obtain the isomorphism $p^{-1}(H)_{\text{red}} \simeq C$. We also note that the fiber $(p|_R)^{-1}(H)$ is equal to $p^{-1}(H)_{\text{red}}$. This is clear set-theoretically. We only explain the scheme-theoretic equality in characteristic 0. Note that the morphism $R \rightarrow G$ is smooth at all points (H, L) such that $H \in W_3 \setminus W_2$: this follows immediately from the description of the fibers in the proof of

Proposition 2.2. This implies that $R \cap p^{-1}(\mathbf{P} \setminus W_2)$ is smooth and by generic smoothness, the map $R \rightarrow W_3$ is smooth over some point $H \in W_3 \setminus W_2$. However, PGL_4 acts on both I and G and p is an equivariant morphism. Since the action is transitive on $W_3 \setminus W_2$, we conclude that the morphism $R \rightarrow W_3$ is smooth over $W_3 \setminus W_2$; in particular, the fiber over every quadric in this set is reduced.

3. A BRIEF REVIEW OF CYCLIC COVERS

For the Artin-Mumford example, we will need to take a certain double cover. In this section we review what we need about double covers (in fact, we discuss more generally cyclic covers).

Suppose that \mathcal{L} is a line bundle on a variety X and we have a nonzero section $\sigma \in H^0(X, \mathcal{L}^m)$, for some $m \geq 2$, not divisible by $\text{char}(k)$. Note that σ induces a morphism $\varphi_\sigma: \mathcal{L}^{-m} \rightarrow \mathcal{O}_X$. Consider the coherent sheaf of \mathcal{O}_X -algebras \mathcal{S} , which is the quotient of $\bigoplus_{i \geq 0} \mathcal{L}^{-i} y^i$ by the ideal generated by $uy^m - \varphi_\sigma(u)$, where u is a local section of \mathcal{L}^{-m} . The corresponding finite morphism

$$f: Y = \text{Spec}(\mathcal{S}) \rightarrow X$$

is the m -cyclic cover corresponding to σ .

If we choose an affine open subset $U \subseteq X$ and a trivialization $\mathcal{L}|_U \simeq \mathcal{O}_U$, then via the corresponding trivialization $\mathcal{L}^m|_U \simeq \mathcal{O}_U$, the section $\sigma|_U$ corresponds to some $h \in \mathcal{O}_X(U)$, and we have an isomorphism over U :

$$(1) \quad f^{-1}(U) \simeq \text{Spec } \mathcal{O}_X(U)[y]/(y^m - h).$$

We collect in the next proposition some well-known properties of this construction.

Proposition 3.1. *Let $f: Y \rightarrow X$ be the m -cyclic cover associated to the nonzero section $\sigma \in H^0(X, \mathcal{L}^m)$.*

- i) *The morphism f is étale of degree m over $X \setminus D$, where $D = Z(\sigma)$ is the effective Cartier divisor defined by σ .*
- ii) *There is a Cartier divisor E on Y such that $f^*(D) = mE$ and f induces an isomorphism $E \simeq D$.*
- iii) *If X is smooth and D is a smooth prime divisor, then Y is smooth.*
- iv) *If X is smooth and D has an irreducible component with multiplicity 1, then Y is irreducible and reduced.*
- v) *If X is smooth and D is reduced, then Y is normal.*

Proof. The assertion in i) follows directly from the local description in (1) and the fact that the characteristic of k does not divide m . Moreover, we deduce from (1) that $\mathcal{O}_Y(f^{-1}(U))$ is free over $\mathcal{O}_X(U)$, hence h is a non-zero-divisor on $\mathcal{O}_Y(f^{-1}(U))$. Since $y^m = h$, it follows that y is a non-zero-divisor, and if E is the Cartier divisor defined on each such U by y , then it is clear that $f^*(D) = mE$ and f induces an isomorphism $E \simeq D$.

Under the assumptions in iii), we see in particular that E is a smooth divisor. This implies that Y is smooth in a neighborhood of E . On the other hand, since X is smooth

and f is étale over $X \setminus D$ by i), we conclude that $Y \setminus E$ is smooth, too, hence Y is smooth.

