Chapter 6: Divisors and line bundles

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Invariant divisors. If X is a normal variety, we denote by $\mathrm{Div}(X)$ the group of Weil divisors on X, i.e. the free abelian group generated by the codimension one irreducible subvarieties. The group of Cartier divisors is denoted by $\mathrm{Cart}(X)$. Each Cartier divisor determines a Weil divisor

$$\sum_{\mathrm{codim}(V)=1} \mathrm{ord}_V(D)\cdot V,$$

where $\mathrm{ord}_V(D)$ is the order of vanishing of an equation for D along the subvariety V. Since X is normal, this gives an embedding $\mathrm{Cart}(X)\hookrightarrow \mathrm{Div}(X)$. The support of a Weil divisor D is the union of the codimension one subvarieties which appear in D with nonzero coefficient.

If f is a rational function on X, then we denote the principal divisor corresponding to f by $\mathrm{div}(f)$. The quotient of $\mathrm{Div}(X)$ by the subgroup of principal divisors is the class group $\mathrm{Cl}(X)$. The inclusion $\mathrm{Cart}(X)\hookrightarrow \mathrm{Div}(X)$ induces an embedding $\mathrm{Pic}(X)\hookrightarrow \mathrm{Cl}(X)$.

Each Weil divisor D determines the subsheaf $\mathcal{O}(D)$ of the constant sheaf $K(X)$, where $K(X)$ is the function field of X. This is defined by

$$\Gamma(U,\mathcal{O}(D))=\{f\in K(X)\smallsetminus\{0\}\mid \mathrm{div}(f)+D\,\text{is effective on}\,U\}\cup\{0\}.$$

Note that D and E define the same sheaf if and only if $D=E$. Moreover $\mathcal{O}(D)\simeq\mathcal{O}(E)$ if and only if D and E have the same class in $\mathrm{Cl}(X)$.

A $\mathbb{Q}$-divisor on X is an element of the $\mathbb{Q}$-vector space $\mathrm{Div}(X)_{\mathbb{Q}}:=\mathrm{Div}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$. By tensoring with $\mathbb{Q}$ we get also $\mathrm{Pic}(X)_{\mathbb{Q}}\hookrightarrow \mathrm{Cl}(X)_{\mathbb{Q}}$. If $D=\sum_{i=1}^r a_iD_i$ is a $\mathbb{Q}$-divisor on X, then $\lfloor D\rfloor=\sum_{i=1}^r\lfloor a_i\rfloor D_i$, where $\lfloor x\rfloor$ denotes the largest integer $\leq x$. Similarly, $\lceil D\rceil=\sum_{i=1}^r\lceil a_i\rceil D_i$, where $\lceil x\rceil$ is the smallest integer $\geq x$. If D and E are $\mathbb{Q}$-divisors on X, then multiplication of rational functions induces a multiplication map

$$\Gamma(X,\mathcal{O}(\lfloor D\rfloor))\otimes\Gamma(X,\mathcal{O}(\lfloor E\rfloor))\to\Gamma(X,\mathcal{O}(\lfloor D+E\rfloor)).$$

A $\mathbb{Q}$-divisor is called $\mathbb{Q}$-Cartier if some multiple is integral and Cartier.

If U is an open subset of X whose complement has dimension at least two, then the natural restriction maps $\mathrm{Div}(X)\to\mathrm{Div}(U)$ and $\mathrm{Cl}(X)\to\mathrm{Cl}(U)$ are isomorphisms. If $i\colon U\to X$ is the inclusion, then we have $i_*\mathcal{O}(D|_U)=\mathcal{O}(D)$ for every Weil divisor D on X.

From now on we suppose that $X=X(\Delta)$ is the toric variety corresponding to a fan Δ in a lattice N of rank n. We put $T=T_N$. Let $\tau_1,\ldots,\tau_d$ be the rays of Δ, and for every i let v_i be the first lattice point along τ_i. Each τ_i determines a prime divisor $D_i = V(\tau_i)$ which is invariant under the torus action. We denote by $\mathrm{Div}_T(X)$ the group of **T-Weil divisors** (or, simply, of T-divisors): this is the subgroup of $\mathrm{Div}(X)$ generated by the D_i.

We start by computing the principal divisor corresponding to a rational function of the form χ^u.

Lemma 1. *If $u \in M$, then*

$$\mathrm{div}(\chi^u) = \sum_{i=1}^{d} \langle u, v_i \rangle D_i.$$

Proof. We may replace X by $U_{\tau_1} \cup \ldots \cup U_{\tau_d}$, as the complement of this set in X has dimension at least two. Moreover, it is enough to prove the assertion on each U_{τ_i}. We may choose a basis $e_1, \ldots, e_n$ of N with $v_i = e_1$. This gives $U_{\tau_i} \simeq \mathrm{Spec}(k[t]) \times (k^*)^{n-1}$ such that χ^u corresponds to $t^{\langle u, v_i \rangle}$, and proves the lemma.

The action of T on X induces an action of T on $K(X)$ such that $(t \cdot f)(x) = f(t^{-1}x)$. We also have an action of T on $\mathrm{Div}(X)$ induced by $D \to tD := \{tx \mid x \in D\}$ for every prime Weil divisor D. Note that we have $\mathrm{div}(tf) = t(\mathrm{div}(f))$.

We say that a subsheaf $\mathcal{F}$ of $K(X)$ is T-invariant if for every invariant open subset U of X, and for every f in $\mathcal{F}(U)$ we have $tf \in \mathcal{F}(U)$ for every $t \in T$. It is clear that it is enough to check this for all $U = U_\sigma$. Moreover, if this is the case, then for all such U and all $t \in T$ we have $tf \in \mathcal{F}(U)$ if and only if $f \in \mathcal{F}(U)$.

If $\mathcal{F} = \mathcal{O}(D)$ for some Weil divisor D, U is an open invariant subset and f is in $K(X)$, then tf is in $\Gamma(U, \mathcal{O}(D))$ if and only if f is in $\Gamma(U, \mathcal{O}(t^{-1}D))$. Therefore $\mathcal{O}(D)$ is T-invariant if and only if for every $t \in T$ we have $tD = D$. This is case if and only if D lies in $\mathrm{Div}_T(X)$.

A **Cartier T-divisor** is a Cartier divisor such that it corresponds to a T-Weil divisor. As we have seen, this is equivalent with the fact that $\mathcal{O}(D) \subseteq K(X)$ is a T-invariant sheaf. We denote by $\mathrm{Cart}_T(X)$ the subgroup of $\mathrm{Cart}(X)$ consisting of Cartier T-divisors. It follows from Lemma 1 that every principal divisor of the form $\mathrm{div}(\chi^u)$ is a Cartier T-divisor. The next lemma shows that on an affine toric variety every Cartier T-divisor is of this form.

Lemma 2. *If σ is a pointed cone in N, then for every Cartier T-divisor D on U_σ there is $u \in M$ such that $D = \mathrm{div}(\chi^u)$.*

Proof. Note first that $\Gamma(U_\sigma, \mathcal{O}(D))$ is contained in $k[M]$. Indeed, since the Weil divisor corresponding to D is supported on the complement of T, the domain of any f in $\Gamma(U_\sigma, \mathcal{O}(D))$ contains T. We have seen that a vector subspace of $k[M]$ is preserved by the action of T if and only if it is an M-graded subspace.

We may write $U_\sigma = U_{\sigma'} \times (k^*)^r$, where σ' is a cone in a lattice N' of rank equal to the dimension of σ'. Since D is a T-divisor, we can find a T-divisor D' on U'_σ such that D is the pull-back of D' via the projection $U_\sigma \to U_{\sigma'}$. Therefore we may reduce to the case when σ has dimension n.

Since the M-graded module $\Gamma(U_\sigma, \mathcal{O}(D))$ is locally free of rank one over $\mathcal{O}(U_\sigma)$ and since σ spans $N_{\mathbb{R}}$, we deduce from Chapter 3 that this module can be generated by a homogeneous element. Therefore $\Gamma(U_\sigma, \mathcal{O}(D))$ is generated by some χ^{-u}, so $D = \operatorname{div}(\chi^u)$.

Remark 1. If u is in M, it follows from Lemma 1 that $\operatorname{div}(\chi^u)$ is zero on U_σ if and only if σ is contained in $u^\perp$. Therefore the class of u in $M/M(\sigma)$ is uniquely determined by $\operatorname{div}(\chi^u)$.

Suppose now that X is a non necessarily affine toric variety. A Cartier T-divisor D is defined by specifying an element $u(\sigma)$ in $M/M(\sigma)$ for each cone $\sigma \in \Delta$, defining divisors $\operatorname{div}(\chi^{-u(\sigma)})$ on U_σ (the minus sign is taken to conform to the literature). Equivalently, $\chi^{u(\sigma)}$ generates the fractional ideal $\mathcal{O}(D)$ on U_σ.

These must agree on overlaps; i.e. when τ is a face of σ, the canonical map from $M/M(\sigma)$ to $M/M(\tau)$ must take $u(\sigma)$ to $u(\tau)$. In short,

$$\begin{aligned}\{T - \text{Cartier divisors}\} &= \lim_{\leftarrow} M/M(\sigma) \\ &= \operatorname{Ker}\Big(\oplus_i M/M(\sigma_i) \to \oplus_{i<j} M/M(\sigma_i \cap \sigma_j)\Big),\end{aligned}$$

with σ_i the maximal cones.

Exercise 1. Show that for each irreducible Weil T-divisor D on an affine toric variety U_σ, there is an effective Cartier T-divisor that contains D with multiplicity one.

Line bundles. The following proposition shows that $\operatorname{Cl}(X)$ and $\operatorname{Pic}(X)$ can be computed by T-divisors and Cartier T-divisors, respectively. In addition, it gives a recipe for calculating these groups.

Proposition 1. *Let Δ be a fan, $X = X(\Delta)$, and $V_\Delta \subseteq N_{\mathbb{R}}$ the subspace spanned by the cones in Δ. Then there is a commutative diagram with exact rows:*

$$\begin{tikzcd} 0 \arrow[r] & M/M\cap V_\Delta^\perp \arrow[r] \arrow[d, equal] & \mathrm{Cart}_T X \arrow[r] \arrow[d, hook] & \mathrm{Pic}(X) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M/M\cap V_\Delta^\perp \arrow[r] & \oplus_{i=1}^d \mathbb{Z}\cdot D_i \arrow[r] & \mathrm{Cl}(X) \arrow[r] & 0 \end{tikzcd}$$

In particular, $\mathrm{rank}(\mathrm{Pic}(X)) \leq \mathrm{rank}(\mathrm{Cl}(X)) = d - \dim V_\Delta$*, where d is the number of edges in the fan. In addition,* $\mathrm{Pic}(X)$ *is free abelian if* Δ *contains a cone of dimension equal to* $\dim V_\Delta$.

Proof. First note that $X \smallsetminus \bigcup_i D_i = T_N$ is affine, with coordinate ring the unique factorization ring $k[M]$, so all Cartier divisors and Weil divisors on T_N are principal. This gives an exact sequence

$$\oplus_{i=1}^d \mathbb{Z}\cdot D_i \to \mathrm{Cl}(X) \to \mathrm{Cl}(T_N) = 0.$$

Next, note that if f is a rational function on X such that $\mathrm{div}(f)$ is a T-divisor then $f = \lambda \cdot \chi^u$ for some $u \in M$ and $\lambda \in k^*$; indeed, this follows by applying Lemma 2 for T_N and using the fact that every invertible function on $k[M]$ is of the form χ^u for some $u \in M$. Moreover, Lemma 1 shows that u is uniquely determined by $\mathrm{div}(f)$ up to addition of an element in in $M \cap V_\Delta^\perp$.

If $\mathcal{L}$ is a line bundle, its restriction to T_N must be trivial, so $\mathcal{L} \simeq \mathcal{O}(D)$ for some Cartier divisor supported on $\bigcup_i D_i$; indeed, writing $\mathcal{L} \simeq \mathcal{O}(E)$ for some Cartier divisor E, take a rational function whose divisor agrees with E on T_N, and set $D = E - \mathrm{div}(f)$. Hence D is T-invariant. The exactness of the upper row follows easily.

The group $\mathrm{Cart}_T(X)$ is torsion free since it is a subgroup of $\oplus M/M(\sigma)$. If Δ contains a cone σ of dimension $\dim(V_\Delta)$, then $M(\sigma)$ is $M \cap V_\Delta^\perp$, the embedding $M/M\cap V_\Delta^\perp \to \mathrm{Cart}_T(X)$ must split, so the cokernel $\mathrm{Pic}(X)$ is torsion free.

Remark 2. It follows from Lemma 2 and Proposition 1 that the Picard group of any affine toric variety is trivial.

Exercise 2. Let Δ be the complete fan in $\mathbb{Z}^2$ with rays along $v_1 = 2e_1 - e_2$, $v_2 = -e_1 + 2e_2$, and $v_3 = -e_1 - e_2$. Show that $a_1D_1 + a_2D_2 + a_3D_3$ is a Cartier divisor on $X = X(\Delta)$ if and only if $a_1 \equiv a_2 \equiv a_3 \pmod 3$. Show that

$$\mathrm{Pic}(X) = \mathbb{Z}\cdot 3D_1 \simeq \mathbb{Z},$$

$$\mathrm{Cl}(X) = (\mathbb{Z}\cdot D_1 + \mathbb{Z}\cdot D_2)/\mathbb{Z}\cdot 3(D_1 - D_2) \simeq \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

In particular, $\mathrm{Cl}(X)$ can have torsion.

Exercise 3. Let Δ be a fan and $V_\Delta \subset N_\mathbb{R}$ the vector subspace spanned by Δ. If $X = X(\Delta)$, then the following are equivalent:

(1) Δ is simplicial;
(2) Every $\mathbb{Q}$-divisor on X is $\mathbb{Q}$-Cartier.
(3) $\mathrm{Pic}(X) \otimes \mathbb{Q} \to \mathrm{Cl}(X) \otimes \mathbb{Q}$ is an isomorphism;
(4) $\mathrm{rank}(\mathrm{Pic}(X)) = d - \dim V_\Delta$.

