Chapter 5: Toric resolution of singularities

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Normalized blow-ups along invariant subschemes. We describe the normalization of the blowing-up of an affine toric variety along an invariant closed subscheme. Let σ be a pointed cone in N. We have seen in Chapter 1 that a proper closed invariant subscheme of the toric variety U_σ is defined by a nonzero ideal $\mathfrak{a} = (\chi^{u_1}, \ldots, \chi^{u_r})$ of $k[\sigma^\vee \cap M]$. Let $P = P_{\mathfrak{a}}$ be the Newton polyhedron of $\mathfrak{a}$. Recall that $P = \mathrm{conv}(u_1, \ldots, u_r) + \sigma^\vee$, so by Proposition 5 in Chapter 4, the normal fan Δ_P to P gives a refinement of σ.

Proposition 1. *The induced morphism of toric varieties $f\colon X(\Delta_P) \to U_\sigma$ is the normalization of the blowing-up of U_σ along the subscheme V defined by $\mathfrak{a}$.*

Proof. We may assume that σ spans $N_{\mathbb{R}}$. Let $g\colon Y \to U_\sigma$ be the normalization of the blowing-up of U_σ along V. Since the action of the torus T_N on $k[\sigma^\vee \cap M]$ preserves the ideal $\mathfrak{a}$, we get an induced action of T_N on $\oplus_{i\in\mathbb{N}}\mathfrak{a}^i$ extending the action on $\mathcal{O}(U_\sigma)$. Therefore we have an action of T_N on the blowing-up of U_σ along V, hence on its normalization Y, such that g is an equivariant morphism. Moreover, since $V \cap T_N = \emptyset$ we see that g is an isomorphism over T_N. We deduce that we have an open immersion $T_N \subseteq Y$ such that the induced action of T_N on itself is the standard one.

By Theorem 1 and Proposition 6 in Chapter 4, there is a fan Δ refining σ such that $Y = X(\Delta)$ and g is induced by the identity on N. We need to show that $\Delta = \Delta_P$.

By the universal property of the blowing-up (see Proposition 7.14 in [Ha]) we see that $g^{-1}(\mathfrak{a})$ is a locally principal sheaf of ideals; moreover, if $h\colon Y' \to U_\sigma$ is a proper, birational morphism, Y' is a normal variety, and $h^{-1}(\mathfrak{a})$ is locally principal, then there is a unique morphism $h'\colon Y' \to Y$ such that $g \circ h' = h$.

The sheaf of ideals $g^{-1}(\mathfrak{a})$ is locally principal if and only if for every maximal cone τ in Δ, the ideal $\mathfrak{a} \cdot k[\tau^\vee \cap M]$ is locally principal. Such τ spans $N_{\mathbb{R}}$, so $\mathfrak{a} \cdot k[\tau^\vee \cap M]$ is locally principal if and only it is generated by one of the χ^{u_i}, where $\chi^{u_1}, \ldots, \chi^{u_r}$ are the generators of $\mathfrak{a}$ (see Chapter 3).

If this is the case, then there is i with $u - u_i \in \tau^\vee$ for every u such that $\chi^u \in \mathfrak{a}$. As P is the convex hull of such u, it follows that $u - u_i$ is in $\tau^\vee$ for every u in P. If v lies in the interior of τ, we see that $\langle u, v\rangle \geq \langle u_i, v\rangle$ for every u in P. Moreover, we have equality if and only if $u = u_i$ (by our choice of v and the fact that τ spans $N_{\mathbb{R}}$). Therefore u_i is a vertex of P, and τ is contained in the corresponding maximal cone of Δ_P. This shows that Δ is a fan refining Δ_P, so we have an induced morphism of toric varieties $f'\colon Y \to X(\Delta_P)$ such that $f \circ f' = g$.

By reversing the above argument, we see that $f^{-1}(\mathfrak{a})$ is a locally principal ideal. The universal property of the blowing-up implies that we have a morphism $g'\colon X(\Delta_P) \to X(\Delta)$ such that $g \circ g' = f$. We see that f' and g' give inverse isomorphisms, as this is the case over T_N; this completes the proof.

Example 1. Suppose that σ is a nonsingular cone which spans $N_{\mathbb{R}}$ and that $\mathfrak{a}$ is the ideal defining $V(\tau)$ for a face τ of σ. Let $e_1, \ldots, e_n$ be a basis of N such that σ is generated by $e_1, \ldots, e_n$ and τ is generated by $e_1, \ldots, e_r$. If we put $t_i = \chi^{e_i^*}$, then the ideal $\mathfrak{a}$ defining $V(\tau)$ is generated by $t_1, \ldots, t_r$.

The Newton polyhedron of this ideal is

$$P = \{(u_1, \ldots, u_n) \in \mathbb{R}_+^n \mid u_1 + \ldots + u_r \geq 1\}.$$

The normal fan has r maximal cones, generated by $\sum_{i=1}^r e_i$ and $e_1, \ldots, \widehat{e_j}, \ldots, e_n$ for $1 \leq j \leq r$. Therefore we recover the description of the smooth blowing-up from Example 6 in Chapter 4.

Example 2. For positive integers $d_1, \ldots, d_n$, consider the **weighted blowing-up** of $\mathbb{A}^n$ with weights $d_1, \ldots, d_n$. This is the morphism of toric varieties $f\colon X(\Delta) \to \mathbb{A}^n$ induced by the identity on $N = \mathbb{Z}^n$, where Δ is the fan with maximal cones $\sum_{i=1}^n d_i e_i, e_1 \ldots, \widehat{e_j}, \ldots, e_n$ for $1 \leq j \leq n$. When $d_1 = \ldots = d_n = 1$, this is the usual blowing-up at the origin. Using Proposition 1, we see that if m is the least common multiple of the d_i, then f is the blowing-up of $\mathbb{A}^n$ along the ideal $(t_1^{m/d_1}, \ldots, t_n^{m/d_n})$.

