LECTURE 4: UNIRATIONALITY OF GENERAL HYPERSURFACES OF SMALL DEGREE

Our main goal in this lecture is to show that for every fixed d, a general hypersurface of degree d in $\mathbf{P}^n$ is unirational, as long as n is large enough. This will be done in §4. We begin with a general discussion of incidence correspondences for linear subspaces contained in hypersurfaces. In §2, motivated by the rationality criterion for even-dimensional cubics in Lecture 2, we compute the dimension of the space of smooth cubic hypersurfaces of dimension $2r$ that contain 2 disjoint r-dimensional linear subspaces. The third section is devoted to a necessary and sufficient condition for all hypersurfaces of degree d in $\mathbf{P}^n$ to contain an r-dimensional linear subspace.

1. Fano varieties of r-planes and the incidence correspondence

For simplicity, in this section we work over an algebraically closed field k. Given a projective space $\mathbf{P}^n$ and $r < n$, let G be the Grassmann variety parametrizing the r-dimensional linear subspaces in $\mathbf{P}^n$ (hence G is a smooth, projective variety of dimension $(r+1)(n-r)$). For $d \geq 2$, let $\mathbf{P}$ be the projective space parametrizing the degree d hypersurfaces in $\mathbf{P}^n$. Note that if $V = H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big)$, so that $\mathbf{P}^n = \mathbf{P}(V)$, then $\mathbf{P} = \mathbf{P}\big((S^dV)^*\big)$, hence $\dim(\mathbf{P}) = \binom{n+d}{d} - 1$.

Consider the incidence correspondence

$$I = \{(X, \Lambda) \in \mathbf{P} \times G \mid \Lambda \subseteq X\}.$$

We begin be describing equations for I in $\mathbf{P} \times G$, extending what we did in Lecture 2. Suppose that we are over the open subset $V \simeq \mathbf{A}^{(r+1)(n-r)}$ of G, where a subspace Λ is described by the linear span of the rows of the matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & a_{0,r+1} & \ldots & a_{0,n} \\ 0 & 1 & \ldots & 0 & a_{1,r+1} & \ldots & a_{1,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & a_{r,r+1} & \ldots & a_{r,n} \end{pmatrix}.$$

The hypersurface corresponding to $c = (c_\alpha)$, where $\alpha = (\alpha_0, \ldots, \alpha_n)$ with $\sum_i \alpha_i = d$, is defined by

$$f_c = \sum_\alpha c_\alpha x^\alpha.$$

This contains the subspace corresponding to the above matrix if and only if

$$f_c\left(x_0, \ldots, x_r, \sum_{0 \leq i \leq r} a_{i,r+1}x_i, \ldots, \sum_{0 \leq i \leq r} a_{i,n}x_i\right) = 0 \quad \text{in} \quad k[x_0, \ldots, x_r].$$

We can write

$$f_c\left(x_0,\ldots,x_r,\sum_{0\leq i\leq r}a_{i,r+1}x_i,\ldots,\sum_{0\leq i\leq r}a_{i,n}x_i\right)=\sum_{\beta}F_{\beta}(a,c)x^{\beta}, \tag{1}$$

with the sum running over those $\beta=(\beta_0,\ldots,\beta_r)$ with $\sum_i\beta_i=d$. With this notation, $I\cap(\mathbf{P}\times V)$ is defined in $\mathbf{P}\times V$ by $(F_\beta)_\beta$. The equations over the other charts in G are similar (in fact, these charts have the same form as V, up to a reordering of the coordinates on $\mathbf{P}^n$).

Let $p\colon I\to\mathbf{P}$ and $q\colon I\to G$ be the morphisms induced by the projections onto the two factors.

Proposition 1.1. *The inclusion $I\hookrightarrow\mathbf{P}\times G$ makes I a projective subbundle over G, of codimension $\binom{r+d}{d}$. In particular, I is smooth and irreducible, with*

$$\dim(I)=(r+1)(n-r)+\binom{n+d}{d}-\binom{r+d}{d}-1.$$

Proof. It is clear that it is enough to prove the first assertion. This can be checked over suitable affine charts of G, hence it is enough to show it over the subset V considered above. We have seen that $I\cap(\mathbf{P}\times V)$ is defined in $\mathbf{P}\times V$ by $\binom{r+d}{d}$ equations that are linear in the coordinates on $\mathbf{P}$. Moreover, for every $\Lambda\in V$, the subset cut out by these equations in $\mathbf{P}\times\{\Lambda\}$ is a linear subspace of codimension $\binom{r+d}{d}$. In order to see this, we may assume that Λ is defined by $x_{r+1}=\ldots=x_n=0$. It is clear that if f defines a hypersurface X, then X contains Λ if and only if all coefficients of the monomials in $x_0,\ldots,x_r$ in f vanish; this gives a linear subspace of codimension $\binom{r+d}{d}$. We thus conclude that $I\cap(\mathbf{P}\times V)$ is a projective subbundle of $\mathbf{P}\times V$ of the claimed codimension. □