Exercise 4. If U_σ is the affine toric variety corresponding to a simplicial cone σ, show that $\mathrm{Cl}(U_\sigma)$ is a finite abelian group of order equal to $\mathrm{mult}(\sigma)$.

Exercise 5. Given a fan Δ and $X = X(\Delta)$, the following are equivalent:

(1) Δ is nonsingular.
(2) Every Weil T-divisor on X is Cartier.
(3) $\mathrm{Pic}(X) = \mathrm{Cl}(X)$.

Exercise 6. Start with the fan over the faces of the cube with the vertices at $(\pm 1, \pm 1, \pm 1)$ in $\mathbb{Z}^3$. Let Δ be the fan with the cones spanned by the same generators, but with the ray through $(1,1,1)$ replaced by the ray through $(1,2,3)$. Show that $\mathrm{Pic}(X(\Delta)) = \{0\}$.

Remark 3. The variety $X = X(\Delta)$ in the previous exercise is complete, but not projective. Note that the only effective Cartier divisor is the zero one. Indeed, every Cartier divisor is principal, hence it is equal to $\mathrm{div}(f)$ for some f. If the divisor is effective, then f is a regular function on X, and therefore it is constant.

By taking a nonzero Cartier divisor on X, we get an example of a Cartier divisor which can't be written as a difference of two effective Cartier divisors.

Divisors and polyhedra. A $\mathbb{Q}$-T-divisor D is an element of

$$T - \mathrm{Div}(X)_{\mathbb{Q}} = \oplus_{i=1}^{d} \mathbb{Q} \cdot D_i.$$

Given such $D = \sum_{i=1}^{d} a_i D_i$, we define the rational polyhedron

$$P_D := \{u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle + a_i \geq 0 \quad \text{for} \quad 1 \leq i \leq d\}.$$

As in the proof of Lemma 2, since D is T-invariant we see that $\Gamma(X, \mathcal{O}(\lfloor D \rfloor))$ is generated by elements of the form χ^u. Moreover, by Lemma 1, u in M lies in P_D if and only if χ^u is in $\Gamma(X, \mathcal{O}(\lfloor D \rfloor))$. Therefore we have

$$\Gamma(X, \mathcal{O}(\lfloor D \rfloor)) = \oplus_{u \in P_D \cap M} k \cdot \chi^u.$$

Exercise 7. Show that $P_{\lambda D} = \lambda P_D$ for every positive rational number λ. If u is in M, then $P_{D + \mathrm{div}(\chi^u)} = P_D - u$. For two $\mathbb{Q}$-T-Weil divisors D and E, we have $P_D + P_E \subseteq P_{D+E}$.

Suppose now that D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor. If m is a positive integer such that mD has integral coefficients and is Cartier, then by Lemma 2 we can find for every maximal cone σ in Δ an element $u(\sigma) \in \frac{1}{m}M$ such that $mD|_{U_\sigma} = \mathrm{div}(\chi^{-mu(\sigma)})$. Moreover, such $u(\sigma)$ is uniquely determined up to adding an element in $\sigma^\perp \cap M$. In particular, $u(\sigma) - u(\tau)$ is in $(\sigma \cap \tau)^\perp$ for every maximal cones σ and τ in Δ.

We define a function $\psi_D \colon |\Delta| \to \mathbb{R}$ by $\psi_D(v) = \langle u(\sigma), v \rangle$ for v in σ. We see that this function is well-defined and it does not depend on m. It is linear on each cone σ and $\psi_D(|\Delta| \cap N_{\mathbb{Q}}) \subseteq \mathbb{Q}$. Note that if $D = \sum_{i=1}^d a_i D_i$, then ψ_D is determined by $\psi_D(v_i) = -a_i$ for every i.

Conversely, given a function ψ on $|\Delta|$ which takes rational values on points in $|\Delta| \cap N_{\mathbb{Q}}$ and is linear on each cone in Δ, there is a unique $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor D such that $\psi = \psi_D$: take $D = -\sum_{i=1}^d \psi(v_i) D_i$. Note that D is in $T - \mathrm{Div}(X)$ if and only if $\psi_D(|\Delta| \cap N) \subseteq \mathbb{Z}$, and it is in $T - \mathrm{Cart}(X)$ if and only if the restriction to each cone in Δ is given by some element in M.

If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor, then we have

$$P_D = \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq \psi_D(v) \quad \text{for every} \quad v \in |\Delta|\}.$$

Exercise 8. Let Δ' and Δ be fans in N' and N, respectively, and $\phi \colon N' \to N$ be a group morphism inducing a morphism of fans. If $f \colon X(\Delta') \to X(\Delta)$ is the corresponding morphism of toric varieties, show that for every $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor D on $X(\Delta)$, the pull-back $f^*(D)$ is defined and $\psi_{f^*(D)} = \psi_D \circ \phi$ on $|\Delta'|$. If $\phi^* \colon M \to M'$ is the dual of ϕ, then $\phi^*_{\mathbb{R}}(P_D) \subseteq P_{f^*(D)}$. Moreover, if $\phi_{\mathbb{R}}(|\Delta'|) = |\Delta|$ (for example, if $\mathrm{coker}(\phi)$ is finite and f is proper), then $P_D = (\phi^*_{\mathbb{R}})^{-1}(P_{f^*(D)})$.

Proposition 2. *If P_D is nonempty and we write $P_D = Q + C$, for a polytope Q and a cone C, then $C = \cap_{\sigma \in \Delta} \sigma^\vee$, so $k[C \cap M] = \mathcal{O}(X)$. In particular, P_D is a polytope for every $\mathbb{Q}$-T-divisor D if and only if this holds for some D with P_D nonempty (for example, for $D = 0$); this is the case if and only if $|\Delta|$ spans N_R as a convex cone.*

Proof. We write $D = \sum_i a_i D_i$. We have seen in Chapter 2 that since P_D is defined by $\langle u, v_i \rangle \geq -a_i$ for every i, C is defined by $\langle u, v_i \rangle \geq 0$ for every i. This condition is equivalent to the fact that u is in $\cap_{\sigma \in \Delta} \sigma^\vee$. The last assertion follows from that fact that $|\Delta|$ generates $N_{\mathbb{R}}$ as a convex cone if and only if $\cap_{\sigma \in \Delta} \sigma^\vee = \{0\}$.

Remark 4. If $|\Delta|$ spans $N_{\mathbb{R}}$ as a convex cone, then P_D is bounded, so there are only finitely many points in $P_D \cap M$. Therefore $\Gamma(X, \mathcal{O}(D))$ is finite-dimensional. In particular, we recover the finite-dimensionality of $\Gamma(X, \mathcal{O}(D))$ in the case of complete toric varieties.

Let D be a $\mathbb{Q}$-T-divisor on the toric variety $X = X(\Delta)$. The multiplication of rational functions induces on

$$R_D := \oplus_{m\in\mathbb{N}}\Gamma(X, \mathcal{O}(\lfloor mD\rfloor))$$

a structure of graded k-algebra.

Exercise 9 ([El]). Show that if D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor on X, then R_D is a finitely generated k-algebra. In fact, if P_D is nonempty and σ is the cone over $P_D \times \{1\}$ in $M_{\mathbb{R}} \times \mathbb{R}$, show that there is a canonical isomorphism

$$R_D = k[\sigma \cap (M \times \mathbb{Z})].$$

Exercise 10. If $\mathcal{L}$ is an invertible sheaf on X, i.e. an element of $\mathrm{Pic}(X)$, let L denote the corresponding geometric line bundle, whose sheaf of sections is $\mathcal{L}^{-1}$.

(1) If D is a Cartier T-divisor on the toric variety X and $\mathcal{L} = \mathcal{O}(D)$, show that L has a natural action of T_N such that the projection $\pi\colon L \to X$ is equivariant.
(2) This action induces an action of T_N on $\Gamma(X, \mathcal{L})$ such that the section χ^u is an eigenvalue corresponding to the character of T_N corresponding to $-u$.
(3) If $\mathcal{L}_1, \ldots, \mathcal{L}_r$ are in $\mathrm{Pic}(X)$ then the geometric vector bundle $L_1 \oplus \ldots \oplus L_r$ and the projective bundle $\mathbb{P}(L_1 \oplus \ldots \oplus L_r)$ have natural structures of toric varieties such that the projections to X are morphisms of toric varieties.

Exercise 11. Show that if $\mathcal{L}_1, \ldots, \mathcal{L}_r$ are line bundles on a toric variety X, then the ring

$$\oplus_{m=(m_i)\in\mathbb{N}^r}\Gamma(X, \mathcal{L}_1^{m_1} \otimes \ldots \otimes \mathcal{L}_r^{m_r})$$

is a finitely generated k-algebra.

Exercise 12. If D and E are $\mathbb{Q}$-T-divisors on a toric variety X, show that the natural multiplication map

$$\Gamma(X, \mathcal{O}(\lfloor D\rfloor)) \otimes \Gamma(X, \mathcal{O}(\lfloor E\rfloor)) \to \Gamma(X, \mathcal{O}(\lfloor D + E\rfloor))$$

is surjective if and only if every lattice point in P_{D+E} is the sum of lattice points in P_D and P_E.

Recall the definition of volume for line bundles (see [Laz] for motivation and details). If $\mathcal{L}$ is a line bundle on a complete variety X of dimension n, then

$$\mathrm{vol}(X, \mathcal{L}) := \limsup_{m\to\infty} \frac{h^0(X, \mathcal{L}^m)}{m^n/n!}.$$

It can be shown that $\mathrm{vol}(X, \mathcal{L}^m) = m^n \cdot \mathrm{vol}(X, \mathcal{L})$. This allows the extension of the definition to $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors: if D is such a divisor and if m is a positive integer such that mD is a Cartier divisor, then $\mathrm{vol}(X, D) := \mathrm{vol}(X, \mathcal{O}(mD))/m^n$.

A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D is called big if $\mathrm{vol}(X, D) > 0$. For example, if D is ample then $\mathrm{vol}(X, D) = (D^n) > 0$.

Exercise 13. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor on a complete toric variety X of dimension n. Show that $\mathrm{vol}(X, D) = n! \cdot \mathrm{vol}_n(P_D)$. In particular, D is big if and only if $\dim(P_D) = n$.

Recall the definition of Iitaka dimension (see [Laz] for details). If $\mathcal{L}$ is a line bundle on the complete variety X, then it is known that if m is divisible enough, then the dimension of the image of the rational map defined by $\Gamma(X, \mathcal{L}^m)$ does not depend on m. This dimension is called the Iitaka dimension of $\mathcal{L}$ and is denoted by $\kappa(X, \mathcal{L})$. By convention, if $\Gamma(X, \mathcal{L}^m) = 0$ for every m, then $\kappa(X, \mathcal{L}) = -\infty$.

The Iitaka dimension of $\mathcal{L}$ can be alternatively described as follows. If $\kappa(X, \mathcal{L}) = r \geq 0$, then there are A, $B > 0$ such that

$$A \cdot m^r < h^0(X, \mathcal{L}^m) < B \cdot m^r$$

for divisible enough m.

We have $\kappa(X, \mathcal{L}^m) = \kappa(X, \mathcal{L})$ for every positive m. This allows us to extend in the obvious way the definition of Iitaka dimension to $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

Exercise 14. If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor on a complete toric variety X, then the Iitaka dimension $\kappa(X, D)$ is equal to the dimension of P_D.

Globally generated line bundles. Recall that a function ψ defined on a subset of $N_{\mathbb{R}}$ is called **concave** (or **upper convex**) if for every v_1 and v_2 and every t with $0 \leq t \leq 1$, we have

$$\psi(tv_1 + (1-t)v_2) \geq t\psi(v_1) + (1-t)\psi(v_2) \tag{1}$$

whenever v_1, v_2 and $tv_1 + (1-t)v_2$ are in the domain of ψ.

We will consider functions defined on $|\Delta|$ which are linear on each of the cones in Δ. We assume that each maximal cone in Δ has dimension n. For every such cone σ, let $u(\sigma)$ in $M_{\mathbb{R}}$ be such that $\psi(v) = \langle u(\sigma), v \rangle$ for v in σ.

Lemma 3. *If all cones maximal in Δ have dimension n, then ψ is concave if and only if for every maximal cone σ in Δ and every $v \in |\Delta|$*

we have

$$\langle u(\sigma), v\rangle \geq \psi(v). \tag{2}$$

Proof. If (2) holds for every maximal cone, then for v_1, v_2 and $tv_1 + (1-t)v_2$ in $|\Delta|$ choose a maximal cone σ in Δ such that $tv_1+(1-t)v_2$ is in σ. Since $\psi(v_1) \leq \langle u(\sigma), v_1\rangle$ and $\psi(v_2) \leq \langle u(\sigma), v_2\rangle$ we deduce

$$\psi(tv_1 + (1-t)v_2) = \langle u(\sigma), tv_1 + (1-t)v_2\rangle \geq t\psi(v_1) + (1-t)\psi(v_2),$$

so ψ is concave.

Conversely, if ψ is concave and if σ is a maximal cone in Δ and v is in $|\Delta|$, choose w in the interior of σ. For some $0 < t < 1$ we have $tv + (1-t)w$ in σ and

$$\langle u(\sigma), tv + (1-t)w\rangle = \psi(tv + (1-t)w) \geq t\psi(v) + (1-t)\psi(w) =$$

$$t\psi(v) + (1-t)\langle u(\sigma), w\rangle.$$

This implies $\langle u(\sigma), v\rangle \geq \psi(v)$.

Suppose now, in addition, that $|\Delta|$ is convex. We say that ψ is **strictly concave** if the inequality in (2) is strict whenever v is not in σ (note that this definition depends on the fan).

