

## LECTURE 5: RATIONALITY AND UNIRATIONALITY FOR SURFACES

Our main goal in this lecture is to prove Castelnuovo's rationality criterion for surfaces (in fact, we will only give the proof in characteristic 0). This gives, in particular, a positive answer to the Lüroth problem in characteristic 0. This is discussed in §2, after we prove in §1 a ruledness criterion due to Noether and Enriques. In both these sections we follow closely the arguments in [Bea78]. In the last section we discuss Shioda's examples of unirational surfaces in  $\mathbf{P}^3$ , over a field of positive characteristic, which give a negative answer to the Lüroth problem in this setting.

### 1. A CRITERION FOR RULEDNESS

We will make use of following theorem in the proof of the rationality criterion, but the result is interesting on its own. We work over an algebraically closed field  $k$ .

**Theorem 1.1.** (Noether-Enriques) *Let  $g: S \rightarrow C$  be a morphism, where  $C$  is a smooth projective curve, and  $S$  is a smooth projective surface. If  $x \in C$  is such that the scheme-theoretic fiber  $g^{-1}(x)$  is isomorphic to  $\mathbf{P}^1$ , then there is an open subset  $U$  of  $x$  such that  $g^{-1}(U)$  is isomorphic to  $U \times \mathbf{P}^1$  over  $U$ .*

We will give the proof following [Bea78, Chapter III] and in a key step we will assume that  $k = \mathbf{C}$ . The extension to the case  $\text{char}(k) = 0$  is standard (see Remark 1.2 below). For a proof in positive characteristic, we refer to [Băd01, Chapter 13].

*Proof of Theorem 1.1.* We divide the proof in 3 steps.

**Step 1.** We show that  $H^2(S, \mathcal{O}_S) = 0$ . Let  $F = g^{-1}(x)$ . By assumption, this is a smooth, irreducible divisor on  $X$  and the adjunction formula gives

$$-2 = 2p_a(F) - 2 = (F^2) + (K_S \cdot F).$$

Since  $F$  is a fiber of  $g$ , it is clear that  $(F^2) = 0$ . Therefore  $(K_S \cdot F) = -2$ .

If  $H^2(S, \mathcal{O}_S) \neq 0$ , then Serre duality implies that  $H^0(S, \omega_S) \neq 0$ . Therefore there is an effective divisor  $D$  with  $\mathcal{O}_S(D) \simeq \omega_S$ . In particular, we have  $(D \cdot F) = -2$ . On the other hand, since  $(F^2) = 0$  and  $(F \cdot C) \geq 0$  for every prime divisor  $C$  on  $S$  different from  $F$ , we deduce that  $(F \cdot D) \geq 0$ , a contradiction. Therefore  $H^2(S, \mathcal{O}_S) = 0$ .

**Step 2.** We show that there is a divisor  $H$  on  $S$  such that  $(H \cdot F) = 1$ . In this step, we assume that  $k = \mathbf{C}$  and use some basic facts of complex geometry, for which we refer to Appendix 1. The exponential sequence

$$0 \rightarrow \underline{\mathbf{Z}} \rightarrow \mathcal{O}_{S^{\text{an}}} \rightarrow \mathcal{O}_{S^{\text{an}}}^* \rightarrow 0$$

induces an exact sequence

$$\mathrm{Pic}(S) \xrightarrow{c^1} H^2(S, \mathbf{Z}) \longrightarrow H^2(S, \mathcal{O}_S) = 0.$$

We use here the fact that  $S$  being projective, we have by GAGA that

$$\mathrm{Pic}(S) \simeq \mathrm{Pic}(S^{\mathrm{an}}) \simeq H^1(S^{\mathrm{an}}, \mathcal{O}_{S^{\mathrm{an}}}^*) \quad \text{and} \quad H^2(S^{\mathrm{an}}, \mathcal{O}_{S^{\mathrm{an}}}) \simeq H^2(S, \mathcal{O}_S).$$

We thus see that  $c^1: \mathrm{Pic}(S) \rightarrow H^2(S, \mathbf{Z})$  is surjective.

Therefore, in order to complete the proof of this step, it is enough to find  $h \in H^2(S, \mathbf{Z})$  such that if  $f = c^1(\mathcal{O}_S(F))$ , then  $(h \cdot f) = 1$ . By Poincaré duality, the mapping

$$H^2(S, \mathbf{Z})/H^2(S, \mathbf{Z})_{\mathrm{tors}} \times H^2(S, \mathbf{Z})/H^2(S, \mathbf{Z})_{\mathrm{tors}} \rightarrow \mathbf{Z}, \quad (\bar{\alpha}, \bar{\beta}) \rightarrow (\alpha \cdot \beta)$$

is a perfect pairing. Consider the set

$$\{(a \cdot f) \mid a \in H^2(S, \mathbf{Z})\} \subseteq \mathbf{Z}.$$

This is an ideal in  $\mathbf{Z}$ , hence it is equal to  $d\mathbf{Z}$  for some  $d > 0$  (the fact that this is not the zero ideal follows from the fact that  $F$  has nonzero intersection number with some divisor, for example with  $K_S$ ). The map

$$H^2(S, \mathbf{Z})/H^2(S, \mathbf{Z})_{\mathrm{tors}} \ni \bar{\alpha} \rightarrow \frac{1}{d}(\alpha \cdot f) \in \mathbf{Z}$$

is linear. We deduce using Poincaré duality that there is  $f' \in H^2(S, \mathbf{Z})$  such that

$$\frac{1}{d}(\alpha \cdot f) = (\alpha \cdot f') \quad \text{for all } \alpha \in H^2(S, \mathbf{Z}).$$