Suppose now that X is smooth and D has a reduced component. If Y is not irreducible, we consider two distinct irreducible components Y_1 and Y_2 of Y . Since f is finite and flat, it follows that both Y_1 and Y_2 surject onto X . On the other hand, we deduce from ii) that $E \subseteq Y_1 \cap Y_2$. Note now that iii) implies that Y is smooth over $X \setminus D_{\text{sing}}$. This contradicts the fact that $f^{-1}(D \setminus D_{\text{sing}})$, which is nonempty by our assumption on D , is contained in $Y_1 \cap Y_2$. Therefore Y is irreducible.

Since X is smooth, it follows from the description in (1) that Y is Cohen-Macaulay. On the other hand, it is generically smooth by part i), and therefore it is reduced. This gives iv).

Finally, suppose that X is smooth and D is reduced. As we have mentioned, in this case Y is Cohen-Macaulay. Moreover, it is smooth in codimension 1, since $Y \setminus f^{-1}(D_{\text{sing}})$ is smooth by iii). Therefore Y is normal. \square

Remark 3.2. Let μ_m denote the group of m^{th} roots of 1 in k , so that $\mu_m \simeq \mathbf{Z}/m\mathbf{Z}$. With the notation in Proposition 3.1, we have an action of μ_m on Y such that $X \simeq Y/\mu_m$. Indeed, we have an action of μ_m on $\bigoplus_{i \geq 0} \mathcal{L}^{-m} y^m$ such that $\eta \in \mu_m$ maps y to ηy . Since $\eta^m = 1$ for all such η , it is clear that we get an induced action on $f_*(\mathcal{O}_Y)$, which induces an action of μ_m on Y . Since f is an affine morphism, in order to check that Y/μ_m is isomorphic to X , it is enough to show that the embedding $\mathcal{O}_X \hookrightarrow f_*(\mathcal{O}_Y)$ identifies \mathcal{O}_X with the sheaf of μ_m -invariant sections of $f_*(\mathcal{O}_Y)$ (for the basics about quotients of varieties by finite groups, we refer to [Mum08, Section II.7]). This is clear.

Example 3.3. A trivial example is that when $\sigma = \tau^m$ for some section $\tau \in H^0(X, \mathcal{L})$. In this case, it is clear that Y is isomorphic to the disjoint union of m copies of X , with the map f restricting to the identity on each component.

Remark 3.4. In general, the m -cyclic cover depends on the choice of section σ and not just on its zero-locus. However, if X is complete, then the m -cyclic cover only depends on $Z(\sigma)$. Indeed, if σ and σ' are two sections of \mathcal{L}^m that define the same divisor, then there is $\lambda \in k^*$ such that $\sigma' = \lambda\sigma$. If $\alpha \in k$ is such that $\alpha^m = \lambda$ and \mathcal{S} and \mathcal{S}' are the algebras corresponding to σ and σ' , respectively, then we have an isomorphism $\mathcal{S} \rightarrow \mathcal{S}'$ that on $\mathcal{L}^{-i} y^i$ is given by multiplication by α^{-i} .

Remark 3.5. The construction of cyclic covers is functorial. More precisely, suppose that $\varphi: X' \rightarrow X$ is a morphism of varieties, $\sigma \in H^0(X, \mathcal{L}^m)$ is as above and $f: Y \rightarrow X$ is the m -cyclic cover corresponding to σ . If $f': Y' \rightarrow X'$ is the m -cyclic cover corresponding to $\varphi^*(\sigma) \in H^0(X', f^*(\mathcal{L})^m)$ (which we assume to be nonzero), then we have a Cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi'} & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{\varphi} & X. \end{array}$$

This follows from the fact that if \mathcal{S} and \mathcal{S}' are the algebras corresponding to σ and $\varphi^*(\sigma)$, then we have a canonical isomorphism $\varphi^*(\mathcal{S}) \simeq \mathcal{S}'$.