Lemma 4. *Let ψ and Δ be as in Lemma 3. If $|\Delta|$ is convex and ψ is concave, then for every maximal cone σ we have*

$$\{v \in |\Delta| \mid \psi(v) = \langle u(\sigma), v\rangle\} = \bigcup_{u(\tau)=u(\sigma)} \tau, \tag{3}$$

the union being over the maximal cones τ in Δ such that $u(\tau) = u(\sigma)$. In particular, ψ is strictly concave if and only if $u(\sigma) \neq u(\tau)$ for every two distinct maximal cones σ and τ in Δ.

Proof. It is enough to prove the first assertion. If C_σ is the set on the left-hand side of (3), we see that for every maximal cone τ such that $u(\tau) = u(\sigma)$ we have $\tau \subseteq C_\sigma$. Since $|\Delta|$ is convex and ψ is concave, we deduce that C_σ is convex.

Moreover, we deduce from the concavity of ψ that for every maximal cones σ and τ we have $u(\sigma) - u(\tau)$ in $\tau^\vee$. This shows that if v in C_σ lies in the interior of a maximal cone τ, then $u(\tau) = u(\sigma)$.

Suppose now that v is an arbitrary point in C_σ. The convex hull of σ and v is contained in C_σ. Since σ is a maximal cone, this convex hull contains points arbitrarily close to v which lie in the interior of some maximal cone in Δ. Therefore v lies in some maximal cone τ such that $u(\tau) = u(\sigma)$.

Remark 5. If $|\Delta|$ is convex, but not necessarily n-dimensional, and if $\psi\colon |\Delta| \to \mathbb{R}$ is linear on each cone in Δ, then we call ψ strictly concave if it is so when we consider Δ as a fan in its linear span.

Proposition 3. *Let D be a Cartier T-divisor on the toric variety $X(\Delta)$. The line bundle $\mathcal{O}(D)$ is generated by its sections if and only if for every maximal cone σ in Δ we can find $u(\sigma)$ in M such that*

$$\psi_D(v) \leq \langle u(\sigma), v\rangle$$

for every v in $|\Delta|$, with equality if v is in σ. If each maximal cone in Δ has dimension n, then this is equivalent with ψ_D being concave.

Proof. $\mathcal{O}(D)$ is globally generated if and only if for every maximal cone σ in Δ there is $u(\sigma)$ in M such that $D + \operatorname{div}(\chi^{u(\sigma)})$ is effective and its restriction to U_σ is zero. Indeed, it is clear that if this is the case, then $\mathcal{O}(D)$ is globally generated. Conversely, if $\mathcal{O}(D)$ is globally generated, then for every maximal cone σ there is a section of $\mathcal{O}(D)$ which does not vanish at x_σ. As $\Gamma(X, \mathcal{O}(D))$ is spanned by elements of the form χ^u, it follows that there is $u(\sigma)$ in M such that $D+\operatorname{div}(\chi^{u(\sigma)})$ is effective on U_σ and its support does not contain x_σ. Therefore its restriction to U_σ is zero, as x_σ lies in every invariant subvariety of U_σ.

The divisor $D+\operatorname{div}(\chi^u)$ is zero on U_σ if and only if $\psi_D(v) = \langle u, v\rangle$ for v in σ. Moreover, $D + \operatorname{div}(\chi^u)$ is effective if and only if $\psi_D(v) \leq \langle u, v\rangle$ for every v in $|\Delta|$. This completes the proof, as the last assertion follows from Lemma 3.

We see that if $\mathcal{O}(D)$ is globally generated, then we can recover ψ_D from the polyhedron P_D:

$$\psi_D(v) = \min_{u \in P_D \cap M} \langle u, v\rangle.$$

From now on we assume that the support of Δ is convex (so it is a rational polyhedral convex cone).

Exercise 15. Given a fan Δ in N, show that $|\Delta|$ is convex if and only if there is a proper morphism of toric varieties $f = \phi_*\colon X \to Y$, where Y is an affine toric variety. Moreover, we may take ϕ to be a surjective group homomorphism.

We will always assume that $|\Delta|$ has maximal dimension n (and therefore every maximal cone in Δ has dimension n). Note that since $|\Delta|$ is convex, we may always reduce to this case, after replacing N by the intersection of N with the linear span of Δ. If D is a $\mathbb{Q}$-T-divisor on $X = X(\Delta)$, by Proposition 2 we can write P_D as the Minkowski sum of a polytope Q and $|\Delta|^\vee$. As $|\Delta|^\vee$ is a pointed cone, we have a canonical choice for Q, namely the convex hull of the vertices of P_D.

We show now that under these assumptions on Δ, we can describe explicitly the vertices of P_D when $\mathcal{O}(D)$ is globally generated.

Lemma 5. *Suppose that $|\Delta|$ is a convex cone of dimension n and let D be a T-Cartier divisor such that $\mathcal{O}(D)$ is globally generated. If for every maximal cone σ in Δ, $u(\sigma)$ in M is such that $\psi_D(v) = \langle u(\sigma), v\rangle$ for every v in σ, then the vertices of P_D are the points $u(\sigma)$ (note that these points might not be distinct).*

Proof. Since all maximal cones in Δ have dimension n, Proposition 3 says that $\mathcal{O}(D)$ is globally generated if and only if for every maximal cone σ in Δ we have $u(\sigma)$ in P_D. For such σ, let v be in the interior of σ. Note that for every u in P_D we have $\langle u, v\rangle \geq \langle u(\sigma), v\rangle$, with equality if and only if $u = u(\sigma)$. Therefore $u(\sigma)$ is a vertex of P_D.

Suppose now that w is a vertex of P_D. We can find v in $N_{\mathbb{R}}$ such that $\langle u, v\rangle \geq \langle w, v\rangle$ for every u in P_D, with equality if and only if $u = w$. In particular, we have $\langle u, v\rangle \geq 0$ for every u in $|\Delta|^\vee$, so v lies in $|\Delta|$. If σ is a maximal cone containing v, then $\langle w, v\rangle \geq \langle u(\sigma), v\rangle$ since w is in P_D, using the definition of P_D. On the other hand, $\langle u(\sigma), v\rangle \geq \langle w, v\rangle$ since $u(\sigma)$ is in P_D and by the choice of v. We deduce that $w = u(\sigma)$.

Remark 6. It follows from the above lemma that if X is complete and $\mathcal{O}(D)$ is globally generated, then P_D is a lattice polytope.

Exercise 16. Show that if $|\Delta|$ is convex, and D and E are $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisors such that for some m both $\mathcal{O}(mD)$ and $\mathcal{O}(mE)$ are globally generated line bundles, then $P_{D+E} = P_D + P_E$.

Exercise 17. If $X = X(\Delta)$ is an arbitrary toric variety and D is a Cartier T-divisor, then the sections $\chi^{u_1}, \ldots, \chi^{u_r}$ in $\Gamma(X, \mathcal{O}(D))$ generate $\mathcal{O}(D)$ if and only if for every maximal cone σ is Δ one can choose $u(\sigma)$ among the u_i as in Proposition 3. If $|\Delta|$ is convex and has dimension n, then this is the case if and only if $\{u_1, \ldots, u_r\}$ contains all the vertices of P_D.

Let D be a Cartier T-divisor on an arbitrary toric variety $X(\Delta)$. Consider a linear system Σ generated by the sections $\chi^{u_1}, \ldots, \chi^{u_r}$ of $\mathcal{O}(D)$ such that Σ generates $\mathcal{O}(D)$. By Exercise 17, for every maximal cone σ in Δ we can choose $u(\sigma)$ among the u_i, as in Proposition 3. The linear system Σ determines a morphism

$$f_\Sigma \colon X(\Delta) \to \mathbb{P}^{r-1}, \quad x \to (\chi^{u_1}(x) : \ldots : \chi^{u_r}(x))$$

to the projective space $\mathbb{P}^{r-1}$.

Consider $\mathbb{P}^{r-1}$ as the toric variety defined by the fan in $N' := \mathbb{Z}^r/\sum_i e_i$ with maximal cones τ_j generated by the classes $\overline{e_i}$ of e_i for $i \neq j$. Note that f_Σ is a morphism of toric varieties. Indeed, if we take $(\mathbb{Z}^r)^* \to M$ such that the element e_i^* of the dual basis of $\mathbb{Z}^r$ is mapped to u_i, then the dual of the composition

$$M' = \{u = \sum_i u_i e_i^* \in (\mathbb{Z}^r)^* \mid \sum_i u_i = 0\} \hookrightarrow (\mathbb{Z}^r)^* \to M$$

gives the group morphism $\phi\colon N \to N'$ which is a fan morphism inducing f_Σ. To see this, note that if σ is a maximal cone in Δ and if $u(\sigma) = u_j$, then $\phi(v) = \sum_{i\neq j}\langle u_i - u(\sigma), v\rangle\overline{e_i}$ for every v in σ. Therefore $\phi(\sigma)$ is contained in τ_j. Moreover, if $T_1, \ldots, T_r$ are the homogeneous coordinates on $\mathbb{P}^{r-1}$ and if V_j is the affine open subset defined by $T_j \neq 0$, we deduce that the induced morphism $U_\sigma \to V_j$ corresponds to

$$k[T_i/T_j \mid 1 \leq i \leq r] \to k[\sigma^\vee \cap M], \quad T_i/T_j \to \chi^{u_i - u(\sigma)}.$$

This shows that $f_\Sigma = \phi_*$.

Ample divisors. Let D be a Cartier T-divisor on $X(\Delta)$. We assume that $|\Delta|$ is convex, of dimension n, and for every maximal cone σ in Δ we choose $u(\sigma)$ in M such that $\psi_D(v) = \langle u(\sigma), v\rangle$ for every v in σ.

Proposition 4. *If $|\Delta|$ is a convex cone of dimension n, then $\mathcal{O}(D)$ is very ample if and only if ψ_D is strictly concave and for every maximal cone σ in Δ, $S_\sigma := \sigma^\vee \cap M$ is generated by $\{u - u(\sigma) \mid u \in P_D \cap M\}$.*

Proof. By Proposition 3, we may assume that $\mathcal{O}(D)$ is globally generated. We have $\mathcal{O}(D)$ very ample if and only if there is a finite-dimensional vector subspace $V \subseteq \Gamma(X, \mathcal{O}(D))$ such that the corresponding morphism $X \to \mathbb{P}(V)$ is a locally closed embedding. Note that after possibly enlarging V, we may assume that it is generated by $\chi^{u_1}, \ldots, \chi^{u_r}$, for some $u_1, \ldots, u_r$ in $P_D \cap M$.

Suppose that $u_1, \ldots, u_r$ are lattice points in P_D and let $f\colon X(\Delta) \to \mathbb{P}^{r-1}$ be the corresponding morphism. We denote by $T_1, \ldots, T_r$ the homogeneous coordinates on $\mathbb{P}^{r-1}$.

We prove first the "if" part. If σ and τ are two maximal cones, let j and ℓ be such that $u_j = u(\sigma)$ and $u_\ell = u(\tau)$. We see that f maps U_σ to the open subset V_j defined by $T_j \neq 0$ and $U_\sigma \cap f^{-1}(V_\ell) = U_{\sigma\cap\tau}$, since $\sigma \cap (u_j - u_\ell)^\perp = \sigma \cap \tau$ by the strong concavity of ψ_D. Letting σ vary, we see that $f^{-1}(V_\ell) = U_\tau$.

The corresponding morphism of affine varieties $U_\tau \to V_\ell$ is induced by

$$k[T_i/T_\ell \mid 1 \leq i \leq r] \to k[S_\tau], \quad T_i/T_\ell \to \chi^{u_i - u(\tau)}.$$

This is a closed immersion if and only if $\{u_i - u(\tau) \mid 1 \leq i \leq r\}$ generates S_τ. Since every S_τ is finitely generated, it follows by hypothesis that after possibly adding some lattice points in P_D, we may assume that every S_τ is generated by the $u_i - u(\tau)$, so $\mathcal{O}(D)$ is very ample.

Conversely, suppose that f is a locally closed embedding. In this case, it follows from the lemma below that for every j the inverse image $f^{-1}(V_j)$ is affine. Since it is an invariant open subset of $X(\Delta)$, it is equal to U_τ for some cone τ in Δ. If σ is a maximal cone in Δ and j is such that $u(\sigma) = u_j$, then $U_\sigma \subseteq f^{-1}(V_j)$, so we must have equality $U_\sigma = f^{-1}(V_j)$. Therefore $\tau \cap (u(\sigma) - u(\tau))^\perp = \sigma \cap \tau$ for every maximal cones σ and τ, which shows that ψ_D is strictly concave.

We deduce the second assertion from the fact that $U_\sigma \to V_j$ is a locally closed embedding. Indeed, this implies that if S is the semigroup generated by the $u_i - u(\sigma)$, then $\mathrm{Spec}(k[S_\sigma]) \to \mathrm{Spec}(k[S])$ is an open immersion. In particular, $\mathrm{Spec}(k[S_\sigma]) \to \mathrm{Spec}(k[S^{\mathrm{sat}}])$ is an open immersion. By Corollary 4 in Chapter 4, we see that $\sigma \cap N$ is a face of $S^\vee$. Since σ has dimension n, it follows that $S_\sigma = S^{\mathrm{sat}}$. We deduce that the normalization morphism $\mathrm{Spec}(k[S^{\mathrm{sat}}]) \to \mathrm{Spec}(k[S])$ is an open immersion, so it is an isomorphism. This shows that S_σ is generated by the $u_i - u(\sigma)$, and completes the proof.

Lemma 6. *Let $f \colon X = X(\Delta) \to Y$ be a morphism of toric varieties which is a locally closed embedding. If Y is affine and the support of Δ is convex, then X is affine.*

Proof. Let ϕ be the morphism of fans such that $f = \phi_*$. If $\sigma = |\Delta|$, then we have a proper, surjective morphism of toric varieties $g \colon X \to U_\sigma$ (note that if σ is not a pointed cone, then we follow our convention for defining the toric variety associated to a degenerate fan). We have $\phi(\sigma \cap (-\sigma)) = 0$, so we get an induced morphism of toric varieties $h \colon U_\sigma \to Y$ such that $f = h \circ g$. Since f is a locally closed embedding, so is g, which implies that g is an isomorphism. Therefore X is an affine variety.