Moreover, Poincaré duality implies that

$$(1) \quad f - df' \in H^2(S, \mathbf{Z})_{\mathrm{tors}}.$$

Let  $\kappa = c^1(\omega_S)$ . Note that

$$(2) \quad (\alpha \cdot \alpha) + (\alpha \cdot \kappa) \quad \text{is even for all } \alpha \in H^2(S, \mathbf{Z}).$$

Indeed, this expression gives a linear map mod 2 (in  $\alpha$ ), it is even when  $\alpha = c^1(\mathcal{O}_S(G))$  for an irreducible curve  $G$  on  $S$  by the adjunction formula, hence it is even when  $\alpha = c^1(L)$  for some  $L \in \mathrm{Pic}(S)$ . The surjectivity of  $c^1: \mathrm{Pic}(S) \rightarrow H^2(S, \mathbf{Z})$  then implies the assertion.

Since  $(f \cdot f) = 0$  and  $(f \cdot \kappa) = -2$ , it follows from (1) that

$$(f' \cdot f') = 0 \quad \text{and} \quad d(f' \cdot \kappa) = -2.$$

We deduce from (2) that  $(f' \cdot \kappa)$  is even, hence  $d = 1$ . In particular, there is  $h \in H^2(S, \mathbf{Z})$  such that  $(h \cdot f) = 1$ .

**Step 3.** Note that  $g$  is flat since it is a morphism between smooth varieties and all fibers have the same dimension. Since  $g^{-1}(x)$  is smooth, it follows that there is an open subset  $U$  of  $x$  such that  $g$  is smooth over  $U$ . We deduce that for every  $y \in U$ , the fiber  $F_y = g^{-1}(y)$  is a smooth curve on  $S$ . Since

$$h^0(F, \mathcal{O}_F) = 1 \quad \text{and} \quad h^1(F, \mathcal{O}_F) = 0,$$

it follows from the semicontinuity theorem that for every  $y \in U$ , the fiber  $F_y$  is isomorphic to  $\mathbf{P}^1$ .

Let  $H$  be given by Step 2 and put  $\mathcal{E} = f_*\mathcal{O}_S(H)$ . Since  $(F \cdot H) = 1$ , we conclude that  $(F_y \cdot H) = 1$  for every  $y \in U$ , hence  $\mathcal{O}_S(H)|_{F_y} \simeq \mathcal{O}_{\mathbf{P}^1}(1)$ , and thus

$$h^0(F_y, \mathcal{O}_S(H)|_{F_y}) = 2.$$

It follows from Grauert's theorem that  $\mathcal{E}|_U$  is a locally free sheaf of rank 2 and the canonical morphism

$$(3) \quad \mathcal{E}_y \otimes k(y) \longrightarrow H^0(F_y, \mathcal{O}_S(H)|_{F_y})$$

is an isomorphism for every  $y \in U$ . This implies that the canonical map  $g^*\mathcal{E} \rightarrow \mathcal{O}_S(H)$  is surjective on  $U$ . Indeed, by Nakayama's lemma it is enough to check surjectivity for the restriction to the fiber  $F_y$ , for  $y \in U$ . This map is identified via the isomorphism (3) to the canonical map

$$H^0(F_y, \mathcal{O}_S(H)|_{F_y}) \otimes \mathcal{O}_{F_y} \rightarrow \mathcal{O}_S(H)|_{F_y},$$

which is surjective since the line bundle  $\mathcal{O}_{\mathbf{P}^1}(1)$  on  $\mathbf{P}^1$  is globally generated. The surjection  $g^*\mathcal{E} \rightarrow \mathcal{O}_S(H)$  on  $U$  defines a morphism

$$\varphi: g^{-1}(U) \rightarrow \mathbf{P}(\mathcal{E}|_U)$$

over  $U$ , and for every  $y \in U$ , the restriction of  $\varphi$  over  $y$  is the morphism  $g^{-1}(y) \rightarrow \mathbf{P}^1$  defined by  $\mathcal{O}_S(H)|_{F_y} \simeq \mathcal{O}_{\mathbf{P}^1}(1)$ , hence it is an isomorphism. This implies that  $\varphi$  is an isomorphism and the assertion in the theorem is clear.  $\square$