Remark 1. There is a normal semigroup algebra R such that the normalized blow-up of U_σ along $\mathfrak{a}$ is given by $\mathrm{Proj}(R)$. Indeed, if we denote by $\overline{\mathfrak{a}^m}$ the integral closure of $\mathfrak{a}^m$, then it is known that the normalized blow-up of U_σ along $\mathfrak{a}$ is given by $\mathrm{Proj}(R)$, where $R = \bigoplus_{m \in \mathbb{N}} \overline{\mathfrak{a}^m} t^m$. In fact, it will follow from the description of R which we give below that R is normal; on the other hand, one can check using the definition of integral closure that R is contained in the normalization of $\bigoplus_{m \in \mathbb{N}} \mathfrak{a}^m t^m$.

If $P = P_{\mathfrak{a}}$ is the Newton polyhedron of $\mathfrak{a}$, then $\overline{\mathfrak{a}^m} = \oplus_u k\chi^u$, where the sum is over those u in $mP \cap M$. If τ is the closed cone over $P \times \{1\}$

in $M \times \mathbb{Z}$, note that we have $\tau \cap (M_{\mathbb{R}} \times \{0\}) = \sigma^{\vee} \times \{0\}$; we deduce

$$R = k[\tau^{\vee} \cap M] = \bigoplus_{m \in \mathbb{N}} \left(\bigoplus_{u \in mP \cap M} k\chi^{(u,m)} \right).$$

Exercise 1. Show that the blowing-up of $\mathbb{A}^2$ along the ideal (x^2, y^2) is not normal.

The following proposition gives criteria in low dimension for the blowing-up along an invariant ideal to be normal.

Proposition 2 (Ishida)**.** *With $\mathfrak{a}$ an ideal as in Proposition 1, let B be the blowing-up of U_{σ} along $\mathfrak{a}$.*

(1) *If* $\dim(\sigma) = 2$ *and $\mathfrak{a}$ is integrally closed, then B is normal.*
(2) *If* $\dim(\sigma) = 3$ *and $\mathfrak{a}$ is reduced, then B is normal.*

Proof. Let P be the Newton polyhedron of $\mathfrak{a}$. In order to show that B is normal, it is enough to show that

$$\bigoplus_{m \in \mathbb{N}} \mathfrak{a}^m = \bigoplus_{m \in \mathbb{N}} \left(\bigoplus_{u \in mP \cap M} k\chi^{(u,m)} \right)$$

(see Remark 1). Since $\mathfrak{a}$ is assumed integrally closed, this is equivalent with the fact that for every $m \geq 2$ and every u in $mP \cap M$ there are $u_1, \ldots, u_m$ in $P \cap M$ such that $u = u_1 + \ldots + u_m$.

Consider first the case when $\dim(\sigma) = 2$. If u is on a ray of $\sigma^{\vee}$ and if we take u_0 to be the first point on the intersection of P with that ray (this is a lattice point since it is a vertex of P), then $u - mu_0$ is in $\sigma^{\vee}$. Therefore $u - (m-1)u_0$ is in P and we write $u = (u - (m-1)u_0) + u_0 + \ldots + u_0$.

Suppose now that u is in the relative interior of σ, and let u_1, u_2 be the vertices of the facet F of P containing $\mathbb{R}_+ u \cap \partial(P)$. We may write $u = \lambda_1 u_1 + \lambda_2 u_2$ for nonnegative λ_1 and λ_2. On the other hand, there is a positive a and v in $N_{\mathbb{R}}$ such that $\langle u', v \rangle \geq a$ for every u' in P, and $\langle u', v \rangle = a$ for every u' in F. We deduce that $\langle u, v \rangle = \lambda_1 a + \lambda_2 a \geq ma$, so $\lambda_1 + \lambda_2 \geq m$. Therefore we may write $u = \alpha_1 u_1 + \alpha_2 u_2 + w$ for nonnegative integers α_1 and α_2 such that $\alpha_1 \leq \lambda_1$, $\alpha_1 \leq \lambda_2$ and $\alpha_1 + \alpha_2 = m - 1$. Since $w = (\lambda_1 - \alpha_1)u_1 + (\lambda_2 - \alpha_2)u_2$ and $(\lambda_1 - \alpha_1) + (\lambda_2 - \alpha_2) \geq 1$, w is in P and our assertion follows in this case.

Suppose now that $\dim(\sigma) = 3$ and $\mathfrak{a}$ is reduced (note that $\mathfrak{a}$ is in particular integrally closed). Using the two-dimensional case, we may assume that u is in the relative interior of $\sigma^{\vee}$. We consider the facet F of P containing $\tilde{u} = \mathbb{R}_+ u \cap \partial(P)$ and let u_1, u_2 and u_3 be elements in $M \cap F$ such that $\tilde{u}$ lies in the convex hull of u_1, u_2 and u_3 (we may assume that $\tilde{u}$ is not in M, as otherwise our assertion is trivial). Furthermore, we may assume that this convex hull contains no other points in M.