Corollary 1. *If $|\Delta|$ is an n-dimensional convex cone, then a $\mathbb{Q}$-Cartier T-divisor on $X(\Delta)$ is ample if and only if ψ_D is strictly concave.*

Proof. Note that both conditions are invariant under multiplying D by a positive integer, so in particular we may assume that D is a Cartier divisor. By Proposition 4, it is enough to show that if ψ_D is strictly concave, then there is a positive integer m such that for every maximal cone σ in Δ, S_σ is generated by

$$\{u - mu_\sigma \mid u \in mP_D \cap M\}.$$

Write $D = \sum_{i=1}^{d} a_i D_i$. If u is in S_σ, then $u + mu(\sigma)$ is in mP_D for $m \gg 0$. Indeed, $\langle u, v_i\rangle \geq 0$ if v_i is in σ and $\langle u(\sigma), v_i\rangle + a_i \geq 0$ for all i, with strict inequality if v_i is not in σ (since $\psi(v_i) < \langle u(\sigma), v_i\rangle$ in this case). All semigroups S_σ are finitely generated, so we may choose a positive integer m with the required property.

Note that if $|\Delta|$ is convex (in particular, if $X(\Delta)$ is complete), then by Corollary 1, every ample line bundle on $X(\Delta)$ is globally generated.

Remark 7. It follows from Lemmas 4 and 5 that if D is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor on $X(\Delta)$ and $|\Delta|$ is an n-dimensional convex cone, then the vertices of P_D are the distinct vectors $u(\sigma)$, where σ varies over the maximal cones of Δ.

Remark 8. If $X = X' \times (k^*)^r$, $p\colon X \to X'$ is the projection, and D' is a divisor on X', then $p^*(D')$ is ample if and only if D' is ample. We deduce that the assertion in Corollary 1 holds for every Δ with $|\Delta|$ convex, not necessarily n-dimensional.

We describe now the connection between ample divisors and the construction of toric varieties from polyhedra. Suppose that D is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor on the complete toric variety $X(\Delta)$. We have associated to D the rational polyhedron P_D in $M_\mathbb{R}$. The dimension of this polyhedron is n. Indeed, if v in $N_\mathbb{R}$ and a in $\mathbb{R}$ are such that $\langle u, v\rangle = a$ for every u in P_D, then we must have $\langle u, v\rangle = 0$ for every u in $|\Delta|^\vee$. Therefore v is in $|\Delta| \cap (-|\Delta|)$. If $-v$ lies in the maximal cone σ, then v is not in σ. On the other hand we have $\psi_D(v) = a = \langle u(\sigma), v\rangle$, a contradiction with the fact that ψ_D is strictly concave.

We claim that Δ is the normal fan to P_D, so $X(\Delta)$ is the toric variety corresponding to P_D. Since the vertices of P_D are the $u(\sigma)$, where σ varies over the maximal cones in Δ, we see that each maximal cone τ in the normal fan to P_D corresponds to precisely one $\sigma \in \Delta$, such that

$$\tau = \{v \in N_\mathbb{R} \mid \langle u(\sigma), v\rangle \leq \langle u, v\rangle \quad \text{for every} \quad u \in P_D\}.$$

Note that if v is in τ, then $\langle u, v\rangle \geq 0$ for u in $|\Delta|^\vee$, so v lies in $|\Delta|$. We deduce that $\tau = \sigma$ by the strict concavity of ψ_D. Therefore Δ is the normal fan to P_D.

Conversely, suppose we start with an n-dimensional rational polyhedron P in $M_\mathbb{R}$ which does not contain any affine subspace of $M_\mathbb{R}$. Equivalently, if we write $P = Q + C$ for a polytope Q and a cone C, then C is a pointed cone. Let $X = X_P$ be the toric variety corresponding to the normal fan Δ_P of P, so $|\Delta_P| = C^\vee$ is an n-dimensional convex cone.

We define $\psi\colon |\Delta_P| \to \mathbb{R}$ by $\psi(v) := \min_{u\in P}\langle u, v\rangle$. Note that $\psi(N_{\mathbb{Q}}) \subseteq \mathbb{Q}$. By definition of the normal fan, if τ is the maximal cone of Δ_P corresponding to the vertex w of P, then $\psi(v) = \langle w, v\rangle$ for every v in τ. This shows that there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor $D = D_P$ on X such that $\psi = \psi_D$. Moreover, if w and τ are as before, then $w = u(\tau)$. It follows from the definition of ψ that ψ_D is concave, hence strictly concave as $u(\sigma) \neq u(\tau)$ for every distinct maximal cones σ and τ. Therefore D is ample and $P = P_D$, as the two polyhedra have the same vertices and the same cone in the Minkowski decomposition.

This gives a bijection between rational n-dimensional polyhedra P in $M_{\mathbb{R}}$ which do not contain an affine subspace of $M_{\mathbb{R}}$ and quasiprojective toric varieties $X = X(\Delta)$ with $|\Delta|$ convex and n-dimensional, together with an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor D on X. If P corresponds to X and D, then $P = P_D$. Note that D is a Cartier divisor if and only if all vertices of P are in the lattice.

Remark 9. Suppose that P is an m-dimensional rational polyhedron in $M_{\mathbb{R}}$, with $m < n$, such that P contains no affine subspace of $M_{\mathbb{R}}$. In this case the normal fan Δ_P is degenerate and the corresponding toric variety $X(\Delta_P)$ has dimension m. Suppose that the affine span W of P is a linear subspace. In this case Δ_P is the inverse image by the projection $N_{\mathbb{R}} \to N_{\mathbb{R}}/W^\perp$ of a nondegenerate fan Δ'_P. If we put $\psi(v) := \min_{u\in P}\langle u, v\rangle$ for v in $|\Delta_P|$, then ψ induces a strictly concave function $\psi'\colon |\Delta'_P| \to \mathbb{R}$. This determines an ample divisor D on X_P such that $P_D = P$.

Remark 10. If P is a rational polyhedron in $M_{\mathbb{R}}$ which contains an affine subspace of $M_{\mathbb{R}}$, write $P = Q + C$ for a polytope Q and a convex cone C. The maximal affine subspace of P is a translate of $W = C \cap (-C)$. If the dimension of W is r, then we get an isomorphism $X(\Delta_P) \simeq X(\Delta_{P'}) \times (k^*)^r$, where P' is the image of P in $M_{\mathbb{R}}/W$. Moreover, if the origin lies in the affine span of P and if we define $\psi\colon |\Delta_P| \to \mathbb{R}$ by $\psi(v) := \min_{u\in P}\langle u, v\rangle$, this gives an ample divisor D_P on $X(\Delta_P)$ which is the pull-back of the divisor corresponding to P' on $X(\Delta_{P'})$.

We use Corollary 1 to characterize f-ample divisors. Recall that if $f\colon X \to Y$ is a morphism of varieties, a $\mathbb{Q}$-Cartier divisor D on X is called f-ample if Y has a cover by affine open subsets U_i such that the restriction of D to each of the $f^{-1}(U_i)$ is ample. In fact, the definition is independent on the affine cover. Moreover, if f is proper, then D is ample if and only if its restriction to each fiber of f is ample (see, for example, Proposition 1.41 in [KM]).

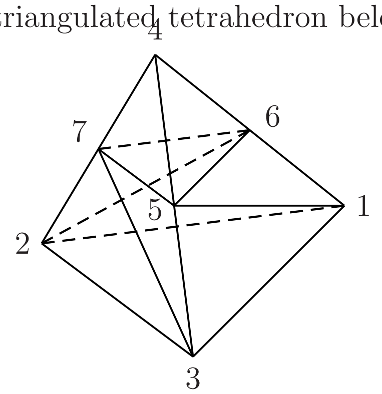
Suppose that $\phi\colon N' \to N$ induces a morphism of fans between Δ' and Δ such that the corresponding morphism of toric varieties $f = \phi_*\colon X(\Delta') \to X(\Delta)$ is proper.

Corollary 2. *With the above notation, if D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-T-divisor on $X(\Delta')$, then D is f-ample if and only if for every maximal cone σ in Δ, the restriction of ψ_D to $\phi^{-1}(\sigma)$ is strictly concave.*

Proof. Let σ be a maximal cone in Δ. Note that since f is proper, for every maximal cone σ in Δ, $f^{-1}(U_\sigma)$ is a toric variety whose fan has support $\phi^{-1}(\sigma)$, so it is convex. Since D is f-ample if and only if its restriction to each $f^{-1}(U_\sigma)$ is ample, the assertion in the corollary follows from Corollary 1 (see also Remark 8).

We have seen an example of a three-dimensional complete toric variety X having no nontrivial line bundles. In particular, X is not projective. Using the description of ample line bundles in Corollary 1 it is easy to give examples of smooth complete toric varieties which are not projective (in this case the variety has many nontrivial line bundles, as its Picard group is isomorphic to $\mathbb{Z}^{d-n}$, by Proposition 1).

Consider the fan Δ in $\mathbb{Z}^3$ with rays passing through $v_1 = -e_1$, $v_2 = -e_2$, $v_3 = -e_3$, $v_4 = e_1 + e_2 + e_3$, $v_5 = v_3 + v_4$, $v_6 = v_1 + v_4$, and $v_7 = v_2 + v_4$. The cones in Δ are the cones through the faces of the triangulated tetrahedron below



Exercise 18. Show that $X = X(\Delta)$ is complete and nonsingular. On the other hand, show that if ψ is a strictly concave function on $N_{\mathbb{R}}$, then

$$\psi(v_1) + \psi(v_5) > \psi(v_3) + \psi(v_6)$$
$$\psi(v_2) + \psi(v_6) > \psi(v_1) + \psi(v_7)$$
$$\psi(v_3) + \psi(v_7) > \psi(v_2) + \psi(v_5)$$

which add to a contradiction. Deduce that X is not projective.

Exercise 19. Prove the toric version of Chow's Lemma, that for any complete toric variety there is a proper, birational morphism of toric varieties $f\colon X' \to X$, such that X' is projective. Carry this out for the preceding example.

Exercise 20. Show that every complete toric surface is projective.

Ample versus very ample line bundles. We discuss now how far is an ample line bundle on a toric variety from being very ample. In this subsection, all toric varieties will be complete.

Proposition 5. *If $\mathcal{L}$ is an ample line bundle a smooth projective toric variety $X = X(\Delta)$, then $\mathcal{L}$ is very ample.*

Proof. Let D be a Cartier T-divisor such that $\mathcal{L} \simeq \mathcal{O}(D)$. If σ is a maximal cone in Δ and $e_1, \ldots, e_n$ are the primitive generators of the rays in σ, then they form a basis of the lattice N. We denote by $e_1^*, \ldots, e_n^*$ the dual basis of M.

For $i \leq n$, let $\sigma_i \neq \sigma$ be the maximal cone in Δ having the convex cone generated by $e_1, \ldots, \widehat{e_i}, \ldots, e_n$ as a facet. Since $\mathcal{O}(D)$ is ample, by Corollary 1 we have

$$\langle u(\sigma_i) - u(\sigma), e_j \rangle = 0 \quad \text{for} \quad j \neq i,$$

$$\langle u(\sigma_i) - u(\sigma), e_i \rangle > 0.$$

Therefore there are positive integers m_i such that $u(\sigma_i) - u(\sigma) = m_i e_i^*$ for every i. We have $u_i := u(\sigma) + \frac{1}{m_i}(u(\sigma_i) - u(\sigma))$ in $P_D \cap M$ and $u_i - u(\sigma) = e_i^*$. Hence $\sigma^\vee$ is generated by $\{u - u(\sigma) \mid u \in P_D \cap M\}$. This holds for every maximal cone σ, so $\mathcal{O}(D)$ is very ample by Proposition 4.

Exercise 21. Let $M = \mathbb{Z}^3$ and P the convex hull of $(0,0,0)$, $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$. Let X be the corresponding toric variety and D the Cartier T-divisor such that $P = P_D$. Show that $\mathcal{O}(D)$ is not very ample; in fact, it determines a morphism $f\colon X \to \mathbb{P}^3$ that is a double cover branched along the four coordinate planes.

Recall that if X is a normal variety, a very ample line bundle $\mathcal{L}$ on X is called normally generated if the homogeneous coordinate ring of $X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{L}))$ is normal. Equivalently, for every $m \geq 1$ the multiplication map $\mathrm{Sym}^m \Gamma(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^m)$ is surjective.

Remark 11. If $\mathcal{L}$ is an ample line bundle on a projective variety X and $\mathcal{L}$ is globally generated, but not very ample, then there is m such that the natural multiplication map $\mathrm{Sym}^m\Gamma(X,\mathcal{L}) \to \Gamma(X,\mathcal{L}^m)$ is not surjective. Indeed, let m be such that $\mathcal{L}^m$ is very ample and suppose that the multiplication map is surjective for m. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f_{\mathcal{L}}} & \mathbb{P}(\Gamma(X,\mathcal{L})) \\ {\scriptstyle f_{\mathcal{L}^m}}\downarrow & & \downarrow{\scriptstyle i} \\ \mathbb{P}(\Gamma(X,\mathcal{L}^m)) & \xrightarrow{j} & \mathbb{P}(\mathrm{Sym}^m\Gamma(X,\mathcal{L})), \end{array}$$

where $f_{\mathcal{L}}$ and $f_{\mathcal{L}^m}$ are the morphisms defined by the complete linear systems of $\mathcal{L}$ and $\mathcal{L}^m$, respectively, j is induced by the multiplication map and i is the Veronese embedding. As $f_{\mathcal{L}^m}$, i and j are closed immersions, so is $f_{\mathcal{L}}$, a contradiction.

Exercise 22. Let D be an ample Cartier T-divisor on the complete toric variety $X = X(\Delta)$. For $P = P_D$, we take σ to be the cone over $P \times \{1\}$ in $M_{\mathbb{R}} \times \mathbb{R}$.