For every hypersurface X of degree d in $\mathbf{P}^n$, the *Fano scheme* of r-planes in X, denoted $F_r(X)$, is the scheme-theoretic fiber $p^{-1}(X)$ of p over the point corresponding to X. Note that in general this is a non-reduced scheme. By definition, the k-points of $F_r(X)$ parametrize the r-dimensional linear subspaces of X, and a similar assertion holds for a field extension K of k if we replace X by X_K and the linear subspaces are considered over K, too.

Our next goal is to describe the tangent spaces to $F_r(X)$; we also want to give a condition for the smoothness of the map $p\colon I\to\mathbf{P}$ at a point (X,Λ). In order to do this, we first introduce some notation. Let $(X,\Lambda)\in I$ be fixed. Let $\ell_1,\ldots,\ell_{n-r}\in V=H^0\big(\mathbf{P}^n,\mathcal{O}_{\mathbf{P}^n}(1)\big)$ be generators for the ideal of Λ. In this case, if $f\in S^d(V)$ is an equation for X, then we can write $f=\sum_{j=1}^{n-r}\ell_jf_j$, with $f_j\in S^{d-1}(V)=H^0\big(\mathbf{P}^n,\mathcal{O}_{\mathbf{P}^n}(d-1)\big)$. Let $g_j=f_j|_\Lambda\in H^0\big(\Lambda,\mathcal{O}_\Lambda(d-1)\big)$ for $1\leq j\leq n-r$. Note that the g_j are independent of the choice of the f_j: if $f=\sum_{j=1}^{n-r}\ell_jf'_j$, since $\ell_1,\ldots,\ell_{n-r}$ form a regular sequence, we have $f_j-f'_j\in(\ell_1,\ldots,\ell_{n-r})$, hence $f_j|_\Lambda=f'_j|_\Lambda$ for all j. If $\mathcal{I}_\Lambda$ is the ideal defining Λ in $\mathbf{P}^n$, then $H^0(\mathbf{P}^n,\mathcal{I}_\Lambda(1))$ is a vector space with basis $\ell_1,\ldots,\ell_{n-r}$ and we consider on the dual vector space the dual basis $\ell_1^*,\ldots,\ell_{n-r}^*$. Consider now the multiplication map

$$\Phi=\Phi_{X,\Lambda}\colon H^0\big(\Lambda,\mathcal{O}_\Lambda(1)\big)\otimes_k H^0\big(\mathbf{P}^n,\mathcal{I}_\Lambda(1)\big)^*\longrightarrow H^0\big(\Lambda,\mathcal{O}_\Lambda(d)\big)$$

mapping $u \otimes \ell_j^*$ to ug_j. We have seen that this is independent of the choice of the f_j, and it is easy to see now that it is also independent of the choice of linear forms $\ell_1, \ldots, \ell_{n-r}$, hence it only depends on Λ and X (actually, on an equation defining X; if we choose a different equation, then the map Φ gets rescaled by a nonzero element of the ground field).

Theorem 1.2. *With the above notation, the following hold:*

i) *The tangent space $T_\Lambda F_r(X)$ to $F_r(X)$ at the point Λ is isomorphic to* $\mathrm{Ker}(\Phi_{X,\Lambda})$.
ii) *If the map $\Phi_{X,\Lambda}$ is surjective, then the morphism $p\colon I \to \mathbf{P}$ is smooth at (X,Λ); in particular, the Fano scheme $F_r(X)$ is smooth at the point Λ.*

Proof. We have to extend to our general setup the computation in the proof of Proposition 1.7 in Lecture 2. We may assume that Λ lies in the affine open subset V of G that we have used before, and in fact, after choosing appropriate coordinates on $\mathbf{P}^n$, we may assume that Λ corresponds to the point with $a_{i,j} = 0$ for $0 \leq i \leq r$ and $r+1 \leq j \leq n$. Let c be such that $f_c = f$ is the equation defining X. We may choose the equations $\ell_1, \ldots, \ell_{n-r}$ for the ideal $\mathcal{I}_\Lambda$ in $\mathbf{P}^n$ to be given by $\ell_i = x_{r+i}$. Note that in this case, if we write $f = \sum_{i=1}^{n-r} \ell_i f_i$, then

(2)
$$g_i(x_0, \ldots, x_r) := f_i(x_0, \ldots, x_r, 0, \ldots, 0) = \frac{\partial f}{\partial x_{r+i}}(x_0, \ldots, x_r, 0, \ldots, 0) \text{ for } 1 \leq i \leq r-n.$$