**Remark 1.2.** In the above proof we used the hypothesis  $k = \mathbf{C}$  in order to find a divisor  $H$  on  $S$  with  $(H \cdot F) = 1$ . It is standard to see that if we have this over  $\mathbf{C}$ , we can extend the assertion to every algebraically closed field  $k$ , with  $\text{char}(k) = 0$ . Indeed, we can find an algebraically closed field  $k_0 \subseteq k$  with  $\text{trdeg}_{\mathbf{Q}}(k_0) < \infty$  and a morphism  $g_0: S_0 \rightarrow C_0$  as in the theorem, with a fiber  $F_0 \simeq \mathbf{P}_{k_0}^1$  of  $g_0$  such that  $g$  and  $F$  are obtained by base-change from  $g_0$  and  $F_0$ . Of course, it is then enough to find a divisor  $H_0$  on  $S_0$  such that  $(H_0 \cdot F_0) = 1$ , since we then obtain  $H$  from  $H_0$  by base-change. On the other hand, there is a field homomorphism  $k_0 \hookrightarrow \mathbf{C}$  and we may consider  $g': S' \rightarrow C'$  and  $F'$  obtained from  $g$  and  $F$  by base-change to  $\mathbf{C}$ . There is a scheme  $\mathcal{H}$  over  $k_0$  such that for every field extension  $K/k_0$ , the  $K$ -valued points of  $\mathcal{H}$  are in bijection with the divisors  $E$  on  $S_K$  such that  $(E \cdot F_K) = 1$  (indeed,  $\mathcal{H}$  is a union of connected components of the Hilbert scheme parametrizing all divisors on  $S$ ). The proof of Theorem 1.1 shows that  $\mathcal{H}(\mathbf{C})$  is nonempty, and therefore  $\mathcal{H}(k_0)$  is nonempty.

## 2. THE RATIONALITY CRITERION

We can now prove the following rationality criterion for surfaces. We keep the assumption that  $k$  is an algebraically closed field. Recall that for a smooth, projective variety, we put  $p_m(X) = h^0(X, \omega_X^{\otimes m})$  for  $m \geq 1$  and  $q(X) = h^1(X, \mathcal{O}_X)$ .

**Theorem 2.1.** (Castelnuovo) *A smooth, projective surface  $S$  is rational if and only if  $p_2(X) = q(X) = 0$ .*

**Corollary 2.2.** *If  $\text{char}(k) = 0$ , then every unirational surface  $X$  over  $k$  is rational.*

*Proof.* Let  $X$  be a unirational surface. We may assume that  $X$  is smooth and projective. We have seen in Corollary 3.4 in Lecture 1 that since  $\text{char}(k) = 0$ , we have  $q(X) = 0$  and  $p_m(X) = 0$  for all  $m \geq 1$ , hence  $X$  is rational by the theorem.  $\square$

**Example 2.3.** Recall that a *Del Pezzo* surface is a smooth, projective surface  $X$  such that  $\omega_X^{-1}$  is ample. In this case, it is clear that  $h^0(X, \omega_X^{\otimes 2}) = 0$  and if  $\text{char}(k) = 0$ , then we obtain by Kodaira vanishing

$$h^1(X, \mathcal{O}_X) = h^1(X, \omega_X \otimes \omega_X^{-1}) = 0.$$

Therefore  $X$  is rational by Theorem 2.1. In fact, this also holds in positive characteristic. Moreover, it is known that in general, a Del Pezzo surface is isomorphic to either  $\mathbf{P}^1 \times \mathbf{P}^1$  or to the blow-up of  $\mathbf{P}^2$  at  $n \leq 8$  points (see [KSC04, Chapter III]).

**Remark 2.4.** The condition in Theorem 2.1 implies that also  $p_1(X) = 0$ . It is natural to ask whether the vanishing of  $p_1(X)$  and of  $q(X)$  implies that  $X$  is rational. However, this is not the case (for example, Enriques surfaces provide counterexamples).

Before giving the proof of the theorem, we need some preparations. Recall that a smooth projective surface is *minimal* if it contains no curve  $C \simeq \mathbf{P}^1$  with  $(C^2) = -1$ .

**Lemma 2.5.** *If  $X$  is a minimal smooth, projective surface with  $(K_X^2) < 0$ , then for every  $a > 0$ , there is an effective divisor  $D$  on  $X$  with*

$$(K_X \cdot D) \leq -a \quad \text{and} \quad |K_X + D| = \emptyset.$$

*Proof. Step 1.* We show that it is enough to find an effective divisor  $E$  on  $X$  such that  $(K_X \cdot E) < 0$ . Note first that after replacing  $E$  by some irreducible component, we may assume that  $E$  is a prime divisor. The adjunction formula gives

$$(E^2) + (K_X \cdot E) = 2p_a(E) - 2 \geq -2,$$

and since  $(K_X \cdot E) < 0$ , it follows that  $(E^2) \geq -1$ . If  $(E^2) = -1$ , then we must also have  $(K_X \cdot E) = -1$ , hence  $p_a(E) = 0$ ; that is,  $E \simeq \mathbf{P}^1$ . This contradicts the fact that  $X$  is a minimal surface. Therefore we have  $(E^2) \geq 0$ .