(1) Show that if we consider on $k[\sigma \cap (M \times \mathbb{Z})]$ the $\mathbb{N}$-grading given by $\deg(\chi^{(u,m)}) = m$, then we have an isomorphism $X \simeq \mathrm{Proj}(k[\sigma \cap (M \times \mathbb{Z})])$.
(2) Show that $\mathcal{O}(D)$ is normally generated if and only if for every positive integer m, any lattice point in mP is the sum of m lattice points in P.
(3) If this is the case, then $X \hookrightarrow \mathbb{P}(\Gamma(X,\mathcal{O}(D)))$ is arithmetically Cohen-Macaulay, i.e. the homogeneous coordinate ring is Cohen-Macaulay.

Remark 12. There are examples of very ample line bundles on projective toric varieties which are not normally generated (see [BG], Example 5.1).

The next proposition gives a uniform bound in terms of dimension for a multiple of an ample line bundle on a toric variety to be very ample (and even normally generated). The following proof is due to Nakagawa (see also [EW] and [BGT]).

Proposition 6. *If $\mathcal{L}$ is an ample line bundle on a projective toric variety X of dimension $n \geq 2$ and if $m \geq n-1$, then the natural multiplication map*

$$\Gamma(X,\mathcal{L}) \otimes \Gamma(X,\mathcal{L}^m) \to \Gamma(X,\mathcal{L}^{m+1})$$

is surjective. In particular, for every $m \geq n-1$ the line bundle $\mathcal{L}^m$ is normally generated, so in particular, it is very ample.

Proof. It is enough to prove the first assertion. Let D be a Cartier T-divisor such that $\mathcal{L} \simeq \mathcal{O}(D)$, and let C be the cone over $P_D \times \{1\}$ in $M_{\mathbb{R}} \times \mathbb{R}$. By Lemma 8 in Chapter 2, we may write C as a union of cones C_i such that each cone C_i is generated by $(n+1)$ linearly independent elements $u_{i,1}, \ldots, u_{i,n+1}$ in $(P_D \cap M) \times \{1\}$. Moreover, we may assume that there is no lattice point different from the $u_{i,j}$ in the interior of the convex hull of $u_{i,1}, \ldots, u_{i,n+1}$. Indeed, if there is a lattice point u in the interior of this convex hull, then we may replace the cone C_i by the maximal cones in the star subdivision $C_i(u)$. As u is in $P_D \cap M$, after repeating this finitely many times we achieve our requirement.

Let $m \geq n-1$ and consider u in $(m+1)P_D \cap M$. We need to show that u lies in $(P_D \cap M) + (mP_D \cap M)$. We can find a cone C_i as above such that $(u, m+1)$ lies in C_i. There are nonnegative real numbers $\lambda_1, \ldots, \lambda_{n+1}$ such that

$$\sum_{j=1}^{n+1} \lambda_j u_{i,j} = u, \quad \sum_{j=1}^{n+1} \lambda_j = m+1.$$

If for some j we have $\lambda_j \geq 1$, then we may write $u = u_{i,j} + u'$, where $u' = \lambda_1 u_{i,1} + \ldots + (\lambda_j - 1)u_{i,j} + \ldots + \lambda_{n+1} u_{i,n+1}$, so u' is in $mP_D \cap M$.

Suppose now that $\lambda_j < 1$ for every j. This implies $m+1 = \sum_j \lambda_j < n+1$, so $m = n-1$. Let $w = \sum_{j=1}^{n+1}(1-\lambda_j)u_{i,j}$, so w is in M. Moreover, as $\sum_j (1-\lambda_j) = 1$, we see that w lies in the interior of the convex hull of $u_{i,1}, \ldots, u_{i,n+1}$, a contradiction. This completes the proof.

Exercise 23. Show that if $\mathcal{L}$ and $\mathcal{M}$ are ample line bundles on a projective n-dimensional toric variety X, then for every $m \geq n$ the multiplication map

$$\Gamma(X, \mathcal{L}) \otimes \Gamma(X, \mathcal{L}^m \otimes \mathcal{M}) \to \Gamma(X, \mathcal{L}^{m+1} \otimes \mathcal{M})$$

is surjective.

It is known that the embeddings of a projective toric variety defined by larger multiples of a given ample line bundle satisfy suitable stronger properties. For example, it is shown in [BGT] that if $\mathcal{L}$ is ample and $m \geq \dim(X)$, then the homogeneous coordinate ring of the embedding given by $\mathcal{L}^m$ is Koszul. In particular, the ideal I_X defining X in the projective space is generated by quadrics. Moreover, it is shown in [He] and [SS] that if $p \geq 1$ and $m \geq \dim(X) + p - 1$, then the embedding given by $\mathcal{L}^m$ satisfies the property N_p, i.e. the first p steps in the minimal free resolution of I_X over the homogeneous coordinate ring S of the projective space are as follows:

$$S(-p-1)^{\oplus \beta_p} \to \ldots \to S(-2)^{\oplus \beta_1} \to I_X \to 0.$$

Note that the property N_1 says precisely that I_X is generated by quadrics.

We have seen that every ample line bundle on a smooth toric variety is very ample. It is not known whether the following stronger assertions are true.

Problem. *Let $\mathcal{L}$ be an ample line bundle on a smooth projective toric variety.*

(1) *(Oda) Is $\mathcal{L}$ normally generated ?*
(2) *(Sturmfels) If $\mathcal{L}$ is normally generated, does it follow that the image of X by the morphism given by $\mathcal{L}$ has the ideal generated by quadrics ?*

Remark 13. Suppose (1) above has a positive answer. If $\mathcal{L}$ and $\mathcal{M}$ are ample line bundles on the smooth projective toric variety X, then the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(L \oplus M)$ is ample. Applying (1) for $\mathcal{O}(1)$ gives the surjectivity of the multiplication map

$$\Gamma(X, \mathcal{L}) \otimes \Gamma(X, \mathcal{M}) \to \Gamma(X, \mathcal{L} \otimes \mathcal{M}).$$

Equivalently, if D and E are T-divisors such that $\mathcal{L} \simeq \mathcal{O}(D)$ and $\mathcal{M} \simeq \mathcal{O}(E)$, then every lattice point in $P_D + P_E$ is a sum of lattice points in P_D and P_E.

Remark 14. If X is a projective toric variety of dimension n and $\mathcal{L}$ is an ample line bundle which is normally generated, then it is easy to give upper bounds on the degree of the equations defining the embedding $X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{L}))$. In fact, the ideal I_X defining X is $(n+1)$-regular in the sense of Castelnuovo-Mumford, so in particular the degrees of its minimal generators are $\leq (n+1)$ (see [Eis], Chapter 20 for equivalent definitions of Castelnuovo-Mumford regularity). Note that I_X is $(n+1)$-regular if and only if the homogeneous coordinate ring of X is n-regular. Since X is projectively normal, this is equivalent with the vanishing of $H^i(X, \mathcal{L}^{n-i})$ for $1 \leq i \leq n$. We will see in the next chapter that a globally generated line bundle on a toric variety has no higher cohomology, which gives our assertion.

Problem. *If $\mathcal{L}$ is a very ample line bundle on a projective toric variety X, give m such that the multiplication map*

$$\mathrm{Sym}^r \Gamma(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^r)$$

is surjective for every $r \geq m$. Given such m, the argument in the above remark shows that the ideal I_X of X in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ is $\min\{n+1, m\}$*-regular.*

More generally, suppose that $X \hookrightarrow \mathbb{P}^N$ is a morphism of toric varieties which is a closed immersion. Find lower bounds for the regularity of the ideal I_X. Recall that a general conjecture of Eisenbud and Goto [EG] *predicts that if $r = \deg(X) - \mathrm{codim}(X, \mathbb{P}^N) - 1$, then I_X is r-regular. Note that if X is projectively normal, then the conjecture is satisfied. Indeed, this follows from the fact that X is arithmetically Cohen-Macaulay; after n successive hyperplane sections, the assertion is reduced to the case of a reduced set of points, when it can be easily checked.*

Restriction of divisors. Let $X = X(\Delta)$ be a toric variety. If D is a Cartier T-divisor on X and τ is a cone in Δ, we describe the restriction of $\mathcal{O}(D)$ to $V(\tau)$. We have previously considered inverse images of Cartier T-divisors by toric morphisms, but recall that the inclusion $V(\tau) \hookrightarrow X$ is not toric, unless it is the identity.

Note first that we may assume that $V(\tau)$ is not contained in the support of D. More precisely, we may replace D by a linearly equivalent divisor with this property. Indeed, let σ be a cone in Δ containing τ. Since D is Cartier, we can find $u(\sigma)$ in M such that the restriction of $D' = D + \mathrm{div}(\chi^{u(\sigma)})$ to U_σ is zero. In particular, we see that no prime divisor containing $V(\tau)$ is in the support of D'.

Therefore we may assume that $V(\tau)$ is not contained in the support of D. In this case we have a restriction $D|_{V(\tau)}$ that is a Cartier divisor. We describe it locally: suppose that σ is a cone containing τ and let $u(\sigma)$ in M be such that $D|_{U_\sigma} = \mathrm{div}(\chi^{-u(\sigma)})|_{U_\sigma}$. The hypothesis on D implies that $u(\sigma)$ lies in $\tau^\perp$. Therefore the restriction of $D|_{V(\tau)}$ to $V(\tau) \cap U_\sigma$ is the principal divisor corresponding to $\chi^{-u(\sigma)}$ (recall that the torus in $V(\tau)$ is $\mathrm{Spec}(k[M \cap \tau^\perp])$).

We see that $P_{D|_{V(\tau)}} = P_D \cap \tau^\perp$. Moreover, if $p\colon N \to N/N_\tau$ is the projection, then we have $\psi_D = \psi_{D|_{V(\tau)}} \circ p$.

We compare now the coefficients of the prime divisors in D and $D|_{V(\tau)}$. A prime T-divisor in $V(\tau)$ is of the form $V(\gamma)$ for some γ in Δ containing τ and such that $\dim(\gamma) = \dim(\tau) + 1$. Recall that we denote by $v_1, \ldots, v_d$ the primitive generators of the rays in Δ.

Let I_γ be the set of those i such that v_i is on a ray in γ but not in τ. We take the generator e of N_γ/N_τ such that the classes $\overline{v_i}$ for $i \in I_\gamma$ are positive multiples of e. Let s_i be such that $\overline{v_i} = s_i e$. We denote by a_i the coefficient of D_i in D and by b_γ the coefficient of $V(\gamma)$ in $D|_{V(\tau)}$. If σ and $u(\sigma)$ are as before, then $\langle u(\sigma), v_i \rangle = -a_i$ and $\langle u(\sigma), e \rangle = -b_\gamma$, and therefore

$$b_\gamma = \frac{a_i}{s_i}$$

for every i in I_γ.

Example 1. If X is smooth, then I_γ consists of only one i. Moreover, the coefficient of $V(\gamma)$ in $D|_{V(\tau)}$ is equal to the coefficient of D_i in D.

Exercise 24. Show that if γ is simplicial, then there is only one i in I_γ and

$$b_\gamma = a_i \cdot \frac{\mathrm{mult}(\tau)}{\mathrm{mult}(\gamma)}.$$

Recall that if D is Cartier divisor on X and C is a complete curve in X, then the intersection number $(D \cdot X)$ is the degree of $D|_C$.

Exercise 25. Suppose that an $(n-1)$-dimensional cone τ is the common face of two nonsingular n-dimensional cones γ' and γ''. Let $v_1, \ldots, v_{n-1}$ be the primitive generators of the rays in τ and let v' and v'' be the primitive generators of the other rays in γ' and γ'', respectively.

(1) Show that there are (unique) integers $a_1, \ldots, a_{n-1}$ such that

$$v' + v'' = a_1 v_1 + \ldots + a_{n-1} v_{n-1}.$$

(2) Show that for $1 \leq k \leq n-1$ we have $(D_k \cdot V(\tau)) = -a_k$.

Exercise 26. Let D be an ample divisor on the n-dimensional complete toric variety X. If D_i is the prime divisor corresponding to the facet Q_i of P_D, then

$$(D^{n-1} \cdot D_i) = (n-1)! \cdot \mathrm{vol}_{n-1}(Q_i).$$

We use the description of restrictions of divisors to invariant subvarieties to prove a toric version of the Nakai criterion. We treat first the criterion for being globally generated.

Suppose that $X = X(\Delta)$, where $|\Delta|$ is convex. A $\mathbb{Q}$-divisor on X is called **numerically effective** (**nef**, for short) if $(D \cdot C) \geq 0$ for every complete irreducible curve C on X. The next proposition shows in particular that unlike on general varieties, every nef divisor on a complete toric variety is globally generated (see also [Re] and [Ma] for different proofs).

Proposition 7. *Let $X = X(\Delta)$ be a toric variety such that $|\Delta|$ is convex, and let $\mathcal{L}$ be a line bundle on X. The following are equivalent:*

(1) *$\mathcal{L}$ is globally generated.*
(2) *$\mathcal{L}$ is nef.*
(3) *For every complete invariant curve $C \subseteq X$, we have $(\mathcal{L} \cdot C) \geq 0$.*

Exercise 27. Let D be a T-Cartier divisor on the toric variety $X = X(\Delta)$, where $|\Delta|$ is convex and of dimension n. The line bundle $\mathcal{O}(D)$ is globally generated if and only if for every maximal cones σ, $\sigma' \in \Delta$

with $\dim(\sigma \cap \sigma') = n-1$, we have $\langle u(\sigma), v\rangle \geq \langle u(\sigma'), v\rangle$ for every $v \in \sigma'$. Similarly, $\mathcal{O}(D)$ is ample if and only if for every σ and σ' as above, $\langle u(\sigma), v\rangle > \langle u(\sigma'), v\rangle$ for every v in the interior of σ'.