We use the equations $(F_\beta)_\beta$ that define $I \cap (\mathbf{P} \times V)$ in $\mathbf{P} \times V$. Note that $F_r(X)$ is thus defined by the equations $F_\beta(c,a)$ in the $a_{i,j}$, hence

$$T_\Lambda F_r(X) = \left\{ u = (u_{i,j}) \mid \sum_{i=0}^{r} \sum_{j=r+1}^{n} \frac{\partial F_\beta}{\partial a_{i,j}}(c,0) u_{i,j} = 0 \text{ for all } \beta \right\}.$$

On the other hand, by differentiating the formula (1) with respect to $a_{i,j}$, we get

$$\sum_\beta \frac{\partial F_\beta}{\partial a_{i,j}}(c,0) x^\beta = x_i \frac{\partial f}{\partial x_j}(x_0, \ldots, x_r, 0, \ldots, 0). \tag{3}$$

We thus see that $u = (u_{i,j})$ lies in $T_\Lambda F_r(X)$ if and only if

$$\sum_{i,j} u_{i,j} x_i \frac{\partial f}{\partial x_j} = 0 \quad \text{in} \quad k[x_0, \ldots, x_r].$$

This says that if we put $\alpha_j = \sum_{i=0}^r u_{i,j} x_i$ for $r+1 \leq j \leq n$, then $u = (u_{i,j})$ lies in $T_\Lambda F_r(X)$ if and only if

$$(\alpha_{r+1}, \ldots, \alpha_n) \in H^0\big(\Lambda, \mathcal{O}_\Lambda(1)\big)^{\oplus(n-r)} \simeq H^0\big(\Lambda, \mathcal{O}_\Lambda(1)\big) \otimes_k H^0\big(\mathbf{P}^n, \mathcal{I}_\Lambda(1)\big)^*$$

lies in the kernel of $\Phi_{X,\Lambda}$. This completes the proof of i).

For ii), note that p is smooth at (X,Λ) if and only if the tangent map

$$dp_{X,\Lambda}\colon T_{(X,\Lambda)} I \to T_X \mathbf{P}$$

is surjective. In other words, we need that the projection onto the first component induces a surjective map between the subspace

$$T_{(X,\Lambda)}I \hookrightarrow T_X\mathbf{P} \oplus T_\Lambda G$$

and $T_X\mathbf{P}$. It follows from Lemma 1.3 below and the description in (3) for the last columns in the Jacobian matrix of p at (X,Λ) that $dp_{X,\Lambda}$ is surjective if the polynomials $x_i g_j(x_0,\ldots,x_r)$, for $0 \le i \le r$ and $1 \le j \le n-r$ span the vector space of degree d polynomials in $k[x_0,\ldots,x_r]$. This is precisely the condition that the map $\Phi_{X,\Lambda}$ is surjective. □

Lemma 1.3. *If $A \in M_{m,m+n}$ is a matrix defining a linear map $k^{\oplus m} \oplus k^{\oplus n} \to k^{\oplus m}$, then the projection onto the first component $p\colon k^{\oplus m} \oplus k^{\oplus n} \to k^{\oplus m}$ induces a surjective map $\mathrm{Ker}(A) \to k^{\oplus m}$ if and only if the first m columns of A lie in the linear span of the other columns. In particular, this holds if the last n columns of A span $k^{\oplus m}$.*

Proof. The proof is straightforward and we omit it. □

Remark 1.4. When Λ is contained in the smooth locus X_{sm} of X, then the two assertions in Theorem 1.2 can be reformulated in terms of the normal bundle $N_{\Lambda/X}$ of Λ in X, as follows. Since Λ, X_{sm}, and $\mathbf{P}^n$ are smooth, we have a short exact sequence of normal bundles

$$0 \longrightarrow N_{\Lambda/X} \longrightarrow N_{\Lambda/\mathbf{P}^n} \xrightarrow{\rho} N_{X/\mathbf{P}^n}|_\Lambda \to 0.$$

With the notation in Theorem 1.2, we have

$$N_{\Lambda/\mathbf{P}^n} = \bigoplus_{j=1}^{n-r} \mathcal{O}_\Lambda(1) \quad \text{and} \quad N_{X/\mathbf{P}^n}|_\Lambda \simeq \mathcal{O}_\Lambda(d),$$

and it is easy to see that the map ρ is defined by $(g_1,\ldots,g_{n-r})$. Therefore the map Φ can be identified with the map induced from ρ by taking global sections. First, assertion i) in the theorem implies that

$$T_\Lambda F_r(X) \simeq H^0(\Lambda, N_{\Lambda/X}).$$

Second, since $H^1\big(\Gamma, \mathcal{O}_\Lambda(1)\big) = 0$, the assertion in ii) is equivalent to saying that if $H^1(\Lambda, N_{\Lambda/X}) = 0$, then p is smooth at (X,Λ).