Remark 3.6. Suppose that X is a complete variety and $f: Y \rightarrow X$ is the m -cyclic cover corresponding to an effective Cartier divisor D such that $\mathcal{O}_X(D) \simeq \mathcal{L}^m$. If $\varphi: X' \rightarrow X$ is a dominant morphism, with X' a complete variety such that $\varphi^*(D) = mR$ for some effective divisor R on X' , then there is a morphism $g: X' \rightarrow Y$ such that $f \circ g = \varphi$. Note first that since we deal with complete varieties, it follows from Remark 3.4 that the cyclic covers are determined, up to isomorphism, by the corresponding divisors. We deduce from Remark 3.5 that if $f': Y' \rightarrow X'$ is the cyclic cover corresponding to $\varphi^*(D)$, then we have a Cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi'} & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{\varphi} & X. \end{array}$$

On the other hand, Example 3.3 shows that since $\varphi^*(D) = mR$, we can identify Y' with m disjoint copies of X' . If $j: X' \rightarrow Y'$ is any section of f' , then $g = \varphi' \circ j$ satisfies the required condition.

Example 3.7. We can apply the previous remark for the morphism $p: I \rightarrow \mathbf{P}$ in Proposition 2.2 to conclude that if $f: Y \rightarrow \mathbf{P}$ is the double cover (that is, the 2-cyclic cover) corresponding to the divisor W_3 on \mathbf{P} , then there is a morphism $g: I \rightarrow Y$ such that $f \circ g = p$. For future reference, we note that for every $y \in Y$ such that $f(y) \notin W_2$, we have $g^{-1}(y) \simeq \mathbf{P}^1$. This is clear if $f(y) \notin W_3$: indeed, in this case it follows from Proposition 3.1 that $f^{-1}(f(y))$ is the disjoint union of two points, while $p^{-1}(f(y))$ is the disjoint union of two \mathbf{P}^1 by Proposition 2.2. Suppose now that $f(y) \in W_3 \setminus W_2$. In this case we have

$$g^{-1}(y) \hookrightarrow (p|_R)^{-1}(f(y))$$

and the two schemes have the same reduced structure. Since $(p|_R)^{-1}(f(y)) \simeq \mathbf{P}^1$ by Remark 2.3, we conclude that $g^{-1}(y) \simeq \mathbf{P}^1$.

4. THE ARTIN-MUMFORD EXAMPLE

We work over an algebraically closed field k with $\text{char}(k) = 0$ and keep the notation in §2. We consider a general 3-dimensional linear system $\Pi \subset \mathbf{P}$ and assume that this has the properties in the following proposition.

Proposition 4.1. *If Π is a general 3-dimensional linear system in the complete linear system \mathbf{P} of quadrics in \mathbf{P}^3 , then Π satisfies the following properties:*

- i) Π is base-point free.
- ii) The intersection $S = \Pi \cap W_3$ is a quartic surface, which is irreducible and reduced.
- iii) The intersection $\Gamma = \Pi \cap W_2$ is a reduced set of 10 points and $\Pi \cap W_1 = \emptyset$.
- iv) The singular locus of S consists precisely of Γ . Moreover, every point $P \in \Gamma$ is a node of S (that is, the projectivized tangent cone to S at P is a smooth conic in \mathbf{P}^2).
- v) The inverse image $I' := p^{-1}(\Pi)$ is smooth and connected, of dimension 4, and the morphism $I' \rightarrow G$ induced by q is birational.

Proof. Since the complete linear system of quadrics in \mathbf{P}^3 is base-point free, any 4 general quadrics have empty intersection. Therefore a general Π is base-point free. Since $\dim(W_1) = 3$, it is clear that for Π general, we have $W_1 \cap \Pi = \emptyset$. Furthermore, it follows from Bertini's theorem that $(\Pi \cap W_2)_{\text{sing}} \subseteq (W_2)_{\text{sing}}$ and $(\Pi \cap W_3)_{\text{sing}} \subseteq (W_3)_{\text{sing}}$. Since W_2 is generically smooth, of dimension 6, it follows that $\Gamma := \Pi \cap W_2$ is a reduced set of points, of cardinality $\deg(W_2) = 10$. Since W_3 is a quartic hypersurface in \mathbf{P}^9 , with $W_3 \setminus W_2$ smooth, it follows that $S := \Pi \cap W_3$ is a quartic surface in $\Pi \simeq \mathbf{P}^3$, with $S_{\text{sing}} \subseteq \Gamma$. In particular, it follows that S is irreducible and reduced.