Proof of Proposition 7. We may assume that $|\Delta|$ has maximal dimension. The implications (1)⇒(2)⇒(3) are trivial, so it is enough to show (3)⇒(1). Let σ and σ' be two maximal cones in Δ with $\dim(\sigma \cap \sigma') = n-1$, and let $\tau = \sigma \cap \sigma'$. Suppose that D is a Cartier T-divisor such that $\mathcal{O}(D) \simeq \mathcal{L}$.

Let $u(\sigma)$ and $u(\sigma')$ be such that $D|_{U_\sigma} = \mathrm{div}(\chi^{-u(\sigma)})$ and similarly on $U_{\sigma'}$. We write $D + \mathrm{div}(\chi^{u(\sigma)} = \sum_i a_i D_i$, where the D_i are the prime T-divisors. Take j such that the primitive vector v_j corresponding to D_j is in $\sigma' \setminus \sigma$. Note that $a_j = \langle u(\sigma) - u(\sigma'), v_j\rangle$.

It follows from the description of $D|_{V(\tau)}$ that there is a positive integer s_j such that $(D \cdot V(\tau)) = a_j/s_j$ (note that $s_j = 1$ if X is smooth). If $(D \cdot V(\tau)) \geq 0$, we deduce $\langle u(\sigma), v_j\rangle \geq \langle u(\sigma'), v_j\rangle$ for all j such that v_j is in $\sigma' \setminus \sigma$. By Exercise 27, we see that $\mathcal{O}(D)$ is globally generated.

A similar argument proves the following proposition (the toric Nakai criterion).

Proposition 8. *If $\mathcal{L}$ is a line bundle on the toric variety $X = X(\Delta)$, with $|\Delta|$ convex, then $\mathcal{L}$ is ample if and only if $(\mathcal{L} \cdot C) > 0$ for every complete invariant curve $C \subseteq X$.*

The following is a consequence of the proof of Proposition 7. We state it for future reference.

Corollary 3. *Let $X = X(\Delta)$ be an n-dimensional smooth toric variety, and let D be a T-divisor on X. If $C = V(\tau)$ is a complete invariant curve in X, and if σ, σ' are the distinct maximal cones in Δ with $\sigma \cap \sigma' = \tau$, then*

$$(D \cdot V(\tau)) = \langle u(\sigma) - u(\sigma'), v'\rangle = \langle u(\sigma') - u(\sigma), v\rangle,$$

where v and v' are the primitive generators outside τ of σ, respectively σ'.

Proposition 7 implies that on a toric variety whose fan has convex support there are no nonzero numerically trivial line bundles. Recall that a line bundle $\mathcal{L}$ on X is **numerically trivial** if $(\mathcal{L} \cdot C) = 0$ for every complete irreducible curve C on X.

Corollary 4. *If $\mathcal{L}$ is a line bundle on the toric variety $X = X(\Delta)$ with $|\Delta|$ convex, and if $(\mathcal{L} \cdot C) = 0$ for every complete invariant curve on X, then $\mathcal{L} \simeq \mathcal{O}_X$.*

Proof. We may assume that $|\Delta|$ has maximal dimension. If D ia a T-Cartier divisor with $\mathcal{L} \simeq \mathcal{O}(D)$, then by Proposition 7 both $\mathcal{O}(D)$ and $\mathcal{O}(-D)$ are globally generated. Therefore both ψ_D and $-\psi_D$ are concave, so ψ_D is linear. This means that D is a principal divisor, and $\mathcal{L} \simeq \mathcal{O}_X$.

Remark 15. It follows from Proposition 7 shows that if we have a proper morphism of toric varieties $f\colon X(\Delta') \to X(\Delta)$, then a line bundle L on $X(\Delta')$ is f-ample if and only if for every invariant curve C on X that lies in a fiber of f we have $(L \cdot C) > 0$. Indeed, it is enough to note that if σ is a cone in Δ and C is a complete curve in $f^{-1}(U_\sigma)$, then C lies in a fiber of f.

Exercise 28. Let $X = X(\Delta)$ be a complete toric variety.

(1) Show that the morphisms of toric varieties $f\colon \mathbb{P}^1 \to X$ are in bijection with the elements v in N. The morphism f is constant if and only if the corresponding v is zero.

(2) If f as above corresponds to v and D is a $\mathbb{Q}$-Cartier T-divisor, then $\deg(f^*D) = -(\psi_D(v) + \psi_D(-v))$.

Nef versus ample line bundles. Let $X = X(\Delta)$ be a toric variety such that $|\Delta|$ is convex. We have seen that every ample line bundle on X is globally generated. Moreover, a line bundle is globally generated if and only if it is nef. We show now that every nef line bundle is the pull-back of a suitable ample line bundle. In fact, we can explicitly describe the morphism involved.

Let D be a Cartier T-divisor on X such that $\mathcal{O}(D)$ is globally generated. We assume that the origin lies in the affine span W of P_D. Note that we can achieve this after replacing D by a suitable linearly equivalent divisor: for example, if D is effective then the origin lies in P_D. By Remark 9, on the toric variety X' defined by the (possibly degenerate) normal fan Δ_P, there is an ample Cartier divisor D' such that $\psi_D = \psi_{D'} \circ \phi$, where $\phi\colon N_{\mathbb{R}} \to N_{\mathbb{R}}/W^{\perp}$.

Proposition 9. *The projection ϕ gives a morphism of fans inducing a morphism of toric varieties $f\colon X \to X'$ which is proper, surjective and $f_*\mathcal{O}_X = \mathcal{O}_{X'}$.*

(1) *We have $f^*(D') = D$ and if $C \subseteq X$ is an irreducible curve, then $f(C)$ is a point if and only if $(D \cdot C) = 0$.*

(2) *Each maximal cone σ in Δ determines a maximal cone σ' in Δ', such that*

$$f^{-1}(U_{\sigma'}) = \bigcup_{u(\tau)=u(\sigma)} U_\tau,$$

where the union is over the maximal cones τ *in* Δ *satisfying* $u(\tau) = u(\sigma)$.

(3) *If* X *is complete, then* f *is the Stein factorization of the morphism* $f_{\mathcal{O}(D)}$ *defined by* $\mathcal{O}(D)$.

Proof. The fan Δ_P is the inverse image by ϕ of a nondegenerate fan Δ' in $N_{\mathbb{R}}/W^{\perp}$. Recall that by Lemma 5 in Chapter 6 the vertices of P_D are the $u(\sigma)$, where σ varies over the maximal cones in Δ. For every such σ, we have a maximal cone σ' in Δ' such that

$$\sigma' = \{v \in N_{\mathbb{R}} \mid \langle u(\sigma), v\rangle = \psi_D(v)\}/W^{\perp}.$$

Moreover, every maximal cone in Δ' arises in this way. We have $\phi(\sigma) \subseteq \sigma'$, so ϕ gives a morphism of fans and induces a morphism of toric varieties $f\colon X \to X'$. Since X is complete, f is proper. Moreover, ϕ is surjective so Corollary 3 in Chapter 4 implies that f is surjective and $f_*\mathcal{O}_X = \mathcal{O}_{X'}$.

We have $\psi_{D'} \circ \phi = \psi_D$ and therefore $f^*(D') = D$ (see Exercise 8). Since D' is ample, if $C \subseteq X$ is an irreducible curve then $f(C)$ is a point if and only if $(D' \cdot f(C)) = 0$. By the projection formula this is equivalent with $(D \cdot C) = 0$.

The inverse image of $V_\sigma = U_{\sigma'}$ is the union of those U_τ such that $\phi(\tau) \subseteq \sigma'$. We have $\phi(v)$ in σ' if and only if $\psi_D(v) = \langle u(\sigma), v\rangle$, so the assertion in (2) follows from Lemma 4.

Suppose now that X is complete. Since $f_*\mathcal{O}_X = \mathcal{O}_{X'}$ and $f^*E = D$, the projection formula gives a canonical isomorphism $\Gamma(X, \mathcal{O}(D)) \simeq \Gamma(X', \mathcal{O}(E))$. We have the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}(\Gamma(X, \mathcal{O}(D))) \\ \big\downarrow & & \big\| \\ X' & \longrightarrow & \mathbb{P}(\Gamma(X', \mathcal{O}(D'))), \end{array}$$

where the horizontal arrows are the maps $f_{\mathcal{O}(D)}$ and $f_{\mathcal{O}(D')}$ defined by the complete linear systems of $\mathcal{O}(D)$ and $\mathcal{O}(D')$. Since D' is ample, $f_{\mathcal{O}(D')}$ is a finite map, which shows that f is the Stein factorization of $f_{\mathcal{O}(D)}$.

Differentials and the Euler exact sequence. We recall first the definition of differentials with logarithmic poles. Let X be a smooth n-dimensional variety and $D = D_1 + \ldots + D_d$ be a reduced divisor with **simple normal crossings**. This means that for every x in X, if $D_{i_1}, \ldots, D_{i_m}$ are those D_i passing through x, then there is a system of algebraic coordinates $y_1, \ldots, y_n$ around x such that D_{i_p} is defined by y_p for $1 \leq p \leq m$. Recall that $y_1, \ldots, y_n$ is a system of coordinates if $dy_1, \ldots, dy_n$ form a basis of Ω^1_X. Our main example will be a smooth toric variety X, with $D = D_1 + \ldots + D_d$ the sum of the prime T-divisors.

The sheaf $\Omega^1_X(\log D)$ of differential forms with **logarithmic poles** along D is the subsheaf of $\Omega^1_X \otimes_{\mathcal{O}_X} K(X)$ defined as follows: in the neighborhood of a point x in X which lies in $D_{i_1}, \ldots, D_{i_m}$ but not in the other divisors, let us choose algebraic coordinates $y_1, \ldots, y_n$ as above. In this neighborhood $\Omega^1_X(\log D)$ is generated over $\mathcal{O}_X$ by

$$\frac{dy_1}{y_1}, \ldots, \frac{dy_m}{y_m}, dy_{m+1}, \ldots, dy_n.$$

Equivalently, in this neighborhood $\Omega^1_X(\log D)$ is generated by Ω^1_X and $\frac{dy_1}{y_1}, \ldots, \frac{dy_m}{y_m}$. Note that if $y'1, \ldots, y'_n$ is another system of coordinates with the same properties, then in the intersection of the two corresponding neighborhoods of x we may write for $1 \leq i \leq m$

$$y'_i = h_i y_i$$

for some invertible function h_i. We have $\frac{dy'_i}{y'_i} - \frac{dy_i}{y_i} = \frac{dh_i}{h_i}$, so this is a section of Ω^1_X. Hence the definition does not depend on the choice of coordinates.

With the above notation, since $dy_1, \ldots, dy_n$ form a basis of Ω^1_X in the neighborhood of x, we see that $\Omega^1_X(\log D)$ is locally free of rank n. Moreover, by construction we have an inclusion $\Omega^1_X \hookrightarrow \Omega^1_X(\log D)$.

Proposition 10. *There is an exact sequence of sheaves*

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \oplus_{i=1}^d \mathcal{O}_{D_i} \to 0.$$

Proof. The second mapping in the sequence is the **residue mapping**. In local coordinates it is defined by

$$\omega = \sum_{p=1}^m f_p \frac{dy_p}{y_p} + \sum_{q=m+1}^n f_q dy_q \to \sum_{p=1}^m f_p|_{D_{i_p}}.$$

With this definition we see that the map is a surjection and the residue is zero precisely when each f_i is divisible by y_i, i.e. when ω is a section of Ω^1_X. In order to complete the proof of the proposition, it is enough to show that the definition of the residue map does not depend on the choice of coordinates: note that as defined, the residue map is characterized by the fact that it is zero on Ω^1_X and it takes dy_p/y_p to the unit section of $\mathcal{O}_{D_{i_p}}$ for $p \leq m$. We have seen that if $y'_1, \ldots, y'_n$ is another such system of coordinates, then $dy_p/y_p - dy'_p/y_p$ is a section of

Ω^1_X for every $p \leq m$, so the definition of the residue map is independent on the choice of coordinates.

Suppose now that X is a smooth toric variety. If $D_1, \ldots, D_d$ are the prime T-divisors, let $D = D_1 + \ldots + D_d$.

Proposition 11. *There is a canonical isomorphism*

$$\alpha\colon M \otimes_{\mathbb{Z}} \mathcal{O}_X \to \Omega^1_X(\log D).$$

Proof. The map $M \to \Omega^1_X \otimes_{\mathcal{O}_X} K(X)$ which takes u to $d(\chi^u)/\chi^u$ is additive, so it induces the $\mathcal{O}_X$-linear map α. To check that α is an isomorphism onto $\Omega^1_X(\log D)$, it is enough to do it over each affine open subset U_σ. Let $e_1, \ldots, e_n$ be a basis of N such that σ is generated by $e_1, \ldots, e_m$. After reordering the D_i we may assume that D_i corresponds to e_i for $i \leq m$. If $t_i = \chi^{e_i^*}$, we see that $\Omega^1_X(\log D)|_{U_\sigma}$ is free with basis

$$\frac{dt_1}{t_1}, \ldots, \frac{dt_m}{t_m}, dt_{m+1}, \ldots, dt_n.$$

Since $t_{r+1}, \ldots, t_n$ are invertible on U_σ, it follows that α is an isomorphism over U_σ. This completes the proof.

We give another description of Ω^1_X as the kernel of a morphism of vector bundles (see [BC]). This generalizes the familiar Euler exact sequence on the projective space.