Remark 1.5. The results in Theorem 1.2 can also be obtained using the theory of Hilbert schemes. If X is a hypersurface in $\mathbf{P}^n$, then the Hilbert scheme parametrizing r-dimensional linear subspaces contained in X is precisely $F_r(X)$. If Λ is contained in the smooth locus of X, then it is a general fact that the tangent bundle to the Hilbert scheme at the point corresponding to Λ is isomorphic to $H^0(\Lambda, N_{\Lambda/X})$. Moreover, if $H^1(\Lambda, N_{\Lambda/X}) = 0$, then the Hilbert scheme is smooth at the point corresponding to Λ. We refer to [Ser06] for the basics about Hilbert schemes, for the description of $F_r(X)$ as a Hilbert scheme, and for the results that we mentioned.

Remark 1.6. For simplicity, we worked over an algebraically closed field, but the results in this section extend without any effort to arbitrary fields. It is clear that for every ground field k, the incidence subscheme $I \hookrightarrow \mathbf{P} \times G$ can be defined over k and it is compatible with field extensions (this follows, for example, from the explicit equations that we gave). In particular, given a field extension K/k, the K-valued points of I are in bijection with pairs (X_K, Λ_K), where X_K is a hypersurface of degree d in $\mathbf{P}^n_K$, and Λ_K is an r-dimensional linear subspace in $\mathbf{P}^n_K$ that is contained in X_K. It is clear that Proposition 1.1 holds in this setting, since the assertions can be checked after passing to the algebraic closure of the ground field. The description of the tangent space, as well as the smoothness criterion in Theorem 1.2 also hold at k-valued points of I.

2. m-DIMENSIONAL PLANES ON $2m$-DIMENSIONAL SMOOTH CUBIC HYPERSURFACES

We keep working over an algebraically closed field k. Motivated by Theorem 2.1 in Lecture 2, we now consider the $2m$-dimensional cubic hypersurfaces containing 2 disjoint linear subspaces, extending some of the results that we proved for cubic surfaces.

We begin with a general proposition, taken together with its proof from [Deb03].

Proposition 2.1. *Let $X \subseteq \mathbf{P}^n$ be a hypersurface of degree $d \geq 2$.*

i) *If $\Lambda \subseteq \mathbf{P}^n$ is a linear subspace of dimension r contained in the smooth locus $X_{\rm sm}$ of X, then $r \leq \frac{n-1}{2}$.*
ii) *If $d \geq 3$ and $r = \frac{n-1}{2}$, then X contains in its smooth locus only finitely many r-dimensional linear subspaces.*

Proof. Let $\ell_1, \ldots, \ell_{n-r}$ be linear forms that generate the ideal of Λ and let S be the homogeneous coordinate ring of Λ. If f is an equation defining X, then the condition that $\Lambda \subseteq X$ implies $f \in (\ell_1, \ldots, \ell_{n-r})$, hence we can write

$$f = \sum_{i=1}^{n-r} \ell_i f_i$$

for some homogeneous polynomials $f_1, \ldots, f_{n-r}$, of degree $d-1 \geq 1$. If $P \in \Lambda$ is a common zero of $f_1, \ldots, f_{n-r}$, then $\mathrm{mult}_P(X) \geq 2$, contradicting the fact that $\Lambda \subseteq X_{\rm sm}$. Since any r homogeneous polynomials of positive degree in S have a common root in $\mathbf{P}^r$, it follows that $n - r \geq r + 1$, giving the assertion in i).

Suppose now that $n = 2r + 1$. Let $g_i \in S$ be the restriction of f_i to Λ, for $1 \leq i \leq n - r = r + 1$. Since $g_1, \ldots, g_{r+1}$ have no common zero in Λ, it follows that they form a regular sequence in S.

We need to show that every point $\Lambda \in F_r(X)$ corresponding to a subspace $\Lambda \subseteq X_{\rm sm}$ is isolated. For this, it is enough to show that $T_\Lambda F_r(X) = \{0\}$. Recall that by Theorem 1.2, if Λ, f, and $g_1, \ldots, g_{r+1}$ are as above, then

$$T_\Lambda F_r(X) \simeq \left\{ (u_1, \ldots, u_{r+1}) \in S_1^{r+1} \mid \sum_{i=1}^{r+1} u_i g_i = 0 \right\}.$$

However, since $g_1, \ldots, g_{r+1}$ form a regular sequence, the Koszul complex constructed on them is exact, hence the module of relations between these polynomials is generated in degree $d-1 \geq 2$. This implies that $T_\Lambda F_r(X) = 0$, completing the proof of the proposition. □

Let $\mathbf{P}$ denote the projective space parametrizing cubic hypersurfaces in $\mathbf{P}^{2m+1}$, hence $\dim(\mathbf{P}) = \binom{2m+4}{3} - 1 = \frac{2}{3}(m+1)(m+2)(2m+3) - 1$. We assume that the ground field is algebraically closed.