Let us show now that for every $P \in \Gamma$, the tangent cone $C_P(S)$ to S at P is the affine cone over a smooth conic in \mathbf{P}^2 . Let $G' = G(4, 10)$ denote the Grassmann variety parametrizing 3-dimensional linear subspaces of \mathbf{P} , hence $\dim(G') = 4 \cdot 6 = 24$. Consider the subset \mathcal{Z} of $W_2 \times G'$ consisting of pairs (P, Π) such that $P \in \Pi$ and the tangent cone to $W_3 \cap \Pi$ at P is *not* the cone over a smooth conic. Note that the fiber of the projection $\mathcal{Z} \rightarrow W_2$ over a point P has dimension ≤ 17 . Indeed, this fiber parametrizes the 3-dimensional vector subspaces W of k^9 with the property that given 3 linear independent linear forms on k^9 , their restriction to W is linearly dependent. Equivalently, W is a 3-dimensional vector subspace of a hyperplane defined by a linear combination of the 3 linear forms. This implies that the dimension of this fiber is $\leq 2 + \dim(G(3, 8)) = 17$. We thus conclude that $\dim(\mathcal{Z}) \leq 17 + 6 = 23$, hence a general element of G' does not lie in the image of \mathcal{Z} , proving the assertion in iv).

Note now that since $p: I \rightarrow \mathbf{P}$ is surjective, it follows from the Kleiman-Bertini theorem (see [Har77, Theorem III.10.8]) that for general Π , the inverse image $p^{-1}(\Pi)$ is smooth, of dimension $\dim(I) - 6 = 4$. The fact that $p^{-1}(\Pi)$ is connected is a general fact and holds for all Π (see [Laz04, Theorem 3.3.3] for the characteristic 0 case and [MNP15, Theorem 1.1] for an argument in arbitrary characteristic). We give an ad-hoc argument in our case for Π general. Recall that by Example 3.7, we can factor p as $I \xrightarrow{g} Y \xrightarrow{f} \mathbf{P}$, where f is the double cover of \mathbf{P} corresponding to the divisor W_3 . Since g has connected fibers, it follows that it is enough to check that $Y' := f^{-1}(\Pi)$ is connected. On the other hand, it follows from Remark 3.5 that Y' is the double cover of Π corresponding to the divisor S . Since S is reduced and irreducible, it follows from Proposition 3.1 that Y' is irreducible.

Finally, we show that for Π general, the morphism $\varphi: I' \rightarrow G$ induced by q is birational. Indeed, suppose that $L \in G$ is a fixed line in \mathbf{P}^3 . The fiber $q^{-1}(L) \subseteq \mathbf{P}$ is a 6-dimensional linear subspace. It follows that for Π general, the fiber of φ over L , which is $\Pi \cap q^{-1}(L)$, consists of one reduced point. Therefore there is an open neighborhood U of L over which φ has finite fibers. Since both G and I' are smooth, this implies that φ is flat over U , and since $\varphi^{-1}(L)$ is a reduced point, we conclude that φ is birational. \square

From now on, we assume that Π satisfies the properties in the proposition and we keep the same notation. Note that the surface $S \subset \Pi$ is a *quartic symmetroid*, that is, there are linear forms $\ell_{i,j}$ on $\Pi \simeq \mathbf{P}^3$, with $1 \leq i, j \leq 4$, and $\ell_{i,j} = \ell_{j,i}$, such that S is defined by the determinant of the symmetric matrix $(\ell_{i,j})$. Indeed, suppose that $h_1, \dots, h_4 \in H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2))$ span the linear subspace corresponding to Π . In this case

we have coordinates $\lambda_1, \dots, \lambda_4$ on Π such that the quadric corresponding to the point $(\lambda_1, \dots, \lambda_4)$ is defined by $\sum_i \lambda_i h_i$. It follows that if we write $h_m = \sum_{i,j} a_{i,j}^{(m)} x_i x_j$, then we may take $\ell_{i,j} = \sum_{m=1}^4 a_{i,j}^{(m)} \lambda_m$.