Corollary 5. *If the linear span of $|\Delta|$ is $N_{\mathbb{R}}$, then there is an exact sequence of vector bundles on X:*

$$0 \to \Omega^1_X \to \oplus_{i=1}^d \mathcal{O}(-D_i) \to \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \simeq \mathcal{O}_X^{\oplus(d-n)} \to 0.$$

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \Omega^1_X \arrow[r] \arrow[d] & M \otimes_{\mathbb{Z}} \mathcal{O}_X \arrow[r] \arrow[d] & \oplus_{i=1}^d \mathcal{O}_{D_i} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \oplus_{i=1}^d \mathcal{O}(-D_i) \arrow[r] \arrow[d] & \mathbb{Z}^d \otimes_{\mathbb{Z}} \mathcal{O}_X \arrow[r] \arrow[d] & \oplus_{i=1}^d \mathcal{O}_{D_i} \arrow[r] & 0 \\
 & \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \arrow[r, equal] \arrow[d] & \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

in which the two rows and the middle column are exact. The first horizontal row is deduced from the proposition, the middle column is obtained by tensoring with $\mathcal{O}_X$ the short exact sequence

$$0 \to M \to \mathbb{Z}^d \to \mathrm{Pic}(X) \to 0,$$

and the definition of the morphisms in the first column is forced by commutativity. The Snake Lemma implies that the first column is exact.

Since X is smooth, $\mathrm{Pic}(X) = \mathrm{Cl}(X)$ so it has rank $d-r$. Moreover, $\mathrm{Pic}(X)$ is free, hence $\mathrm{Pic}(X) \simeq \mathbb{Z}^{\oplus(d-n)}$.

Corollary 6. *If X is a smooth toric variety and if $D_1, \ldots, D_d$ are the prime T-divisors on X, then the total Chern class of X is given by*

$$c(T_X) = \prod_{i=1}^{d}(1 + c^1(\mathcal{O}(D_i))).$$

Proof. If $X = X' \times (k^*)^r$ and if p_1 and p_2 denote the projections onto the two factors, then $T_X \simeq (p_1)^*(T_{X'}) \oplus \mathcal{O}_X^{\oplus r}$, since $T_{k^*} \simeq \mathcal{O}_{k^*}$. Therefore $c(T_X) = (p_1)^* c(T_{X'})$, so we can reduce the assertion to the case when the linear span of $|\Delta|$ is $N_{\mathbb{R}}$, when it follows from the exact sequence in Corollary 5.

Corollary 7. *The sheaf of top differential forms on a smooth toric variety X is given by $\Omega_X^n \simeq \mathcal{O}(-D_1 - \ldots - D_d)$.*

Proof. We may reduce the statement to the case when the linear span of $|\Delta|$ is $N_{\mathbb{R}}$. In this case the isomorphism follows by taking determinants in the exact sequence in Corollary 5.

Remark 16. If $u_1, \ldots, u_n$ are in M, then $d\chi^{u_1} \wedge \ldots \wedge d\chi^{u_n}$ is a rational n-form on X. Let $e_1, \ldots, e_n$ be a basis for N, and let us denote $t_i = \chi^{e_i^*}$. If we write $u_i = \sum_{j=1}^n a_{i,j} e_j^*$, then we see that

$$\frac{d\chi^{u_1}}{\chi^{u_1}} \wedge \ldots \wedge \frac{d\chi^{u_n}}{\chi^{u_n}} = \det(a_{i,j}) \cdot \frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n}.$$

Using the commutativity of the upper left square in the diagram in the proof of Corollary 5, we see that $\frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n}$ corresponds to the section ± 1 in $\Gamma(T_N, \mathcal{O}(-D_1 - \ldots - D_d))$. Therefore the n-form $d\chi^{u_1} \wedge \ldots \wedge d\chi^{u_n}$ corresponds by the isomorphism in Corollary 7 to the rational section $\pm \det(a_{i,j}) \cdot \chi^{\sum_i u_i}$ of $\mathcal{O}(-D_1 - \ldots - D_d)$.

For a general normal variety X of dimension n, if U is a smooth open subset such that $\mathrm{codim}(X \smallsetminus U, X) \geq 2$ (for example, if U is the smooth locus of X), then a **canonical divisor** on X is a Weil divisor K_X on X such that $\mathcal{O}(K_X)|_U \simeq \Omega_U^n$. Equivalently, if $i \colon U \to X$ is the inclusion, then $i_*\Omega^n_U = \mathcal{O}(K_X)$. It is clear that such a divisor is uniquely defined up to rational equivalence, and the definition does not depend on the choice of U.

If X is an arbitrary toric variety, it follows from Corollary 7 that we may take $K_X = -D_1 - \ldots - D_d$. It is known that in general, if in addition X is Cohen-Macaulay and projective, then $\mathcal{O}(K_X)$ is the dualizing sheaf in the sense of [Ha], Chapter III.7 (see [KM], Proposition 5.75). This applies in particular in the case of a projective toric variety.

The following is the toric version of Fujita's Conjecture. For generalizations to singular varieties, see [Fu] and [Pa].

Exercise 29. Let $X = X(\Delta)$ be an n-dimensional, smooth projective toric variety, and let D be a Cartier T-divisor such that $(D \cdot V(\tau)) \geq n$ (respectively $\geq n+1$) for every $(n-1)$-dimensional cone τ in Δ. If $\mathcal{O}(K_X + D)$ is not globally generated (respectively very ample), then $X \simeq \mathbb{P}^n$ and $\mathcal{O}(D) \simeq \mathcal{O}(n)$ (respectively $\mathcal{O}(n+1)$).

Toric Fano varieties. Let $X = X(\Delta)$ be a complete toric variety. X is called $\mathbb{Q}$-**Fano** if $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor, and it is called **Fano** if, in addition, it is Gorenstein, i.e. K_X is Cartier. Our goal is to describe those rational polytopes P in M such that X_P is a $\mathbb{Q}$-Fano or Fano variety and $D_P = -K_{X_P}$.

Suppose first that $X = X(\Delta)$ is a complete toric variety which is $\mathbb{Q}$-Gorenstein, i.e. K_X is $\mathbb{Q}$-Cartier. If $v_1, \ldots, v_d$ are the primitive generators of the rays in Δ, let Q be the convex hull of the v_i. Note that the origin lies in the interior of Q: otherwise there are u in $M_{\mathbb{R}}$ and a in $\mathbb{R}$ such that $\langle u, v_i \rangle \geq a$ for all i, and $0 \leq a$. This implies that all v_i lie in the half-space $\{v \mid \langle u, v \rangle \geq 0\}$, hence so does $|\Delta|$, a contradiction.

By definition, the polytope associated to $-K_X$

$$P_{-K_X} = \{u \,\text{in}\, M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -1\}$$

is the polar of Q. We deduce the following

Proposition 12. *Let P a rational polytope in M of dimension $n = \mathrm{rk}(M)$. The corresponding toric variety X_P is $\mathbb{Q}$-Fano and $D_P = -K_{X_P}$ if and only if there is a polytope Q containing the origin in its interior and such that its vertices are primitive lattice vectors, and $P = Q^\circ$.*

Proof. Suppose that Q is such a polytope and let P be the polar of Q. If $w_1, \ldots, w_r$ are the vertices of Q, we know that the rays of the normal fan to P are the rays through the w_i. Since each w_i is a primitive lattice vector, and since

$$\psi_{D_P}(w_i) = \min\{\langle u, w_i\rangle \mid u \in P\} = -1$$

for every i, we see that $D_P = -K_{X_P}$. In particular, X_P is $\mathbb{Q}$-Fano. As we have already seen the converse, the proof is complete.

If X is a $\mathbb{Q}$-Gorenstein toric variety, the smallest positive integer r such that rK_X is Cartier is called the **index** of X.

Exercise 30. Let X be a $\mathbb{Q}$-Fano toric variety and let $P = P_{-K_X}$. Show that if Q is the polar of P and r is the index of X, then the origin is the only lattice point in the interior of Q which lies in rN. In particular, if X is Fano, then the only interior lattice point of P is the origin.

We turn now to Fano varieties. If Q and P are as in Proposition 12, then the corresponding toric variety X_P is Gorenstein if and only if P is a lattice polytope. This suggests the following definition. An n-dimensional polytope P in $M_{\mathbb{R}}$ is **reflexive** if it contains the origin in its interior, and both P and its polar P° are lattice polytopes. Note that if P is reflexive, then so is P°.

Proposition 13. *Let P be an n-dimensional rational polytope in $M_{\mathbb{R}}$. The corresponding variety X_P is Fano and $D_P = -K_{X_P}$ if and only if P is a reflexive polytope.*

Proof. The assertion follows from Proposition 12. The only thing that needs to be proved is that if P is reflexive, then the vertices of P° are primitive lattice vectors. Indeed, the vertices lie in N since by definition P° is a lattice polytope. Moreover, if v is such a vertex of P° and if u is a vertex of the corresponding face of P, then $\langle u, v\rangle = -1$. Since u is in M, we see that v is primitive.

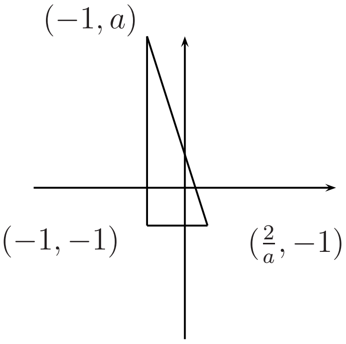
Exercise 31. Let $X = X(\Delta)$ be a complete toric variety and let $v_1, \ldots, v_d$ be the primitive generators of the rays in Δ.

(1) X is $\mathbb{Q}$-Gorenstein (or Gorenstein) if and only if for every maximal cone σ in Δ there is $u(\sigma)$ in $M_{\mathbb{Q}}$ (respectively, M) such that $\langle u(\sigma), v_i\rangle = -1$ for every v_i that lies in σ.

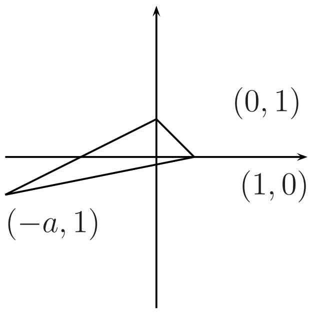
(2) X is $\mathbb{Q}$-Fano (or Fano) if and only if for every maximal cone σ in Δ there is $u(\sigma)$ in $M_{\mathbb{Q}}$ (respectively, M) such that $\langle u(\sigma), v_i\rangle = -1$ if v_i is in σ, and $\langle u(\sigma), v_j\rangle > -1$ if v_j is not in σ.

If we do not put any conditions on the singularities of $\mathbb{Q}$-Fano toric varieties, then there are infinitely many non-isomorphic Fano varieties in a given dimension.

Example 2. For $a \geq 2$, consider the polar polytopes



P_a



Q_a

We see that Q_a contains the origin in its interior and its vertices are primitive lattice vectors, so the toric variety X_a corresponding to P_a is a $\mathbb{Q}$-Fano toric variety. It is clear that all these varieties are pairwise non-isomorphic: the singular locus of X_a consists of one point of multiplicity a.

Note that if to the normal fan of P_a we add the ray through $(-1, 0)$, then the resulting toric variety is isomorphic to the Hirzebruch surface $\mathbb{F}_a$. Moreover, the induced proper birational morphism $\mathbb{F}_a \to X_a$ is the contraction of the unique irreducible curve C_a on $\mathbb{F}_a$ with $(C_a^2) = -a$.

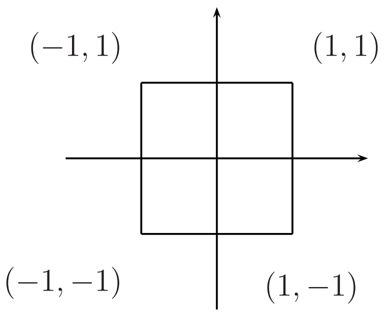
Direct computation shows that the index of X_a is equal to $a/2$ if a is even, and it is equal to a if a is odd.

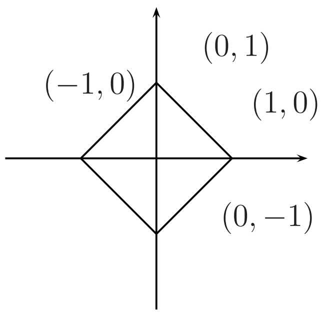
Theorem 1. ([BB]) *Given n and r, there are only finitely many isomorphism classes of $\mathbb{Q}$-Fano toric varieties of dimension n and index bounded by r.*

The idea of the proof is to use the assertion in Exercise 30 to give an upper bound for the volume of Q in terms of r and n (we use the notation in that exercise). Finiteness then follows from the fact that there are only finitely many lattice polytopes with bounded volume. In fact, the result in [BB] is stronger. There is an invariant of the singularities of X (see Chapter 8 for the definition) called the discrepancy of X and denoted $\operatorname{discrep}(X)$ which is always greater than -1 for toric varieties. Moreover, if r is the index of X, then $r \cdot \operatorname{discrep}(X)$ is an integer. It can be shown that if $\operatorname{discrep}(X) > -1 + \frac{1}{r}$ (which is the case, for example, if r is the index of X), then the conclusion of Exercise 30 still holds, and the same argument shows that there are only finitely many isomorphism classes of such varieties in a given dimension. We refer for proofs to [BB].

The above theorem shows in particular that in every dimension there are only finitely many reflexive polytopes (up to lattice isomorphisms). As we have indicated, finding explicit bounds for this number depends on having explicit bounds on the volume of a reflexive polytope (or equivalently, on having bounds for $((-K_X)^n)$ if X is a Fano toric variety of dimension n). For such bounds and for conjectures improving them we refer to [De].

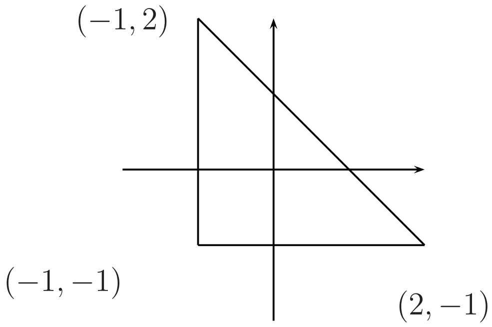
For example, in dimension 2 there are 16 reflexive polytopes, of which 5 are nonsingular (i.e. they correspond to nonsingular toric varieties). In dimensions 3 and 4 the numbers are $4,319$ and respectively $473,800,776$ for arbitrary reflexive polytopes, of which 18 and respectively 124 are nonsingular. Below are the 5 nonsingular two-dimensional polytopes, and their polars.