Proposition 2.2. *The subset of* $\mathbf{P}$ *consisting of cubic hypersurfaces that contain two disjoint* m*-dimensional linear subspaces is an irreducible, constructible subset of* $\mathbf{P}$*, whose closure has dimension* $(m+1)^2(m+4)-1$*. Moreover, it contains a constructible, dense subset corresponding to smooth cubic hypersurfaces.*

Proof. Let G be the Grassmann variety parametrizing m-dimensional subspaces in $\mathbf{P}^{2m+1}$, hence G is a smooth, projective variety, of dimension $(m+1)^2$. Consider the open subset U of $G \times G$ consisting of disjoint pairs of subspaces. We also denote by W the open subset of $\mathbf{P}$ parametrizing smooth hypersurfaces. We define the incidence correspondence

$$Z = \{(X, \Lambda_1, \Lambda_2) \in \mathbf{P} \times U \mid \Lambda_1 \subseteq X, \Lambda_2 \subseteq X\}$$

and consider the two maps induced by projections $p\colon Z \to \mathbf{P}$ and $q\colon Z \to U$. We also consider the open subset Z_W of Z consisting of triples $(X, \Lambda_1, \Lambda_2)$ such that X is a smooth hypersurface.

Note that the fiber of q over a pair (Λ_1, Λ_2) is a projective space of dimension $(m+1)^2(m+2)-1$. Indeed, after a suitable change of coordinates, we may assume that

$$\Lambda_1 = (x_0 = \ldots = x_m = 0) \quad \text{and} \quad \Lambda_2 = (x_{m+1} = \ldots = x_{2m+1} = 0).$$

If f defines the cubic hypersurface X, then $\Lambda_1, \Lambda_2 \subseteq X$ if and only if

$$f \in (x_i x_j \mid 0 \leq i \leq m, m+1 \leq j \leq 2m+1)$$

and it is easy to see that the space of homogeneous polynomials of degree 3 that lie in this ideal has dimension $(m+1)^2(m+2)$.

Since q is proper and all its fibers are irreducible and of the same dimension, it follows from Proposition 1.4 in Lecture 2 that Z is irreducible and

$$\dim(Z) = \dim(U) + (m+1)^2(m+2) - 1 = (m+1)^2(m+4) - 1.$$

The set in the proposition is $q(Z)$. It is constructible by Chevalley's theorem and its closure is irreducible since Z is irreducible.

Let us show now that that the open subset Z_W of Z is nonempty. For this, we assume as usual that $\mathrm{char}(k) \neq 3$ and consider the Fermat cubic Q given by $\sum_{i=0}^{2m+1} x_i^3 = 0$. Note that this is smooth and if

$$\Gamma_\alpha = \{x \in \mathbf{P}^{2m+1} \mid x_i = \alpha x_{m+1+i} \text{ for } 0 \leq i \leq m\},$$

then by taking two different α and β with $\alpha^3 = -1 = \beta^3$, we see that Γ_α and Γ_β are disjoint m-dimensional linear subspaces of Q.

Since Z is irreducible and Z_W is open in Z and nonempty, it follows that $q(Z_W)$ is dense in $q(Z)$. Moreover, it follows from Proposition 2.1 that the fibers of q over $q(Z_W)$ are finite. We thus conclude that $\overline{q(Z)} = \overline{q(Z_W)}$ has the same dimension as Z, completing the proof of the proposition. ∎

Remark 2.3. As we have mentioned in Lecture 2, when $n = 2$, we deduce from Proposition 2.2 that in the space of smooth cubic hypersurfaces in $\mathbf{P}^5$, we have an irreducible, codimension 2 closed subset F, whose general element consists of a hypersurface that contains two disjoint 2-planes, and therefore is rational. Arguing as in the proof of Proposition 2.2, we see that the set of smooth cubic hypersurfaces in $\mathbf{P}^5$ containing *one* 2-plane is an irreducible divisor D in the space of all smooth cubics, hence F is a divisor in D.

3. r-planes on all hypersurfaces of degree d in projective space

In this section we do a small detour, applying the results from §1 to give a necessary and sufficient condition for having an r-dimensional linear subspace on every hypersurface of degree d in $\mathbf{P}^n$. In doing this, we follow closely the approach in [PS92]. Let k be an arbitrary field.