Note that

$$\lim_{n \rightarrow \infty} ((aE + nK_X) \cdot E) = -\infty.$$

This implies that  $|aE + nK_X| = \emptyset$  for  $n \gg 0$  (otherwise, if  $B \in |aE + nK_X|$ , we have  $(B \cdot E) < 0$ , and since  $B$  is effective, we see that  $(E^2) < 0$ , contradiction). Since  $|aE| \neq \emptyset$ , we can choose  $n \geq 0$  such that

$$|aE + nK_X| \neq \emptyset \quad \text{and} \quad |aE + (n+1)K_X| = \emptyset.$$

If  $D \in |aE + nK_X|$ , then it is clear that  $|D + K_X| = \emptyset$ . Moreover, we have

$$(K_X \cdot D) = a(K_X \cdot E) + n(K_X^2) \leq -a,$$

hence  $D$  satisfies the conclusion of the lemma.

**Step 2.** We now show that we can find  $E$  as described in the previous step. Let  $H$  be an effective, very ample divisor on  $X$ . If  $(K_X \cdot H) < 0$ , then we are done: we can simply take  $E = H$ .

If  $(K_X \cdot H) = 0$  and if  $E \in |K_X + nH|$ , for  $n \gg 0$ , then

$$(K_X \cdot E) = (K_X^2) < 0,$$

hence we are done.

Suppose now that  $(K_X \cdot H) > 0$ . We choose  $r_0$  such that  $(H + r_0 K_X) \cdot K_X = 0$ , that is,  $r_0 = \frac{(K_X \cdot H)}{-(K_X^2)}$ . Note that  $r_0 > 0$ . Moreover, we have

$$(H + r_0 K_X)^2 = (H^2) + 2r_0(K_X \cdot H) + \frac{(K_X \cdot H)^2}{(K_X^2)} = (H^2) - \frac{(K_X \cdot H)^2}{(K_X^2)} > 0.$$

If  $r > r_0$  is a rational number such that  $r$  is close to  $r_0$ , then

$$(H + r K_X)^2 > 0, (H + r K_X) \cdot K_X < 0, \text{ and } (H + r K_X) \cdot H > 0.$$

We write  $r = p/q$ , with  $p$  and  $q$  positive integers. If we put  $D_m = mq(H + r K_X)$ , then using Riemann-Roch, we obtain

$$\begin{aligned} \chi(\mathcal{O}_X(D_m)) &= \frac{1}{2} \cdot ((D_m^2) - (D_m \cdot K_X)) + \chi(\mathcal{O}_X) \\ &= \frac{m^2 q^2}{2} (H + r K_X)^2 - \frac{mq}{2} ((H + r K_X) \cdot K_X) + \chi(\mathcal{O}_X), \end{aligned}$$

which goes to infinity when  $m$  goes to infinity. We thus have

$$(4) \quad \lim_{m \rightarrow \infty} (h^0(X, \mathcal{O}_X(D_m)) + h^0(X, \mathcal{O}_X(K_X - D_m))) = \infty.$$

Since  $(H + r K_X) \cdot H > 0$ , we have  $\lim_{m \rightarrow \infty} (K_X - D_m) \cdot H = -\infty$ , hence

$$h^0(X, \mathcal{O}_X(K_X - D_m)) = 0 \quad \text{for } m \gg 0.$$

Therefore (4) implies

$$\lim_{m \rightarrow \infty} h^0(X, \mathcal{O}_X(D_m)) = \infty.$$

If we choose  $E \in |D_m|$  for  $m \gg 0$ , then  $(K_X \cdot E) < 0$ , as required. This completes the proof of the lemma.  $\square$

The following proposition will be used in the proof of Castelnuovo's criterion.

**Proposition 2.6.** *If  $X$  is a minimal smooth, projective surface with  $q(X) = p_2(X) = 0$ , then there is a curve  $C \simeq \mathbf{P}^1$  on  $X$  such that  $(C^2) \geq 0$ .*

One should note that the assertion in the proposition is not clear even if we know that  $X$  is rational.

*Proof of Proposition 2.6. Step 1.* We first show that it is enough to find an effective divisor  $D$  on  $X$  such that  $(K_X \cdot D) < 0$  and  $|K_X + D| = \emptyset$ . Indeed, in this case there is an irreducible curve  $C$  that appears in  $D$  such that  $(K_X \cdot C) < 0$ . Moreover, since  $D - C$  is effective, we see that  $|K_X + C| = \emptyset$ .