As above, we write $u = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ and we see that $\lambda_1 + \lambda_2 + \lambda_3 \geq m$. Moreover, using the same argument as in the two-dimensional case, we see that it is enough to consider the case when the sum of the integral parts of the λ_i is $\leq m-2$. We get real numbers α_1, α_2 and α_3 in $(0,1)$ with $\alpha_1 + \alpha_2 + \alpha_3 \geq 2$ such that $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$ is in M.

Note that $w := (1-\alpha_1)u_1 + (1-\alpha_2)u_2 + (1-\alpha_3)u_3$ is in M and some multiple of w is in P. Since $\mathfrak{a}$ is reduced, we deduce that w is in P. As $\sum_i (1-\alpha_i) \leq 1$, we see that either w is not in P, or w is in the relative interior of the convex hull of u_1, u_2 and u_3. As both these cases are impossible, this completes the proof.

Example 3 (Ishida)**.** One could use the following construction to give examples of toric varieties in dimension at least four such that the blowing-up at the fixed point is not normal. Suppose that $R = \bigoplus_{i \in \mathbb{N}} R_i$ is a graded algebra of finite type over $R_0 = k$, such that R is integral over $k[R_1]$ but $R \neq k[R_1]$. Let $S = R[s]$ which is an $\mathbb{N}^2$-graded ring with $\deg R_j s^i = (i, j)$. We show that the blowing-up of $\mathrm{Spec}(S)$ at the ideal I generated by s and $R_+ := \bigoplus_{i>0} R_i$ is not normal.

This blow-up is equal to $\mathrm{Proj}(\bigoplus_{m \in \mathbb{N}} I^m t^m)$. Consider the affine open subset $U = \mathrm{Spec}(\widetilde{S})$ defined by the nonvanishing of the section $s \in I$. We have

$$\widetilde{S} = R[s] \oplus \bigoplus_{m \geq 1} \frac{1}{s^m} (R_+)^m$$

which is $\mathbb{Z}^2$-graded and its component of degree $(-i, i)$ is $\frac{1}{s^i}(R_1)^i$. This gives

$$\bigoplus_{i \in \mathbb{N}} \widetilde{S}_{-i,i} \simeq k[R_1].$$

On the other hand, we have

$$\bigoplus_{i \in \mathbb{N}} (\widetilde{S}_t)_{(-i,i)} = \bigoplus_{i \in \mathbb{N}} \frac{1}{s^i} R_i \simeq R.$$

It follows from our hypothesis that there are elements in $\widetilde{S}_t$ which are not in $\widetilde{S}$ but are integral over $\widetilde{S}$, so $\widetilde{S}$ is not normal.

In order to get non-normal blow-ups of toric varieties, consider a full-dimensional lattice polytope P in N such that for some $m \geq 2$ we have

$$mP \cap N \neq (P \cap N) + ((m-1)P \cap N)$$

(consider Example 4 in Chapter 2 for a concrete example). If $v_1, \ldots, v_r$ are the elements of $P \cap N$, it follows from the exercise in Chapter 2, p. 16 that the semigroup S generated by the $(v_i, 1)$ is not saturated. If on $R = k[S^{\mathrm{sat}}]$ we consider the grading given by $\deg(\chi^{(v,m)}) = m$, then R satisfies our conditions. We deduce that if σ is the cone over $P \times \{1\}$, then the blowing-up of $U_{\sigma^\vee} \times \mathbb{A}^1$ at the fixed point is not normal.

Resolution of toric surface singularities. Recall that a resolution of singularities of a variety X is a proper, birational morphism $f \colon Y \to X$, with Y smooth. In the case when Y is a toric variety, we will construct a resolution of singularities such that f is a morphism of toric varieties. This corresponds to a nonsingular fan that refines the fan defining X. We start by considering the case of surfaces, when we describe the minimal such resolution.

Let σ be a pointed cone in a two-dimensional lattice N with $\dim(\sigma) = 2$. Let (m, ℓ) be a pair of integers such that $0 \leq \ell \leq m$ and $\ell = 0$ if and only if $m = 1$. If this is not the case we assume that m and ℓ are relatively prime.

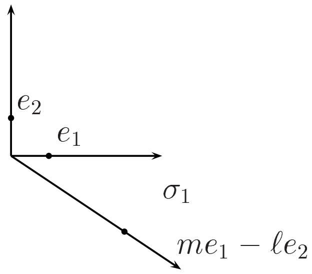
We have seen that we can find a basis e_1, e_2 of N and a pair (m, ℓ) as above, such that σ is generated by $v = e_2$ and $v' = me_1 - \ell e_2$. In this case we say that σ has **type** (m, ℓ). Note that σ has typer $(1, 0)$ if and only if it is nonsingular. The type of σ is not uniquely determined, but almost; in fact, the following holds.

Exercise 2. If σ_1 and σ_2 are two-dimensional pointed cones of types (m_1, ℓ_1) and (m_2, ℓ_2), respectively, then we have an isomorphism of toric varieties $U_{\sigma_1} \simeq U_{\sigma_2}$ if and only if $m_1 = m_2$, and either $\ell_1 = \ell_2$ or $\ell_1 \ell_2 \equiv 1 \pmod{m}$.

Exercise 3. If σ has type (m, ℓ) with $m > 1$, then $\sigma^\vee$ has type $(m, m - \ell)$.

We describe now the algorithm for computing the minimal resolution of U_σ. If σ has type (m, ℓ) with $m > 1$, i.e. σ is singular, let e_1 and e_2 be a basis of N such that σ is generated by $v = e_2$ and $v' = me_1 - \ell e_2$.

We first add the ray through $v_1 = e_1$ to get the fan with two maximal cones, one generated by e_1 and e_2 (which is smooth), and the other one $\sigma_1 = \mathrm{pos}(e_1, me_1 - \ell e_2)$.