We consider the factorization $I \xrightarrow{g} Y \xrightarrow{f} \mathbf{P}$ of p given in Example 3.7 and the induced surjective morphisms $f': Y' \rightarrow \Pi$ and $g': I' \rightarrow Y'$, where $Y' = f^{-1}(\Pi)$. Note that by Remark 3.5, we know that f' is the double cover of Π corresponding to the quartic surface S . Since S is reduced and irreducible, it follows from Proposition 3.1 that so is Y' . Moreover, since $S \setminus \Gamma$ is smooth, it follows from the same proposition that $Y' \setminus f'^{-1}(\Gamma)$ is smooth.

Lemma 4.2. *With the above notation, every $P \in f'^{-1}(\Gamma)$ is a node of Y' , that is, the projectivized tangent cone to Y' at P is a smooth quadric in \mathbf{P}^3 .*

Proof. It follows from the definition of the double cover that there is an affine open neighborhood U of $Q = f'(P)$ and $u \in R = \mathcal{O}_\Pi(U)$ defining S such that $\mathcal{O}(f'^{-1}(U)) \simeq R[y]/(y^2 - u)$. By Proposition 4.1, we know that $\text{mult}_Q(S) = 2$, and in fact, Q is a node of S . It follows that the tangent cone of Y' at P is given by $h + y^2 \in \mathcal{O}(\mathbf{A}^3 \times \mathbf{A}^1)$, where $h \in \mathcal{O}(\mathbf{A}^3)$ defines the tangent cone of S at Q . We thus see that P is a node of Y' . \square

We conclude using the lemma and Proposition 4.1 that Y' has precisely 10 singular points, all of which are nodes. Let $V \rightarrow Y'$ be the blow-up of the 10 nodes. By construction, this is an isomorphism over the smooth locus Y'_0 of Y' . If E is the exceptional divisor, then E is the disjoint union of 10 components, with each of these isomorphic to the projectivized tangent cone at the corresponding point of Y' , hence isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. We thus see that E is smooth, hence V is smooth in a neighborhood of E . Since $V \setminus E$ is clearly smooth, we conclude that V is smooth. The smooth, projective variety V is the *Artin-Mumford threefold*. Unirationality of V follows from our discussion so far.

Proposition 4.3. *With the above notation, the variety V is unirational.*

Proof. Since V and Y' are birational, it is enough to see that Y' is unirational. On the other hand, Proposition 4.1 implies that we have a birational morphism $I' \rightarrow G$, hence I' is rational. We thus deduce the unirationality of Y' from the existence of the surjective morphism $g': I' \rightarrow Y'$. \square

The following is the main result of this chapter. For simplicity, we assume that the ground field is \mathbf{C} (but see Remark 4.7 below). The proof will make use of some basic results about Brauer groups, for which we refer to Appendix 2. Recall that a \mathbf{P}^1 -fibration is a proper, flat morphism $f: X \rightarrow Y$, such that $f^{-1}(y) \simeq \mathbf{P}^1$ for all (closed) points $y \in Y$. Note that if X and Y are smooth, the flatness follows from the condition on the fibers.

Theorem 4.4. *The variety V is not stably rational.*

Proof. Step 1. Consider the morphism $\psi: g'^{-1}(Y'_0) \rightarrow Y'_0$ induced by g' . We have seen that all its fibers are isomorphic to \mathbf{P}^1 , and since both Y'_0 and $g'^{-1}(Y'_0)$ are smooth, we conclude that ψ is a \mathbf{P}^1 -fibration. We now show that ψ has no rational sections.

Recall that since $f': Y' \rightarrow \Pi$ is a cyclic 2-cover, we have an action of μ_2 on Y' such that $\Pi \simeq Y'/\mu_2$ (see Remark 3.2). Let $\sigma: Y' \rightarrow Y'$ be the automorphism over Π given by the action of -1 , hence over $\Pi \setminus S$, this interchanges the fibers of f' .