Note that since  $p_2(X) = 0$ , we also have  $p_1(X) = 0$ , and since  $q(X) = 0$ , we deduce  $\chi(\mathcal{O}_X) = 1$ . By Riemann-Roch and the adjunction formula, we have

$$\chi(\mathcal{O}_X(-C)) = \chi(\mathcal{O}_X) + \frac{1}{2} \cdot ((C^2) + (C \cdot K_X)) = 1 + \frac{1}{2}(C^2) + \frac{1}{2}(C \cdot K_X) = p_a(C).$$

On the other hand,  $h^0(X, \mathcal{O}_X(-C)) = 0$  and by Serre duality

$$h^2(X, \mathcal{O}_X(-C)) = h^0(X, \mathcal{O}_X(K_X + C)) = 0,$$

hence

$$p_a(C) = \chi(\mathcal{O}_X(-C)) = -h^1(X, \mathcal{O}_X(-C)) \leq 0.$$

Therefore  $C \simeq \mathbf{P}^1$ . Since  $(K_X \cdot C) < 0$  and the adjunction formula gives

$$(K_X \cdot C) + (C^2) = -2,$$

we conclude that  $(C^2) \geq 0$  (otherwise  $(C^2) = -1$ , contradicting the assumption that  $X$  is minimal).

We now proceed to find  $D$  as above, by treating separately 3 cases, depending on the sign of  $(K_X^2)$ .

**Step 2.** If  $(K_X^2) < 0$ , then the existence of  $D$  follows directly from Lemma 2.5.

**Step 3.** If  $(K_X^2) = 0$ , then by applying Riemann-Roch for  $\mathcal{O}(-K_X)$ , we obtain, using the fact that  $h^2(X, \mathcal{O}(-K_X)) = h^0(X, \mathcal{O}(2K_X)) = 0$ , the following equality

$$h^0(X, \mathcal{O}(-K_X)) - h^1(X, \mathcal{O}(-K_X)) = 1 + (K_X^2) = 1.$$

In particular, we have  $|-K_X| \neq \emptyset$ .

Let  $H$  be an effective, very ample divisor. Since  $|-K_X| \neq \emptyset$  and  $-K_X \not\sim 0$  (otherwise we would have  $p_2(X) = 1$ ), we deduce  $(H \cdot (-K_X)) > 0$ . This gives  $((H + nK_X) \cdot H) < 0$  for  $n \gg 0$ , hence  $|H + nK_X| = \emptyset$ . Therefore there is  $n \geq 0$  such that

$$|H + nK_X| \neq \emptyset \quad \text{and} \quad |H + (n+1)K_X| = \emptyset.$$

If  $D \in |H + nK_X|$ , then  $|K_X + D| = \emptyset$  and

$$(K_X \cdot D) = (K_X \cdot H) < 0.$$

Therefore  $D$  satisfies the requirements in Step 1.

**Step 4.** Finally, let us consider the case  $(K_X^2) > 0$ . As above, the Riemann-Roch formula for  $\mathcal{O}(-K_X)$  gives

$$h^0(X, \mathcal{O}(-K_X)) - h^1(X, \mathcal{O}(-K_X)) = 1 + (K_X^2) \geq 2,$$

hence  $h^0(X, \mathcal{O}(-K_X)) \geq 2$ .

If there is  $G \in |-K_X|$  which is not a prime divisor, then we write  $G = A + B$ , with  $A$  and  $B$  nonzero effective divisors. Since  $(G \cdot K_X) < 0$ , without any loss of generality we may assume that  $(A \cdot K_X) < 0$ . Since  $|K_X + A| = |-B| = \emptyset$ , we see that  $A$  satisfies the requirement in Step 1, hence we are done.

We assume now that every divisor in  $|-K_X|$  is a prime divisor and choose  $G \in |-K_X|$ . If  $H$  is any effective divisor, then we can find  $n \geq 0$  such that  $|H + nK_X| \neq \emptyset$  and  $|H + (n+1)K_X| = \emptyset$  (indeed, we see that  $|H + nK_X| = \emptyset$  for  $n \gg 0$  by intersecting with a fixed ample divisor and using the fact that  $|-K_X|$  contains a nonzero effective divisor). We now consider separately two cases.

**Case 1.** We suppose that we can find  $H$  and  $n$  as above, such that  $H + nK_X \not\sim 0$ . We choose  $E \in |H + nK_X|$ , and write  $E = \sum_i n_i C_i$ , with the  $C_i$  mutually distinct prime divisors. Note that we have  $(K_X \cdot E) = -(G \cdot E)$  and since  $G$  is a prime divisor with  $(G^2) = (K_X^2) > 0$ , we have  $(G \cdot E) \geq 0$ . We deduce that there is  $i$  such that  $(K_X \cdot C_i) \leq 0$  and put  $C = C_i$ .

Since  $|K_X + E| = \emptyset$  and  $E - C$  is effective, we have  $|K_X + C| = \emptyset$ . If  $(K_X \cdot C) < 0$ , then  $C$  satisfies the requirement in Step 1 and we are done. We may thus assume that  $(K_X \cdot C) = 0$  and we will see that in this case we get a contradiction. Note first that the adjunction formula gives

$$2p_a(C) - 2 = (C^2).$$

On the other hand, the Riemann-Roch formula for  $\mathcal{O}_X(-C)$  gives

$$\chi(X, \mathcal{O}_X(-C)) = 1 + \frac{1}{2}(C^2) = p_a(C),$$

and since  $h^0(X, \mathcal{O}_X(-C)) = 0$  and

$$h^2(X, \mathcal{O}_X(-C)) = h^0(X, \mathcal{O}_X(C + K_X)) = 0,$$

we conclude that  $p_a(C) = 0$  and  $h^1(X, \mathcal{O}_X(-C)) = 0$ . Therefore  $C \simeq \mathbf{P}^1$  and  $(C^2) = -2$ .