In order to compute the type of σ_1, we first rotate it counter-clockwise by $90°$: this corresponds to taking the basis $e'_1 = -e_2$ and $e'_2 = e_1$ such that $v_1 = e'_2$ and $v' = \ell e'_1 + me'_2$. There is a unique integer a_1 such that $v' = \ell(e'_1 + a_1 e'_2) - (a_1\ell - m)e'_2$ and $0 \leq a_1\ell - m < \ell$ (note that we have $a_1 \geq 2$). We see that σ_1 is singular if and only if $\ell > 1$, and in this case σ_1 has type $(m_1, \ell_1) = (\ell, a_1\ell - m)$. We put $(m_0, \ell_0) = (m, \ell)$.

We repeat the above algorithm for σ_1. Since $1 \leq m_1 < m_0$, we see that this procedure finishes after $r \leq m - 1$ steps. This gives a refinement Σ of the cone σ such that the corresponding morphism $f \colon X(\Sigma) \to U_\sigma$ is a resolution of singularities.

We encode the data given by the above process as follows. If r is the number of steps we need to perform until the resulting cone is nonsingular, we denote the primitive generators on the rays we add by $v_1, \ldots, v_r$ (we put also $v' = v_{r+1}$). We denote by σ_i the cone generated by v_i and v_{r+1} for $0 \leq i \leq r$, and by (m_i, ℓ_i) the type of σ_i determined as above. Note that $(m_r, \ell_r) = (1, 0)$.

It follows by induction on i that we have $m_i = \ell_{i-1}$ and $k_i = a_i k_{i-1} - m_{i-1}$ for $1 \leq i \leq r$, where the a_i are integers with $a_i \geq 2$. Note that $a_r = m_{r-1}$.

We have $\frac{m}{\ell} = a_1 - \frac{1}{m_1/\ell_1}$, and repeating this we get the following expression as a continued fraction, but with alternating signs

$$\frac{m}{\ell} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_r}}}.$$

This is called the **Hirzebruch-Jung continued fraction** of m/ℓ.

The resolution $f \colon X(\Sigma) \to U_\sigma$ is an isomorphism over the complement of the fixed point x_σ of U_σ. Moreover, if $\tau_1, \ldots, \tau_r$ are the rays through $v_1, \ldots, v_r$ and if $E_i = V(\tau_i) \subseteq X(\Sigma)$, then $f^{-1}(x_\sigma) = \bigcup_{i=1}^r E_i$. Exercise 5 below shows that $(E_i^2) \leq -2$ for all $1 \leq i \leq r$. It is a consequence of Castelnuovo's Contraction Theorem (see [Ha], Chapter V) that f is the **minimal resolution** of U_σ: if $g \colon X' \to U_\sigma$ is another resolution of singularities, then there is a (unique) morphism $g' \colon X' \to X(\Sigma)$ such that $g = f \circ g'$. We will not need this; in fact, the corresponding minimality assertion in the toric category can be shown directly (see Exercise 8 below).

Exercise 4. Let Σ be a nonsingular fan in a rank two lattice, with maximal cones σ_1 and σ_2 whose intersection τ has dimension one. Let v be the primitive generator of τ, and let v_1 and v_2 be the other primitive generators of σ_1 and σ_2, respectively.

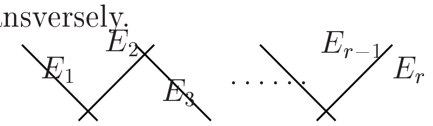
(1) There is an integer a such that $v_1 + v_2 = av$.
(2) If $E = V(\tau) \subseteq X(\Sigma)$, then $E \simeq \mathbb{P}^1$ and the normal bundle of E in $X(\Sigma)$ is $\mathcal{O}_{\mathbb{P}^1}(-a)$. Therefore $(E^2) = -a$.

We return now to the setup and the notation introduced for the resolution of U_σ.

Exercise 5. (1) Show that for every i with $0 \leq i \leq r-1$ we have $\ell_i v_i + v_{r+1} = m_i v_{i+1}$.
(2) Deduce that for every i with $1 \leq i \leq r$, we have

$$(E_i^2) = -a_i \leq -2.$$

(3) The divisors $E_1, \ldots, E_r$ form a chain of rational curves intersecting transversely.



We give now more intrinsic descriptions of the rays in the fan Σ.

Exercise 6. Let P_σ be the convex hull of the nonzero points in $\sigma \cap N$. Show that the boundary of P_σ can be written as

$$\partial(P_\sigma) = \{\lambda v_0 \mid \lambda \geq 1\} \cup \overline{v_0 v_1} \cup \ldots \overline{v_r v_{r+1}} \cup \{\lambda v_{r+1} \mid \lambda \geq 1\},$$

where $\overline{v_i v_{i+1}}$ denotes the line segment joining v_i and v_{i+1}. Moreover, v_i and v_{i+1} are the only points in N on $\overline{v_i v_{i+1}}$. Therefore the rays in Σ are the rays through the points in $\partial(P_\sigma) \cap N$.

Exercise 7. Show that $v_0, v_1, \ldots, v_{r+1}$ is the Hilbert basis of $\sigma \cap N$.

Exercise 8. Show that if Σ' is a nonsingular fan refining the cone σ, then Σ' is a refinement of Δ.