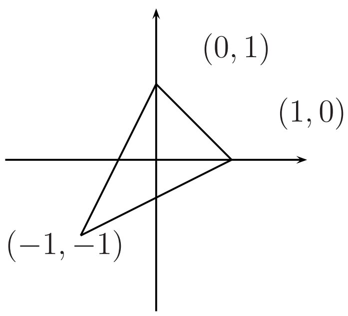




$P;\ X_P \simeq \mathbb{P}^1 \times \mathbb{P}^1$

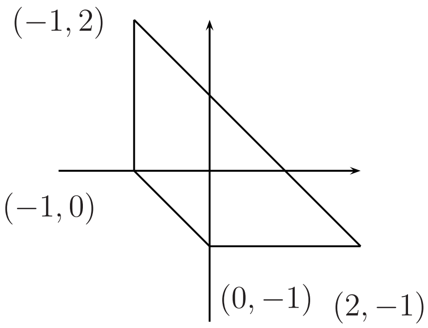
$Q = P^\circ$



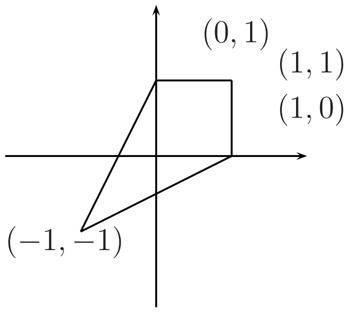


$P;\ X_P \simeq \mathbb{P}^2$

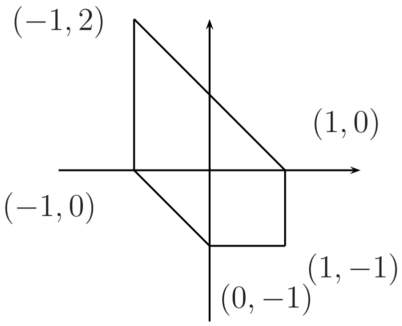
$Q = P^\circ$



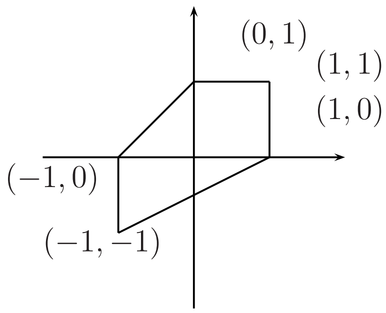
$P;\ X_P \simeq \mathrm{Bl}_{x_1}\mathbb{P}^2$



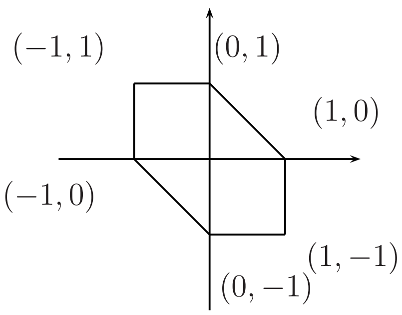
$Q = P^\circ$



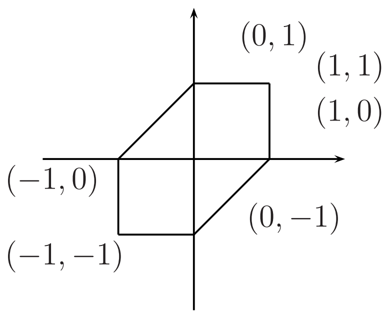
$P;\ X_P \simeq \mathrm{Bl}_{x_1,x_2}\mathbb{P}^2$



$Q = P^\circ$



$P;\ X_P \simeq \mathrm{Bl}_{x_1,x_2,x_3}\mathbb{P}^2$



$Q = P^\circ$

Note that in the last case, P° and P differ by an automorphism of the lattice.

Solutions and hints for selected exercises

Exercise 3. The equivalence (2)$\Leftrightarrow$(3) is obvious, while (3)$\Leftrightarrow$(4) follows from Proposition 1. Moreover, it follows from this proposition that in order to check that every $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier, it is enough to consider $\mathbb{Q}$-T-divisors. In this case the assertion is equivalent with saying that given any σ in Δ, if $v_1, \ldots, v_r$ are the primitive generators of the rays in σ then the map $M_{\mathbb{Q}} \to \oplus_{i=1}^r \mathbb{Q} \cdot D_i$ given by $u \to \sum_i \langle u, v_i \rangle D_i$ is surjective. This is equivalent with the fact that the v_i are linearly independent over $\mathbb{Q}$, which is the case for every σ in Δ if and only if Δ is simplicial. Therefore we have (1)$\Leftrightarrow$(2).

Exercise 4. We may assume that σ spans $N_{\mathbb{R}}$; the assertion follows from Proposition 1 for $X = U_\sigma$ and the fact that if $\phi\colon N' \to N$ is an injective morphism of free abelian groups, then the cokernels of ϕ and ϕ^* have the same order.

Exercise 5. The equivalence (2)$\Leftrightarrow$(3) follows from Proposition 1 and (1)$\Rightarrow$(3) is obvious. By Exercise 3, if (3) holds, then X is simplicial. Moreover, for every cone σ in Δ we have $\mathrm{Cl}(U_\sigma) = \{0\}$, so σ is nonsingular by the previous exercise.

Exercise 8. If σ' and σ are cones in Δ' and Δ, respectively, and if $D|_{U_\sigma} = \mathrm{div}(\chi^{-u(\sigma)})|_{U_\sigma}$, then $f^*D|_{U_{\sigma'}} = \mathrm{div}(\chi^{-\phi^*(u(\sigma))})$, so $\psi_{f^*(D)} = \psi_D \circ \phi$. We deduce that if u' is in $M'_{\mathbb{R}}$, then u' is in $P_{f^*(D)}$ if and only if $\langle u', v' \rangle \geq \psi_D(\phi(v'))$ for every v' in $|\Delta'|$, and the last two assertions follow.

Exercise 9. If $m \geq 1$, then (u, m) lies in σ if and only if u is in $mP_D = P_{mD}$. Therefore we get the identification of R_D and $k[\sigma \cap (M \times \mathbb{Z})]$ in degrees ≥ 1.

If $D = \sum_i a_i D_i$, then P_D is defined by $\langle u, v_i \rangle \geq -a_i$ for all i. Therefore $(u, 0)$ is in σ if and only if $\langle u, v_i \rangle \geq 0$ for every i. This is equivalent with the fact that χ^u is in $\cap_{\sigma \in \Delta} \mathcal{O}(U_\sigma) = \mathcal{O}(X)$, so $(R_D)_0 = \mathcal{O}(X)$.

Exercise 11. If $\mathcal{O}(1)$ is the tautological quotient bundle on $Y = \mathbb{P}(L_1 \oplus \ldots \oplus L_r)$, then

$$\Gamma(Y, \mathcal{O}(m)) = \Gamma(X, \mathrm{Sym}^m(\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r)) = \oplus_{i_1 + \ldots + i_r = m} \Gamma(X, \mathcal{L}_1^{i_1} \otimes \ldots \otimes \mathcal{L}_r^{i_r}),$$

and the assertion follows from the previous two exercises.

Exercise 13. We may assume that D is Cartier. Since $h^0(X, \mathcal{O}(mD)) = |mP_D \cap M|$ and

$$\lim_{m \to \infty} \frac{|mP_D \cap M|}{m^n} = \mathrm{vol}_n(P_D),$$

the assertion follows.

Exercise 14. If $\dim(P_D) = r$, then it is enough to use the fact that if D is Cartier, then

$$\lim_{m \to \infty} \frac{h^0(X, \mathcal{O}(mD))}{m^r} = \mathrm{vol}_r(P_D) > 0.$$

Exercise 16. It is enough to show that every vertex of P_{D+E} is in $P_D + P_E$. For a maximal cone σ in Δ we denote by $u(\sigma)$ and $w(\sigma)$ the elements in M such that $\psi_D(v) = \langle u(\sigma), v\rangle$ and $\psi_E(v) = \langle w(\sigma)v\rangle$ for v in σ. By Proposition 5 every vertex of P_{D+E} is of the form $u(\sigma)+w(\sigma)$ for some maximal cone σ in Δ, hence it is in $P_D + P_E$.

Exercise 17. The first assertion follows as in the proof of Proposition 3. If X is complete, this condition says that for every maximal cone σ in Δ, $u(\sigma)$ is in $\{u_1, \ldots, u_r\}$. Since the vertices of P_D are precisely the $u(\sigma)$, we get the second assertion.

Exercise 22. Since $\mathcal{O}(D)$ is ample, we have an isomorphism $X \simeq \mathrm{Proj}(R_D)$, so (1) follows from Exercise 9. (2) follows from the definition of normally generated line bundles and Exercise 12. If $\mathcal{O}(D)$ is normally generated, then (1) shows that the affine cone over $X \subseteq \mathbb{P}(\Gamma(X, \mathcal{O}(D)))$ is an affine toric variety, hence it is Cohen-Macaulay by Theorem 1 in Chapter 3.

Exercise 23. Since $\mathcal{L}$ and $\mathcal{M}$ are ample, the vector bundle $\mathcal{L} \oplus \mathcal{M}$ is ample, i.e. the line bundle $\mathcal{O}(1)$ is ample on $Y = \mathbb{P}(L \oplus M)$. Applying Proposition 6 for $\mathcal{O}(1)$ and the fact that $\Gamma(Y, \mathcal{O}(\ell)) \simeq \oplus_{i+j=\ell}\Gamma(X, \mathcal{L}^i \otimes \mathcal{M}^j)$, we see that we have a surjective map induced by multiplication of sections

$$(\Gamma(X, \mathcal{L}) \otimes \Gamma(X, \mathcal{L}^m \otimes \mathcal{M})) \oplus \big(\Gamma(X, \mathcal{M}) \otimes \Gamma(X, \mathcal{L}^{m+1})\big) \to \Gamma(X, \mathcal{L}^{m+1} \otimes \mathcal{M}).$$

On the other hand, another application of Proposition 6 for $\mathcal{L}$ shows that the map

$$\Gamma(X, \mathcal{L}) \otimes \Gamma(X, \mathcal{L}^m) \to \Gamma(X, \mathcal{L}^{m+1})$$

is surjective. These two facts imply our statement.

Exercise 24. Use Exercise 14 in Chapter 5.

Exercise 27. Given two points v and v' in $|\Delta|$ that do not lie in the same maximal cone, we can find $v_1, \ldots, v_r$ on the line segment $\overline{vv'}$ such that each segment $\overline{v_i v_{i+1}}$ is contained in the union of two maximal cones whose intersection has dimension $n-1$ (we put $v_0 = v$ and $v_{r+1} = v'$). The assertion follows from the fact that the restriction of ψ_D to $\overline{vv'}$ is (strictly) concave if and only if so is the restriction to each of the segments $\overline{v_i v_{i+1}}$.

Exercise 29. The following argument is due to S. Payne. We give the proof for the globally generated case, the very ample case being analogous. If σ is a maximal cone in Δ, we denote by $u(\sigma)$ and $w(\sigma)$ the elements in M corresponding to D and K_X, respectively. It follows from Exercise 27 that if $\mathcal{O}(K_X+D)$ is not globally generated, then there are n-dimensional cones σ and σ' spanned by the primitive vectors $v, v_1, \ldots, v_{n-1}$ and $v', v_1, \ldots, v_{n-1}$, respectively, such that

$$\langle u(\sigma)+w(\sigma), v'\rangle < \langle u(\sigma')+w(\sigma'), v'\rangle. \tag{4}$$

We can write $v+v'=\sum_{i=1}^{n-1} a_i v_i$, where the a_i are integers. Let $a=\min_i\{a_i\}$, and suppose that $a=a_\ell$. There is a cone $\sigma''=\sigma''(\ell)$ in Δ such that $\sigma'\cap\sigma''$ is spanned by $v_1,\ldots,v_{\ell-1},v_{\ell+1},\ldots,v_{n-1},v'$. For simplicity, we denote $u(\sigma)$, $u(\sigma')$ and $u(\sigma'')$ by u, u' and u'', and similarly for w. Let $b=(D\cdot V(\sigma\cap\sigma'))$ and $c=(D\cdot V(\sigma'\cap\sigma''))$, so by hypothesis we have b, $c\geq n$. By Corollary 3 we have $b=\langle u-u', v'\rangle=\langle u'-u, v\rangle$ and $c=\langle u''-u', v_\ell\rangle$. D is ample by Proposition 8, so $\langle u''-u, v\rangle\geq 0$, with equality if and only if v is in σ''. Via the above formulas for b and c, we rewrite this as $b+ac\geq 0$. On the other hand, (4) gives

$$b+(n-1)a\leq b+\sum_{i=1}^{n-1} a_i\leq 1.$$

Since $b\geq n$ we get $a\leq -1$, and if $a=-1$, then $b=n$ and $a_i=a$ for all i. We deduce also

$$0\leq b+ca\leq b+na\leq b+a+\sum_{i=1}^{n-1} a_i\leq 1+a.$$

Therefore $a=-1$, $b=c=-n$ and $a_i=a$ for all i. Moreover, since $ac+b=0$ we must have v in σ''. We deduce that the cone spanned by any n of $v_1,\ldots,v_{n-1},v,v'$ is in Δ, so $X\simeq\mathbb{P}^n$. The isomorphism $\mathcal{O}(D)\simeq\mathcal{O}(n)$ is a direct consequence.

Exercise 30. Suppose that v is a point in rN which lies in the interior of Q. If σ is a maximal cone in the fan over the proper faces of Q such that v is in σ, let $u(\sigma)$ be such that for all i we have $\langle u(\sigma), v_i\rangle\geq -1$, with equality if and only if v_i is in σ. Note that $ru(\sigma)$ is in N and therefore $\langle ru(\sigma), v\rangle$ is an integer. On the other hand, $u(\sigma)$ lies in Q° and v is in the interior of Q, so $\langle u(\sigma), v\rangle\geq 0$. Since v lies in σ, we deduce that $v=0$.

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