The Riemann-Roch formula for  $\mathcal{O}_X(-C - K_X)$  gives

$$\begin{aligned} h^0(X, \mathcal{O}_X(-C - K_X)) - h^1(X, \mathcal{O}_X(-C - K_X)) + h^2(X, \mathcal{O}_X(-C - K_X)) \\ = 1 + \frac{1}{2} \cdot ((-K_X - C) \cdot (-2K_X - C)) \\ = 1 + \frac{1}{2} \cdot (2(K_X^2) + 3(K_X \cdot C) + (C^2)) = (K_X^2). \end{aligned}$$

On the other hand, we have by Serre duality

$$h^2(X, \mathcal{O}_X(-K_X - C)) = h^0(X, \mathcal{O}_X(C + 2K_X)) \leq h^0(X, \mathcal{O}_X(C + K_X)) = 0,$$

where we used the fact that  $|-K_X| \neq \emptyset$ . By combining these, we deduce

$$h^0(X, \mathcal{O}_X(-C - K_X)) \geq (K_X^2) \geq 1.$$

If  $-K_X - C \sim 0$ , then

$$0 = (C \cdot K_X) = -(K_X^2) < 0,$$

a contradiction. Therefore we have  $-K_X - C \sim A$ , where  $A$  is a nonzero effective divisor. In this case, we have  $-K_X \sim C + A$ , contradicting our assumption that every element of  $|-K_X|$  is a prime divisor.

**Case 2.** We suppose that for every effective divisor  $H$ , if we choose  $n$  as before, then  $H + nK_X \sim 0$ . This implies that  $\text{Pic}(X)$  is generated by  $\mathcal{O}(K_X)$  (in particular, we have  $\text{Pic}(X) \simeq \mathbf{Z}$  since  $\text{Pic}(X)$  can't be torsion). We will show that this leads to a contradiction. In order to complete the argument, we assume that  $X = \mathbf{C}$  and make use of transcendental arguments (we refer to Appendix 1 for the facts that we need).

The exponential short exact sequence:

$$0 \rightarrow \underline{\mathbf{Z}} \rightarrow \mathcal{O}_{X^{\text{an}}} \rightarrow \mathcal{O}_{X^{\text{an}}}^* \rightarrow 0$$

induces, using GAGA, an exact sequence

$$0 = H^1(X, \mathcal{O}_X) \rightarrow \text{Pic}(X) \rightarrow H^2(X, \mathbf{Z}) \rightarrow H^2(X, \mathcal{O}_X) = 0,$$

where the vanishings follow from the fact that  $q(X) = 0 = p_2(X)$ . We thus conclude that  $H^2(X, \mathbf{Z}) \simeq \mathbf{Z}$  and it is generated by  $c^1(\mathcal{O}(K_X))$ .

On the other hand, Poincaré duality gives a perfect pairing

$$H^2(X, \mathbf{Z})/H^2(X, \mathbf{Z})_{\text{tors}} \times H^2(X, \mathbf{Z})/H^2(X, \mathbf{Z})_{\text{tors}} \rightarrow \mathbf{Z},$$

which maps  $(c^1(\mathcal{O}(K_X)), c^1(\mathcal{O}(K_X)))$  to  $(K_X^2)$ . Therefore we have  $(K_X^2) = 1$ .

We now make use of Noether's formula:

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}((K_X^2) + \chi_{\text{top}}(X)).$$

Since  $b_2(X) = 1$  and  $b_3(X) = b_1(X) = 2q(X) = 0$ , we have  $\chi_{\text{top}}(X) = 3$ , so that we get  $\chi(X, \mathcal{O}_X) = \frac{1}{3}$ , a contradiction.  $\square$

**Remark 2.7.** In the proof of Proposition 2.6 we assumed in a key step that  $k = \mathbf{C}$ . However, by arguing as in Remark 1.2, we see that if we know the assertion in the proposition when  $k = \mathbf{C}$ , then we obtain the same assertion when  $k$  is any algebraically closed field with  $\text{char}(k) = 0$ .

We can now prove Castelnuovo's criterion.

*Proof of Theorem 2.1.* If  $X$  is rational, then  $p_2(X) = 0 = q(X)$  by Corollary 3.4 in Lecture 1. The interesting implication is the converse. Suppose that  $p_2(X) = 0 = q(X)$  and we want to show that  $X$  is rational. Recall that there is a finite sequence of morphisms

$$X = X_r \rightarrow X_{r-1} \rightarrow \dots \rightarrow X_1,$$

each of them being the blow-up of a smooth point on a smooth, projective surface, such that  $X_1$  is minimal<sup>1</sup>. Since both  $q$  and  $p_2$  are birational invariants, it follows that after replacing  $X$  by  $X_1$ , we may assume that  $X$  is minimal.