Example 4. For $m \geq 2$ consider the cone σ generated by e_2 and $me_1 - (m-1)e_2$, so U_σ has a singularity of type A_{m-1}. We see that $a_1 = 2$ and the cone σ_1 has type $(m-1, m-2)$. It follows by induction on m that $r = m-1$ and $a_1 = \ldots = a_r = 2$. Therefore the exceptional divisor in the minimal resolution consists of a chain of $(m-1)$ rational curves of self-intersetion (-2).

Using the fact that $v_m+(m-i-1)v_i = (m-i)v_{i+1}$ for $0 \le i \le m-1$, we deduce $v_i = ie_1 - (i-1)e_2$ for $0 \le i \le m$. Therefore $P_\sigma = \{(\lambda, \mu) \in \sigma \mid \lambda + \mu \ge 1\}$.

Exercise 9. Show that the minimal resolution of U_σ can be obtained by repeatedly blowing-up the singular fixed points, as follows.

(1) Show that if σ is a singular cone, then every face of P_σ which is not contained in the boundary of σ is of the form $P_\sigma \cap \{v \mid \langle u, v\rangle = 1\}$ for some u in M such that $\langle u, v\rangle \ge 1$ for every v in P_σ.

(2) Given a singular cone σ, use the previous assertion for $P_{\sigma^\vee}$ to show that the minimal resolution of U_σ factors through the blowing-up of the fixed point on U_σ.

Toric resolutions in arbitrary dimensions. We consider now toric varieties of arbitrary dimension. In this case there might not exist a minimal resolution of singularities. A typical example is that of the affine cone over a quadric in $\mathbb{P}^2$.

Example 5. Consider the cone σ in $\mathbb{R}^3$ generated by the vectors $v_1, \ldots, v_4$ which span $\mathbb{R}^3$ and such that $v_1 + v_2 = v_3 + v_4$. Suppose that N is the lattice generated by the v_i. A nonsingular fan Δ_1 refining σ has maximal cones generated by $\{v_1, v_3, v_4\}$ and $\{v_2, v_3, v_4\}$. We have another such fan Δ_2 having maximal cones $\{v_1, v_2, v_3\}$ and $\{v_1, v_2, v_4\}$.

There is a smooth toric variety $X(\Delta)$ dominating both $X(\Delta_1)$ and $X(\Delta_2)$; it is obtained by adding the ray through $v := v_1 + v_2 = v_3 + v_4$ and having the maximal cones generated by the facets of σ and by v. We have the following diagram, in which all morphisms are proper and birational:

$$
\begin{tikzcd}
 & X(\Delta) \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "\pi"] & \\
X(\Delta_1) \arrow[dr, "\pi_1"'] & & X(\Delta_2) \arrow[dl, "\pi_2"] \\
 & U_\sigma &
\end{tikzcd}
$$

Note that both π_1 and π_2 are *small* resolutions, i.e. they are isomorphisms outside codimension two closed subsets. Using Proposition 1 one can check that π is the blowing-up of U_σ at the fixed point x_σ. Moreover, the fiber $\pi^{-1}(x_\sigma)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, such that the restrictions of f_1 and f_2 to $f^{-1}(x_\sigma)$ are identified to the two projections onto $\mathbb{P}^1$. The birational morphism $\pi_2^{-1} \circ \pi_1$ is an example of a *flop*.

Let $X(\Delta)$ be the toric variety associated to a fan Δ in N. We want to construct a fan refinement Σ of Δ which is nonsingular. Moreover, we want the nonsingular cones in Δ to be in Σ. In this case the resolution $X(\Sigma) \to X(\Delta)$ will be an isomorphism over the smooth locus of $X(\Delta)$.

There are two steps in constructing a resolution of singularities for $X(\Delta)$. The first step consists in the construction of a simplicial refinement of Δ. The second step starts with a simplicial fan and gives an inductive procedure for improving an invariant of the cones that we define now.

Let σ be a simplicial cone in N and let $v_1, \ldots, v_r$ be the primitive generators of its rays. The **multiplicity** of σ is the index of the lattice generated by the v_i in N_σ

$$\mathrm{mult}(\sigma) := |N_\sigma / \sum_{i=1}^{r} \mathbb{Z} v_i|.$$

It follows from definition that σ is nonsingular if and only if the multiplicity of σ is one.

Exercise 10. If $e_1, \ldots, e_r$ is a basis of N_σ, and if we write $v_i = \sum_{i=1}^{r} a_{i,j} e_j$, then $\mathrm{mult}(\sigma) = |\det(a_{i,j})|$. More generally, if $e_1, \ldots, e_n$ is a basis of N and if we write $v_i = \sum_{j=1}^{n} b_{i,j} e_j$, then $\mathrm{mult}(\sigma)$ is the positive integer generating the ideal of $\mathbb{Z}$ generated by the r by r minors of the matrix $(b_{i,j})$.

Note that if σ is a two-dimensional pointed cone of type (m, ℓ), then m is the multiplicity of σ.

Exercise 11. Show that if σ is the cone in $\mathbb{Z}^3$ generated by e_1, e_2 and $e_1 + e_2 + 3e_3$, then $\mathrm{mult}(\sigma) = 3$ and $\mathrm{mult}(\sigma^\vee) = 9$.

The lattice N induces a measure on $N_\mathbb{R}$. More precisely, the choice of an isomorphism $N \simeq \mathbb{Z}^n$ induces an isomorphism $N_\mathbb{R} \simeq \mathbb{R}^n$. The Lebesgue measure on $\mathbb{R}^n$ induces the **n-dimensional lattice volume** vol_n on $N_\mathbb{R}$. Since two such isomorphisms differ by a matrix with determinant ± 1, we see that this measure is well-defined. On every sublattice of N of rank r, we have a corresponding r-dimensional volume.