Suppose that ψ has a rational section φ defined on some open subset U of Y'_0 . After possibly replacing U by a smaller subset, we may assume that U lies over $\Pi \setminus S$ and that $\sigma(U) \subseteq U$. Note that for every $y \in Y$, we can write $\varphi(y) = (f'(y), L_y)$, where L_y is a line contained in the quadric $f'(y)$. Moreover, the line $L_{\sigma(y)}$ lies on the same quadric, but on the other family of lines than L_y . Therefore $L_y \cap L_{\sigma(y)} \neq \emptyset$, and it is clear that the map $U \rightarrow \mathbf{P}^3$, given by $y \rightarrow L_y \cap L_{\sigma(y)}$ is a morphism. Moreover, this maps y and $\sigma(y)$ to the same point in \mathbf{P}^3 , and since $\Pi \simeq Y'/\mu_2$, we conclude that there is a rational map $\Pi \dashrightarrow \mathbf{P}^3$ that maps each $Q = f'(y)$ to $L_y \cap L_{\varphi(y)}$, a point lying on the quadric Q . Since this contradicts the general Lemma 4.5 below, we conclude that in fact ψ has no rational section.

Step 2. By applying Proposition 3.7 in Appendix 2, we deduce from Step 1 that ψ defines a nonzero element in $\text{Br}(Y'_0)$. We now show that the first Chern class map $\text{Pic}(Y'_0) \rightarrow H^2(Y'_0, \mathbf{Z})$ is surjective: since Y'_0 is smooth, Proposition 3.10 in Appendix 2 then implies that $\text{Br}(Y'_0) \simeq H^3(Y'_0, \mathbf{Z})_{\text{tors}}$, and therefore $H^3(Y'_0, \mathbf{Z})_{\text{tors}} \neq 0$.

Recall that Y'_0 is isomorphic to the open subset $V_0 = V \setminus E$ of the smooth, projective 3-fold V , where E is a disjoint union of smooth, irreducible divisors. Consider the following commutative diagram

$$\begin{array}{ccc} \text{Pic}(V) & \longrightarrow & H^2(V, \mathbf{Z}) \\ \downarrow & & \downarrow \\ \text{Pic}(V_0) & \longrightarrow & H^2(V_0, \mathbf{Z}). \end{array}$$

We have seen in Proposition 4.3 that V is unirational, hence

$$H^2(V, \mathcal{O}_V) = 0 = H^1(V, \mathcal{O}_V)$$

by Corollary 3.4 in Lecture 1. We deduce by using GAGA and the exponential sequence that the top horizontal map in the diagram is an isomorphism. In order to finish the proof of this step it is thus enough to show that the map

$$(2) \quad H^2(V, \mathbf{Z}) \rightarrow H^2(V_0, \mathbf{Z})$$

is surjective.

The long exact sequence in cohomology for the pair (V, V_0) gives, via the Thom isomorphism (see Remark 2.4 in Appendix 1)

$$H^2(V, \mathbf{Z}) \rightarrow H^2(V_0, \mathbf{Z}) \rightarrow H^1(E, \mathbf{Z}).$$

Since E is a disjoint union of varieties isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and

$$H^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathbf{Z}) = 0$$

by the Künneth theorem and the fact that $H^1(\mathbf{P}^1, \mathbf{Z}) = 0$, we conclude that $H^1(E, \mathbf{Z}) = 0$. Therefore the map (2) is surjective.

Step 3. We now deduce that $H^3(V, \mathbf{Z})_{\text{tors}} \neq 0$. We can then apply Corollary 1.3 to obtain that V is not stably rational.

We have seen in Step 2 that $H^3(V_0, \mathbf{Z})_{\text{tors}} \neq 0$. We consider again the long exact sequence in cohomology for the pair (V, V_0) , that gives

$$H^1(E, \mathbf{Z}) \rightarrow H^3(V, \mathbf{Z}) \rightarrow H^3(V_0, \mathbf{Z}) \rightarrow H^2(E, \mathbf{Z}).$$

Recall that $H^1(E, \mathbf{Z}) = 0$ and using again the description of the components of E and the Künneth theorem, we get

$$H^2(E, \mathbf{Z}) \simeq H^2(\mathbf{P}^1 \times \mathbf{P}^1, \mathbf{Z})^{\oplus 10} \simeq \mathbf{Z}^{20}.$$

In particular, we see that $H^2(E, \mathbf{Z})_{\text{tors}} = 0$, and we thus conclude from the above exact sequence that

$$H^3(V, \mathbf{Z})_{\text{tors}} \simeq H^3(V_0, \mathbf{Z})_{\text{tors}} \neq 0.$$