We deduce using Proposition 2.6 that there is a curve  $C \simeq \mathbf{P}^1$  on  $X$  such that  $(C^2) = d \geq 0$ . Note that we have an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{O}_{\mathbf{P}^1}(d) \rightarrow 0.$$

Since  $H^1(X, \mathcal{O}_X) = 0$ , the long exact sequence in cohomology gives

$$h^1(X, \mathcal{O}_X(C)) = h^0(X, \mathcal{O}_X) + h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)) \geq 2.$$

Let  $D \in |C|$ ,  $D \neq C$ , and consider the linear system  $|V|$  generated by  $C$  and  $D$ . Since  $C$  is irreducible, it follows that  $|V|$  has no base curves. We consider the map  $f: X \dashrightarrow \mathbf{P}^1$  defined by  $|V|$ . If  $|V|$  is base-point free, then we are done:  $f$  has a scheme-theoretic fiber equal to  $C \simeq \mathbf{P}^1$ , hence Theorem 1.1 implies that there is an open subset  $U$  of  $\mathbf{P}^1$  such that  $f^{-1}(U)$  is isomorphic to  $U \times \mathbf{P}^1$ ; therefore  $X$  is rational.

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<sup>1</sup>The reason such a sequence terminates is that the rank of the Néron-Severi group decreases under a blow-down; since this rank is finite, there can be only finitely many such morphisms.



Suppose now that  $p \in X$  is a base-point of  $|V|$  and consider the blow-up  $g: \tilde{X} \rightarrow X$  of  $X$  at  $p$ , with exceptional divisor  $E$ . By assumption,  $p$  lies on every divisor in  $|V|$ . Since  $C$  is a smooth curve, if  $\tilde{C}$  is the strict transform of  $C$ , we have  $g^*(C) = \tilde{C} + E$  and  $\tilde{C} \simeq C \simeq \mathbf{P}^1$ . Since  $E$  is contained in the inverse image of every divisor in  $|V|$ , it follows that the composition  $f \circ g_1$  is defined by a 1-dimensional linear system, one of whose elements is  $\tilde{C}$ . If this composition is a morphism, we conclude as above that  $\tilde{X}$  is rational, hence  $X$  is rational. Otherwise, we iterate. It follows that in order to complete the proof, it is enough to show that after finitely many blow-ups of base points, the rational map induced by  $f$  becomes a morphism. This is standard, but we give an argument below.

By taking a resolution of singularities for the graph of  $f$ , we obtain a birational morphism  $\pi: Y \rightarrow X$ , with  $Y$  a smooth, projective surface, such that  $\varphi = f \circ \pi$  is a morphism. Like every birational morphism between smooth, projective surfaces,  $\varphi$  factors as a composition of point blow-ups. We note that if  $f$  is not a morphism and  $g$  is as above, then  $p$  does not lie in the domain of  $\pi^{-1}$ . In this case, it follows from [Har77, Proposition V.5.3] that  $\pi$  factors through  $g$ . Since this can happen only finitely many times<sup>2</sup>, it follows that after finitely many such blow-ups the rational map induced by  $f$  becomes a morphism. This completes the proof of the theorem.  $\square$

### 3. SHIODA'S EXAMPLES

In this section we discuss the following family of examples due to Shioda [Shi74].

**Proposition 3.1.** *If  $k$  is a field of characteristic  $p > 2$ , then the surface in  $\mathbf{P}_k^3$  given by the equation*

$$x_1^{p+1} - x_2^{p+1} = x_3^{p+1} - x_4^{p+1}$$

*is a smooth, unirational surface.*

**Remark 3.2.** It is shown in [Shi74] that, assuming  $k$  is algebraically closed and with  $\text{char}(k) = p > 2$ , the surface in  $\mathbf{P}_k^3$  defined by

$$x_1^n + x_2^n + x_3^n + x_4^n = 0$$

is unirational if there is  $m \geq 1$  such that  $p^m \equiv -1 \pmod{n}$ .

*Proof of Proposition 3.1.* Let  $X$  be the surface in the proposition. It is clear that this is smooth. Consider now the following change of variable:

$$\begin{aligned} y_1 &= x_1 + x_2, & y_2 &= x_1 - x_2, & \text{and} \\ y_3 &= x_3 + x_4, & y_4 &= x_3 - x_4. \end{aligned}$$

Note that we have

$$x_1^{p+1} - x_2^{p+1} = \frac{1}{2} y_1 y_2 (y_1^{p-1} + y_2^{p-1}).$$

Indeed, we have

$$y_1 y_2 (y_1^{p-1} + y_2^{p-1}) = y_1^p y_2 + y_2^p y_1$$

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<sup>2</sup>Recall that every birational morphism  $h: S_1 \rightarrow S_2$  between smooth, projective surfaces is a composition of smooth blow-ups, and the number of such blow-ups is  $(K_{S_2}^2) - (K_{S_1}^2)$ .