Recall the following basic and elementary fact that relates the volume to the number of lattice points: if P is an n-dimensional polytope in $N_\mathbb{R}$, then

$$\mathrm{vol}_n(P) = \lim_{\lambda \to \infty} \frac{|N \cap \lambda P|}{\lambda^n}.$$

Exercise 12. Let σ be a simplicial cone and let $v_1, \ldots, v_r$ be the primitive generators of the rays.

(1) We have $\mathrm{mult}(\sigma) = \mathrm{vol}_r(Q_\sigma)$, where
$$Q_\sigma = \left\{ \sum_{i=1}^r \lambda_i v_i \mid 0 \leq \lambda_i \leq 1 \quad \text{for all} \quad i \right\}.$$
(2) The multiplicity of σ is equal to the number of elements in
$$N \cap \left\{ \sum_{i=1}^r \lambda_i v_i \mid 0 \leq \lambda_i < 1 \quad \text{for all} \quad i \right\}.$$

Exercise 13. Let σ be a simplicial cone in N. For every face τ of σ, consider the local ring $\mathcal{O}_{x_\tau}$ of U_σ at x_τ. Show that the Samuel multiplicity of $\mathcal{O}_{x_\tau}$ with respect to the maximal ideal (see [BH], Chapter 4.6) is bounded above by the multiplicity of the cone $\tau^\vee/\tau^\perp$.

Take $\sigma = \tau$ be a two-dimensional cone of type $(m, m-1)$. Show that if $m \geq 2$, then the Samuel multiplicity of $\mathcal{O}_{x_\sigma}$ with respect to the maximal ideal is two, while $\mathrm{mult}(\sigma) = \mathrm{mult}(\sigma^\vee) = m$. In particular, the Samuel multiplicity is strictly smaller than $\mathrm{mult}(\sigma^\vee)$ if $m \geq 3$.

Problem. *Is there an algebraic interpretation of* $\mathrm{mult}(\sigma)$*, as a multiplicity-type invariant associated to either* $k[\sigma \cap N]$ *or* $k[\sigma^\vee \cap M]$ *? We will see in the next chapter that* $\mathrm{mult}(\sigma)$ *is the order of the class group of* U_σ*.*

Exercise 14. Let σ be a simplicial cone with primitive generators of the rays $v_1, \ldots, v_r$. If τ is a face of σ, then
$$\mathrm{mult}(\sigma) = \mathrm{mult}(\tau) \cdot |N_\sigma/N_\tau + \sum_{i=1}^r \mathbb{Z}v_i|.$$
In particular, we have $\mathrm{mult}(\tau) \leq \mathrm{mult}(\sigma)$.

Both steps in the construction of a resolution use the following construction of fan refinements. Let Σ be a fan in N and v a nonzero point in $|\Sigma|$. Define the fan $\Sigma(v)$ to consists of

(1) Cones σ in Σ which do not contain v.
(2) The cone $\sigma(v)$ generated by σ and v, where σ is a cone in Σ which does not contain v, but such that there is a cone γ in Σ containing both σ and v.

Note that if $\sigma(v)$ is as in (2), then $\dim(\sigma(v)) = \dim(\sigma) + 1$. Indeed, if we take γ in Σ containing both v and σ, then σ is equal to the intersection of γ with the linear span of σ. Therefore v is not in the linear span of σ.

The next lemma shows that this is a fan refinement of Δ. It is called a **star subdivision**.

Lemma 1. *For every fan Σ and every nonzero v in $|\Sigma|$, $|\Sigma(v)|$ is a fan refining $|\Sigma|$.*

Proof. It is clear that a face of a cone in (1) is again a cone in (1). If τ is a face of a cone $\sigma(v)$ as in (2), then τ is generated by $\tau \cap \sigma$ and by $\tau \cap \{v\}$. Therefore τ is either of face of σ (if v is not in τ) or it is a cone as in (2). Note that the converse is true: if $\sigma(v)$ is as in (2) and τ is a face of σ, then both τ and $\tau(v)$ are faces of $\sigma(v)$. Indeed, we can write $\tau = \sigma \cap u^{\perp}$ for some u in $\sigma^{\vee}$ such that $\langle u, v\rangle = 0$. If $\tilde{u} \in \sigma^{\vee}$ is such that $\langle \tilde{u}, v\rangle > 0$, then $\tau = \sigma(v) \cap (u + \tilde{u})^{\perp}$ and $\tau(v) = \sigma(v) \cap u^{\perp}$.

The intersection of two cones in (1) is a face of both of the cones, as this is true for all cones in Σ. We show that if $\sigma_1(v)$ and $\sigma_2(v)$ are as in (2), then their intersection is equal to $(\sigma_1 \cap \sigma_2)(v)$, so it is a face of both of the cones. To see this, consider an element in the intersection $w_1 + \lambda_1 v = w_2 + \lambda_2 v$, where w_1 is in σ_1 and w_2 is in σ_2. Suppose, for example, that $\lambda_1 \geq \lambda_2$. If γ is a cone in Σ containing both σ_1 and v, then $w_2 = w_1 + (\lambda_1 - \lambda_2)v$ lies in $\gamma \cap \sigma_2$, which is a face of σ_2. It follows that if $\lambda_1 \neq \lambda_2$, then v is in σ_2, a contradiction. Therefore $\lambda_1 = \lambda_2$, so $w_1 = w_2$ lies in $\sigma_1 \cap \sigma_2$.