Theorem 3.1. *Given $1 \leq r < n$ and $d \geq 2$, the morphism $p\colon I \to \mathbf{P}$ is surjective*[1] *if and only if the following condition holds:*

i) *If $d \geq 3$, then $(r+1)(n-r) \geq \binom{r+d}{d}$.*
ii) *If $d = 2$, then $n \geq 2r + 1$.*

Proof. It is clear that if p is surjective, then $\dim(I) \geq \dim(\mathbf{P})$. Using the formula for $\dim(I)$ in Proposition 1.1, we obtain the inequality in i). On the other hand, Proposition 2.1 implies that if we have an r-dimensional linear subspace on a smooth hypersurface of degree $d \geq 2$ in $\mathbf{P}^n$, then $n \geq 2r + 1$.

Therefore we only need to prove that if the numerical conditions in the theorem are satisfied, then the morphism p is surjective. In fact, since p is proper, we only need to prove that it is dominant. After extending the ground field, we may assume that it is algebraically closed. It is enough to show that every smooth hypersurface X of degree d contains an r-dimensional linear subspace. This is straightforward in the case $d = 2$. Suppose, for simplicity, that $\mathrm{char}(k) \neq 2$ (we leave the characteristic 2 case as an exercise). In this case, an equation defining the smooth quadric hypersurface X can be written in suitable coordinates $x_0, \ldots, x_n$ as

$$f = \sum_{i=0}^{d} x_{2i}x_{2i+1} \quad if \quad n = 2d + 1 \text{ is odd, and}$$

$$f = x_{2d}^2 + \sum_{i=0}^{d-1} x_{2i}x_{2i+1} \quad if \quad n = 2d \text{ is even.}$$

If L is the linear subspace defined by $(x_0, x_2, \ldots, x_{2d})$, then L is contained in X and $\dim(L) = n-(d+1)$. It is then a straightforward computation to see that since $2r+1 \leq n$ and r is an integer, we have $\dim(L) \geq r$ in both cases.

[1] When the ground field is algebraically closed, then this condition says that every hypersurface of degree d in $\mathbf{P}^n$ contains an r-dimensional linear subspace

The interesting case is when $d \geq 3$. In order to show that p is dominant, it is enough to exhibit one pair $(X, L) \in I$ such that p is smooth at (X, L). Let L be the linear subspace of $\mathbf{P}^n$ defined by the ideal $(x_{r+1}, \ldots, x_n)$ and let $W = H^0\big(L, \mathcal{O}_L(1)\big)$ and $V = H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big)$. If we can find a linear subspace $U \subseteq S^{d-1}(W)$ spanned by $n - r$ elements and such that the canonical multiplication map

$$W \otimes U \to S^d(W)$$

is surjective, then we are done: if $g_{r+1}, \ldots, g_n \in S^{d-1}(V)$ lift a system of generators of U and if

$$f = \sum_{i=r+1}^{n} x_i g_i,$$

then it follows from Theorem 1.2 that p is smooth at (X, L). The existence of U is trivial if $n - r \geq \dim_k S^{d-1}(W)$; otherwise, it follows from the following lemma. ☐

Lemma 3.2. *Let W be a linear space over the algebraically closed field k, with $\dim_k W = r+1$. If $d \geq 3$ and $\ell \leq \dim_k S^{d-1}(W)$ are such that $(r+1)\ell \geq \binom{r+d}{d}$, then there is a linear subspace $U \subseteq S^{d-1}(W)$ of dimension ℓ, such that the natural multiplication map*

$$W \otimes U \to S^d(W)$$

is surjective.

Proof. For a given U, the map in the lemma fails to be surjective if and only if there is a nonzero map $\varphi\colon S^d(W) \to k$ such that φ vanishes on the image of $W \otimes U$. Note that for every nonzero map φ, by composing it with the multiplication map in the symmetric algebra, we obtain a map $W \otimes S^{d-1}(W) \to k$, and therefore a map $\alpha_\varphi\colon W \to \big(S^{d-1}(W)\big)^*$. Then φ vanishes on the image of $W \otimes U$ if and only if the composition

$$W \xrightarrow{\alpha_\varphi} \big(S^{d-1}(W)\big)^* \to U^*$$

is 0; in other words, if and only if U^* is a rank ℓ quotient of $\mathrm{Coker}(\alpha_\varphi)$. Moreover, consider $W_\varphi := \mathrm{Ker}(\alpha_\varphi)$. By definition, φ vanishes on the image of

$$W_\varphi \otimes S^{d-1}(W) \to S^d(W).$$

Therefore it induces a nonzero map $S^d(W/W_\varphi) \to k$.