$$= (x_1^p + x_2^p)(x_1 - x_2) + (x_1^p - x_2^p)(x_1 + x_2) = 2(x_1^{p+1} - x_2^{p+1}).$$

We similarly have

$$x_3^{p+1} - x_4^{p+1} = \frac{1}{2}y_3y_4(y_3^{p-1} + y_4^{p-1})$$

and therefore in these new coordinates the equation of  $X$  can be written as

$$y_1y_2(y_1^{p-1} + y_2^{p-1}) = y_3y_4(y_3^{p-1} + y_4^{p-1}).$$

We consider the chart  $(y_4 \neq 0)$  in  $\mathbf{P}^3$ , and we have the corresponding open subset of  $X$ , which is the surface in  $\mathbf{A}^3$  with the equation

$$y_1y_2(y_1^{p-1} + y_2^{p-1}) = y_3(y_3^{p-1} + 1).$$

Consider the morphism

$$\varphi: \mathbf{A}^3 = \operatorname{Spec}(k[y_1, u, v]) \rightarrow \mathbf{A}^3 = \operatorname{Spec}(k[y_1, y_2, y_3])$$

induced by

$$\varphi^*(y_1) = y_1, \varphi^*(y_2) = y_1u, \quad \text{and} \quad \varphi^*(y_3) = uv.$$

Since we can write

$$u = \frac{\varphi^*(y_2)}{\varphi^*(y_1)} \quad \text{and} \quad v = \frac{\varphi^*(y_3)}{u},$$

it follows that  $\varphi$  is birational. Moreover, we have

$$\begin{aligned} \varphi^*(y_1y_2(y_1^{p-1} + y_2^{p-1}) - y_3(y_3^{p-1} + 1)) &= y_1^2u(y_1^{p-1} + y_1^{p-1}u^{p-1}) - uv(u^{p-1}v^{p-1} + 1) = \\ &= u(y_1^{p+1}(1 + u^{p-1}) - v(u^{p-1}v^{p-1} + 1)), \end{aligned}$$

hence  $X$  is birational to the hypersurface  $Y$  defined in  $\mathbf{A}^3$  by

$$(5) \quad y_1^{p+1}(1 + u^{p-1}) = v(u^{p-1}v^{p-1} + 1).$$

Let  $t = y_1^{1/p}$ , so that

$$y_1^{p+1}(1 + u^{p-1}) - v(u^{p-1}v^{p-1} + 1) = u^{p-1}(t^{p+1} - v)^p + (t^{p(p+1)} - v).$$

Note that in  $k(Y)$  we have

$$(6) \quad y_1^{p+1} = \frac{v(u^{p-1}v^{p-1} + 1)}{1 + u^{p-1}}.$$

Since  $\operatorname{trdeg}_k k(Y) = 2$ , it follows that  $u$  and  $v$  are algebraically independent over  $k$ , and (6) implies that  $y_1^{p+1} \notin k(u^p, v^p)$ , hence  $t \notin k(Y)$ . Therefore the extension

$$k(Y) \hookrightarrow k(Y)(t) = K$$

is a purely inseparable extension of degree  $p$ . Note that  $K = k(t, u, v)$ .

Let  $s = u(t^{p+1} - v) \in K$ . We have

$$s^p = u(v - t^{p(p+1)}) = \frac{s(v - t^{p(p+1)})}{t^{p+1} - v}$$

and after cross-multiplying we obtain

$$v(s^p + s) = t^{p+1}(s^p - t^{p^2-1}).$$

Note that  $s^p + s \neq 0$ : otherwise  $s$  is algebraic over  $k$  and since then also  $t^{p+1}(s^p - t^{p^2-1}) = 0$ , we deduce that also  $t$  is algebraic over  $k$ ; this implies that  $y_1$  is algebraic over  $k$  and using for example equation (5), we deduce that  $\text{trdeg}_k k(Y) = 1$ , a contradiction. Therefore

$$v = \frac{t^{p+1}(s^p - t^{p^2-1})}{s(s^{p-1} + 1)} \in k(s, t),$$

and since we also have  $u = s/(t^{p+1} - v) \in k(s, t, v)$ , we conclude that  $K = k(s, t)$ . Since  $\text{trdeg}_k K = 2$ , it follows that  $K$  is a purely transcendental extension of  $k$ , and the inclusion  $k(Y) \hookrightarrow K$  gives a purely inseparable rational, dominant map  $\mathbf{P}^2 \dashrightarrow X$ .  $\square$

**Remark 3.3.** It follows from Corollary 3.5 in Lecture 1 that a smooth surface in  $\mathbf{P}^3$  of degree  $d \geq 4$  is not rational. We deduce that the examples in Proposition 3.1 are unirational surfaces that are not rational, giving a negative answer to the Lüroth problem for surfaces over fields of positive characteristic  $p > 2$ .

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