One can show similarly that if σ_1 is as in (1) and $\sigma_2(v)$ is as in (2), then their intersection is equal to $\sigma_1 \cap \sigma_2$, so it is a face of both of them. This shows that $\Sigma(v)$ is a fan. As it follows from the definition that every cone in $\Sigma(v)$ is contained in a cone of Σ, it is enough to show that the support of Σ is contained in the support of $\Sigma(v)$.

Let w be in $|\Sigma|$, and let $\sigma \in \Sigma$ be such that w is in the relative interior of σ. If v is not in σ, then σ is in $\Sigma(v)$ and therefore w is in $|\Sigma(v)|$.

Suppose now that v is in σ. We may assume that w does not lie on $\mathbb{R}_+ v$ (otherwise note that $\mathbb{R}_+ v = \{0\}(v)$). Consider the half-line starting from v and passing through w. After possibly replacing w by a multiple, we may assume that this half-line intersects the boundary of σ. If τ is a facet of σ which contains the intersection point, then $\tau(v)$ is a cone in $\Sigma(v)$ containing w.

Proposition 3. *Given a fan Δ, there is a sequence of successive star subdivisions not changing any simplicial cone of Δ, arriving at a simplicial fan Σ.*

Proof. We do a double induction, first by the smallest dimension k of a non-simplicial cone in Δ, and then by the number of non-simplicial cones of dimension k. By the induction hypothesis, we may assume that all cones in Δ of dimension $\leq (k-1)$ are simplicial. Let $\sigma \in \Delta$

be a non-simplicial cone of dimension k, and let v be a point in the relative interior of σ. We consider the star subdivision $\Delta(v)$.

Note that if τ is a simplicial cone in Δ, then it does not contain σ, so it is in $\Delta(v)$. Therefore this star-division preserves the simplicial cones, as required by the proposition. If a cone in $\Delta(v)$ of dimension $\leq k$ is of the form $\tau(v)$, then $\dim(\tau) \leq k-1$ so τ is simplicial. Since v does not lie in the linear span of τ, we deduce that $\tau(v)$ is simplicial. Therefore the only non-simplicial cones in $\Delta(v)$ of dimension $\leq k$ are cones in Δ. In particular, all cones in $\Delta(v)$ of dimension $\leq (k-1)$ are simplicial. Moreover, as σ is not in $\Delta(v)$, we see that the number of non-simplicial cones of dimension k in $\Delta(v)$ is smaller than then the corresponding number for Δ. This completes the proof by induction.

This proposition reduces the construction of a desingularization to the case of a simplicial fan.

Lemma 2. *Let Δ be a simplicial fan, and let $v_1, \ldots, v_r$ be primitive generators of a cone in Δ. Suppose that $v = \sum_{i=1}^{r} t_i v_i$ is in N, with $0 < t_i < 1$ for all i. If m is the maximal multiplicity of a cone in Δ containing v, then every cone in $\Delta(v)$ and not in Δ has multiplicity strictly less than m.*

Proof. Any such cone is of the form $\sigma(v)$ for some cone σ in Δ that does not contain v, and such that there is γ in Δ containing both v and σ. Suppose that the generators of γ are $v_1, \ldots, v_r, w_1, \ldots, w_s$. There is $i \leq r$ such that σ is contained in the cone τ generated by $v_1, \ldots, \widehat{v_i}, \ldots, v_r, w_1, \ldots, w_s$. In this case $\sigma(v) \subseteq \tau(v)$, so $\mathrm{mult}(\sigma(v)) \leq \mathrm{mult}(\tau(v))$ by Exercise 14. On the other hand, it follows from the determinantal formula in Exercise 10 that $\mathrm{mult}(\tau(v)) = t_i \cdot \mathrm{mult}(\gamma) \leq m t_i < m$, which completes the proof.

Theorem 1. *If Δ is a fan in N, then there is a sequence of star subdivisions arriving to a nonsingular fan Σ, such that every nonsingular cone in Δ is in Σ.*

Proof. We use Proposition 3 to reduce to the case when Δ is a simplicial fan. Now we do first induction on the largest multiplicity m of a cone in σ, and then on the number of cones in Δ of multiplicity m.

If $m = 1$, then the fan is nonsingular and there is nothing to prove. Suppose that $m > 1$, and choose a cone σ in Δ of multiplicity m. If $v_1, \ldots, v_r$ are the generators of σ, then there is a nonzero $v = \sum_{i=1}^{r} t_i v_i$ in N such that $0 \leq t_i < 1$ for all i. This follows from Exercise 12, but it can be seen also directly; if there is no such point, then $\sigma \cap N$ is generated by the v_i, so it is nonsingular, a contradiction. Applying

Lemma 2 to the cone generated by those v_i such that $t_i > 0$, we see that every cone in $\Delta(v)$ which is not in Δ has multiplicity strictly less than m. In particular, the multiplicity of every cone in $\Delta(v)$ is no larger than m, and the number of cones in $\Delta(v)$ of multiplicity m is strictly smaller than the one for Δ (as σ is not in $\Delta(v)$). Note also that if τ is a nonsingular cone in Δ, then it does not contain v, so it is in $\Delta(v)$. This completes the proof by induction.

The following is the geometric counterpart.

Corollary 1. *If $X(\Delta)$ is a toric variety, then there is a proper, birational morphism of toric varieties $f\colon X(\Sigma) \to X(\Delta)$, such that $X(\Sigma)$ is smooth and f is an isomorphism over the smooth locus of $X(\Delta)$.*

Remark 2. Bierstone and Milman [BM] have made explicit a different approach to toric resolutions, following the method of Hironaka. This is achieved by successively blowing-up certain smooth subvarieties. The method works also for non necessarily normal toric varieties, and it gives a stronger version of resolution, the so-called embedded resolution of singularities.