We now formalize this as saying that the set of subspaces U that do not satisfy the condition in the theorem is the union of the images of certain algebraic varieties Z_m, where Z_m corresponds to those φ as above such that $\dim_k W_\varphi = m$. Let m be fixed, with $0 \leq m \leq r+1$. Let $A = A_m$ be the Grassmann variety parametrizing the m-dimensional linear subspaces of W. On A we have a short exact sequence

$$0 \to S \to W \otimes \mathcal{O}_A \to Q \to 0,$$

with S the tautological subbundle and Q the corresponding quotient. We consider

$$B = B_m := \mathbf{P}_A\big(S^d(Q)\big) \xrightarrow{q} A.$$

Note that a point in B corresponds to an m-dimensional linear subspace W' of W and to a nonzero linear map $\overline{\varphi}\colon S^d(W/W') \to k$. On B we have a morphism of locally free sheaves

$$\alpha\colon W \otimes \mathcal{O}_B \longrightarrow S^{d-1}(W)^* \otimes \mathcal{O}_B$$

that over a point corresponding to W' and $\overline{\varphi}$ is equal to α_φ, where φ is the composition of $\overline{\varphi}$ with the canonical surjection $S^d(W) \to S^d(W/W')$. Let B_0 be the open subset of B where α has rank equal to $(r+1-m)$ (in which case the kernel of α is precisely $q^*(S)$). Suppose that B_0 is nonempty. The cokernel $\mathcal{F}$ of α on B_0 is locally free of rank $\binom{r+d-1}{d-1} - (r+1-m)$ and let $\pi\colon C_m \to B_0$ be the Grassmann bundle over B_0 that parametrizes the rank ℓ quotients of $\mathcal{F}$. Finally, consider the Grassmann variety G that parametrizes the ℓ-dimensional subspaces of $S^{d-1}(W)$ (equivalently, the ℓ-dimensional quotients of $S^{d-1}(W)^*$). We have a morphism $g_m\colon C_m \to G$ that maps a point in C_m corresponding to a triple $(W', \overline{\varphi}, \rho)$, where ρ maps $\mathcal{F}_{(W',\overline{\varphi})}$ onto an ℓ-dimensional vector space R, to the surjective map

$$S^{d-1}(W)^* \longrightarrow \mathcal{F}_{(W',\overline{\varphi})} \xrightarrow{\rho} R.$$

It follows from our discussion at the beginning of the proof that the set we are interested in is the complement in G of the union of the images of the maps $C_m \to G$, over those m such that C_m is nonempty. The assertion in the theorem thus follows if we show that $\dim(C_m) < \dim(G)$ for all m with C_m nonempty. Since

$$\dim(C_m) = m(r+1-m) + \binom{r-m+d}{d} - 1 + \ell \cdot \left(\binom{r+d-1}{d-1} - (r+1-m) - \ell\right)$$

whenever C_m is nonempty and $\dim(G) = \ell \cdot \left(\binom{r+d-1}{d-1} - \ell\right)$, a straightforward computation shows that $\dim(C_m) < \dim(G)$ if and only if

$$(*)_m \qquad (\ell - m)(r+1-m) \geq \binom{r-m+d}{d}.$$

This clearly holds if $m = r+1$, hence we only need to consider m, with $0 \leq m \leq r$. Note that the assumption of the proposition says that $(*)_0$ holds. Moreover, since $d \geq 3$, an easy induction on d allows us to deduce from the hypothesis that $\ell \geq r$.

We now show that for every integers ℓ, r, and d, with $r \geq 1$ and $d \geq 3$, if $(*)_0$ holds, then $(*)_1$ holds. For this, it is enough to show that

$$\frac{1}{(r+1)}\binom{r+d}{d} \geq 1 + \frac{1}{r}\binom{r+d-1}{d}.$$

This is equivalent to

$$(r+2)\cdots(r+d) \geq d! + (r+1)\cdots(r+d-1),$$

and it is easy to check this by induction on $d \geq 3$.

Applying this in our setting with ℓ replaced by $\ell - m$ and r replaced by $r - m$, where $m \leq r-1$, we see that if $(*)_m$ holds, then $(*)_{m+1}$ holds. Since $(*)_0$ holds by assumption, we deduce that $(*)_m$ holds for all $m \leq r$, completing the proof of the lemma. □

Remark 3.3. Since the Grassmann varieties are covered by affine spaces and since any open subset in an affine space has k-rational points whenever the ground field k is infinite, we see that the assertion in Lemma 3.2 also holds if we assume that k is infinite, instead of algebraically closed.

4. Unirationality of hypersurfaces of small degree and large dimension

Our goal in this section is to show that if X is a general hypersurface of degree $d \geq 4$ in $\mathbf{P}^n$, with n large enough, then X is unirational. This is a result due to Morin [Mor42], which was was extended by Predonzan [Pre49] to complete intersections. We follow the presentation in [PS92], but for the sake of simplicity, we stick to the case of hypersurfaces.