This completes the proof of the theorem. \square

Lemma 4.5. *Let $|V|$ be a base-point free linear system of hypersurfaces of degree $d \geq 2$ in \mathbf{P}^n . If $\mathcal{H} \hookrightarrow |V| \times \mathbf{P}^n$ is the universal hypersurface in $|V|$, then the map $p: \mathcal{H} \rightarrow |V|$ induced by the first projection has no rational section.*

Proof. We identify throughout homology and cohomology via Poincaré duality. It is clear that there is a linear subsystem of $|V|$ that is base-point free and of dimension n . Since it is enough to prove the assertion in the lemma for this subsystem, we may and will assume that $\dim |V| = n$. The map p has a rational section if and only if there is a subvariety W of \mathcal{H} that maps birationally onto $|V|$ (given the section, we take W to be the closure of its image). We will show that the existence of such W leads to a contradiction.

Let $W \xrightarrow{i} \mathcal{H} \xrightarrow{j} |V| \times \mathbf{P}^n$ be the inclusions. Consider the following classes $\alpha = c^1(\mathcal{O}(1, 0))$ and $\beta = c^1(\mathcal{O}(0, 1))$ in $H^2(|V| \times \mathbf{P}^n, \mathbf{Z})$. It follows from the description of the cohomology of \mathbf{P}^n (see Proposition 4.1 in Appendix 1) and the Künneth formula that $H^*(|V| \times \mathbf{P}^n, \mathbf{Z})$ has a \mathbf{Z} -basis given by $\alpha^k \cup \beta^\ell$, with $0 \leq k, \ell \leq n$. Note that if $\pi: |V| \times \mathbf{P}^n \rightarrow |V|$ is the projection, then $\pi_*(\alpha^k \cup \beta^\ell) = \alpha^k$ if $\ell = n$ and $\pi_*(\alpha^k \cup \beta^\ell) = 0$, otherwise.

On the other hand, \mathcal{H} is a divisor in $|V| \times \mathbf{P}^n$ defined by a section in $\mathcal{O}(1, d)$. Moreover, since $|V|$ is base-point free, we see that \mathcal{H} is a subbundle of $|V| \times \mathbf{P}^n$ over \mathbf{P}^n . The formula for the cohomology of a vector bundle (see Corollary 4.2 in Appendix 1) implies that a basis for the cohomology of \mathcal{H} is given by $j^*(\alpha^k) \cup j^*(\beta^\ell)$, for $0 \leq k \leq n-1$ and $0 \leq \ell \leq n$. In particular, we may write

$$i_*(\mu_W) = j^*(\delta) \quad \text{with} \quad \delta = \sum_{k=0}^{n-1} c_k \alpha^k \cup \beta^{n-1-k} \in H^{2n-2}(|V| \times \mathbf{P}^n, \mathbf{Z}).$$

We deduce that

$$p_*(\mu_W) = \pi_*(j_* j^*(\delta)) = \pi_*(\delta \cup c^1(\mathcal{O}(1, d))) = \pi_*(\delta \cup (\alpha + d\beta)) = dc_0 \mu_{|V|}.$$

However, if p induces a birational morphism $W \rightarrow |V|$, then $p_*(\mu_W) = \mu_{|V|}$, contradicting the fact that $d \geq 2$. \square

Remark 4.6. While the proof of the above lemma made use of singular cohomology, one can give a similar argument using Chow groups. Therefore the assertion holds true over any algebraically closed field.

Remark 4.7. While we assumed in the proof of Theorem 4.4 that we work over \mathbf{C} , it is standard to deduce from this case that the assertion works over any algebraically closed field k , with $\text{char}(k) = 0$. Indeed, if V is stably rational, then we can find an algebraically closed subfield k_1 of k , with $\text{trdeg}_{k_1} k < \infty$, and an Artin-Mumford 3-fold V_1 over k_1 such that V_1 is stably rational and V is obtained from V_1 by base-change. We can find an embedding $k_1 \hookrightarrow \mathbf{C}$ and the Artin-Mumford 3-fold obtained by base-changing V_1 to \mathbf{C} is again stably rational, contradicting the assertion over \mathbf{C} .

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