Solutions and hints for selected exercises

Exercise 6. One shows first that for every nonzero $(\lambda, \mu) \in \sigma \cap N$, we have $\lambda + \mu \geq 1$. In particular, the line segment $\overline{v_0 v_1}$ is contained in the boundary of P_σ. The next step is to show that if σ_1 is the cone generated by v_1 and v_r, then

$$P_\sigma = P_{\sigma_1} + \{(\lambda, \mu) \mid \lambda + \mu \geq 1\}.$$

These two facts allow the proof by induction on r.

Exercise 7. The cone τ_i generated by v_i and v_{i+1} is nonsingular, so $\tau_i \cap N$ is generated by v_i and v_{i+1}. Hence $v_0, \ldots, v_{r+1}$ generate $\sigma \cap N$. If this system of generators is not minimal, then there is a relation

$$v_i = \sum_{j \neq i} a_j v_j,$$

for some nonnegative integers a_j. We have $a := \sum_{j \neq i} a_j \geq 2$ and $(1/a)v_i$ lies in P_σ. This contradicts the fact that v_i lies on the boundary of P_σ.

Exercise 8. The primitive generators of the rays of Σ' generate $\sigma \cap N$, and therefore the v_i, with $0 \leq i \leq r+1$ are among these vectors by the previous exercise.

Exercise 9. Prove (1) by induction on r; note that the first face $\overline{v_0 v_1}$ is equal to $P_\sigma \cap \{v \mid \langle u, v \rangle = 1\}$, where $u = (1, 1)$. In order to show (2), consider a facet Q of $P_{\sigma^\vee}$ that is not contained in the boundary of $\sigma^\vee$. Applying (1) to $\sigma^\vee$, we can find v in N such that $Q = \{u \mid \langle u, v\rangle = 1\}$ and $\langle u, v\rangle \geq 1$ for every u in P. This v generates the normal to Q. We see that v is irreducible: since σ is singular, one of the end points u_0 of Q lies in the interior of σ, and we have $\langle u_0, v\rangle = 1$. Therefore the ray generated by v is in Δ by Exercise 7.

Exercise 10. Since the matrix giving the change of basis in N has determinant ± 1, it follows that the ideal of $\mathbb{Z}$ generated by the $r \times r$ minors of the matrix $(b_{i,j})$ does not depend on the given basis for N. If we take a basis $e_1, \ldots, e_n$ such that $m_1e_1, \ldots, m_ke_k$ is a basis of N_σ for positive integers m_i, then the assertion follows.

Exercise 12. If $e_1, \ldots, e_r$ is a basis of N_σ, then the map $e_i \to v_i$ induces a diffeomorphism of $(N_\sigma)_\mathbb{R}$ with itself that takes the unit cube to Q_σ. The formula in (1) is a consequence of the change of variable formula.

Let q be the number of elements in the set in (2). If we denote by a_m the number of points in

$$N \cap \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid 0 \leq \lambda_i < m \quad \text{for all} \quad i \right\},$$

we see that $a_m = qm^r$. On the other hand, if a'_m is the number of points in

$$N \cap \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid 0 \leq \lambda_i \leq m \quad \text{for all} \quad i \right\},$$

then $a'_m = |N \cap mQ_\sigma|$, so $\lim_{m\to\infty} \frac{a'_m}{m^r} = \mathrm{vol}_k(Q_\sigma)$. We have $a_m \leq a'_m \leq a_m + \sum_{i=1}^{r} b^i_m$, where b^i_m is the number of elements in

$$N \cap \left\{ \sum_{j\neq i} \lambda_j v_j \mid 0 \leq \lambda_j \leq m \quad \text{for all} \quad j \right\}.$$

Since $\lim_{m\to\infty} b^i_m/m^r = 0$ for all i, we deduce $q = \mathrm{vol}_k(Q_\sigma)$.

Exercise 13. After replacing U_σ by U_τ we may assume that $\sigma = \tau$. Since taking the product with a smooth variety does not change the Samuel multiplicity, it is enough to consider the case when $\dim(\sigma) = \mathrm{rank}(N) = n$. The point x_σ is defined by the ideal I that has a k-basis χ^u for $u \in \sigma^\vee \cap M$, $u \neq 0$. It follows that $\dim_k(k[\sigma^\vee \cap M]/I^m)$ is the number b_m of elements v in $\sigma^\vee \cap M$ which can not be written as a sum on m nonzero elements in $\sigma^\vee \cap M$. If $v_1, \ldots, v_n$ are the primitive generators of the rays of $\sigma^\vee$, then any v as above can be written as $v = \sum_{i=1}^{n} \lambda_i v_i$, where the sum of the integral parts of the λ_i

is less than m. Using the description of $\mathrm{mult}(\sigma^\vee)$ in Exercise 12(2), we get $b_m \le \mathrm{mult}(\sigma^\vee) \cdot \binom{m+n-1}{n}$. As the Samuel multiplicity is equal to $\lim_{m\to\infty} \frac{n!\cdot b_m}{m^n}$, we deduce the required inequality.

Exercise 14. If $v_1, \ldots, v_s$ are the generators of σ which lie in τ, then we have an exact sequence

$$0 \to N_\tau / \sum_{i=1}^{s} \mathbb{Z}v_i \to N_\sigma / \sum_{i=1}^{r} \mathbb{Z}v_i \to N_\sigma / N_\tau + \sum_{i=1}^{r} \mathbb{Z}v_i \to 0.$$

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