We work over an arbitrary field k with $\mathrm{char}(k) = 0$. The idea is to proceed as in the proof of unirationality for cubic hypersurfaces. Given a smooth hypersurface X that contains a suitable linear subspace Λ, the projection with center Λ induces a rational map $X \dashrightarrow \mathbf{P}(W)$. If $\widetilde{X}$ is the blow-up of X along Λ and Y is the exceptional divisor, then the generic fiber of $\widetilde{X} \to \mathbf{P}(W)$ is a hypersurface of degree $d-1$ in a suitable projective space over $k\big(\mathbf{P}(W)\big)$. Using induction on d, we would be done if we could guarantee that this generic fiber contains a linear subspace as prescribed by induction. This is nontrivial, since $k\big(\mathbf{P}(W)\big)$ is not algebraically closed. However, we will see that the condition can be ensured (at least, after a suitable extension of $k\big(\mathbf{P}(W)\big)$) if the general fibers of the morphism $Y \to \mathbf{P}(W)$ are general hypersurfaces of degree $d-1$ in Λ. This will be guaranteed by the following strong condition on the pair (X, Λ).

Suppose that X is a hypersurface in $\mathbf{P}^n$ and Λ is an r-dimensional linear subspace contained in X. Recall from §1 that if Λ is defined by $(\ell_1, \ldots, \ell_{n-r})$ and if we write $f = \sum_{i=1}^{n-r} \ell_i f_i$, then the restrictions g_i of the f_i to $H^0\big(\Lambda, \mathcal{O}_\Lambda(d-1)\big)$ span a linear subspace that only depends on X and Λ. We denote this subspace by $U_{X,\Lambda}$. We say that the pair (X, Λ) satisfies the condition $(*)$ if $U_{X,\Lambda} = H^0\big(\Lambda, \mathcal{O}_\Lambda(d-1)\big)$. Note that by Theorem 1.2, this condition implies that the morphism p is smooth at (X, Λ), but the condition $(*)$ is much stronger than the condition in the theorem. Note that in order for condition $(*)$ to be satisfied, we need

$$n - r \geq \binom{r+d-1}{d-1}.$$

This is what motivates the following inductive definition. For every $d \geq 3$, we put $n_d = n_{d-1} + \binom{n_{d-1}+d-1}{d-1}$, starting with $n_3 = 3$. We also make the convention $n_2 = 1$.

Remark 4.1. Let G be the Grassmann variety that parametrizes r-dimensional linear subspaces in $\mathbf{P}^n$ and $\mathbf{P}$ the projective subspace parametrizing hypersurfaces of degree d in $\mathbf{P}^n$. If $I \hookrightarrow \mathbf{P} \times G$ is the corresponding incidence correspondence, it is easy to see that there is an open subset U of I such that for every field K, the K-valued points of U consist precisely of those pairs (X_K, Λ_K), with X_K a hypersurface of degree d in $\mathbf{P}^n_K$, Λ_K an r-dimensional linear subspace in $\mathbf{P}^n_K$, with $\Lambda_K \subseteq \Lambda_K)$ and such that (X_K, Λ_K) satisfy condition $(*)$. Moreover, U is nonempty if and only if $n - r \geq \binom{r+d-1}{d-1}$.

Remark 4.2. Suppose that X is a degree d hypersurface in $\mathbf{P}^n$ and Λ is an r-dimensional subspace contained in X. If $L \subseteq \mathbf{P}^n$ is another linear subspace such that $\Lambda \subseteq L$ and $X \not\subseteq L$, and if we consider the hypersurface $Y = X \cap L$ in L, then (X, Λ) satisfies $(*)$ if and only if (Y, Λ) satisfies $(*)$. In fact, it follows from the definition that in general, the two subspaces of $H^0\big(\Lambda, \mathcal{O}_\Lambda(d-1)\big)$ corresponding to X and Y, coincide.

Theorem 4.3. *Let $X \subseteq \mathbf{P}^n$ be a smooth hypersurface of degree d and $\Lambda \subseteq X$ be a linear subspace of dimension n_{d-1}. If (X, Λ) satisfies condition $(*)$ and $n \geq n_d$, then X is unirational.*

Corollary 4.4. *If k is an algebraically closed field of characteristic 0, then a general hypersurface of degree $d \geq 4$ in $\mathbf{P}^n$, with $n \geq n_d$, is unirational.*

Proof. Let U be the open subset in Remark 4.1, where we take $r = n_{d-1}$. Note that U is nonempty since $n \geq n_d$. Since the projection p is smooth at the points of U, it follows that p is dominant. Therefore $p(U)$ contains an open subset of $\mathbf{P}$, and the assertion in the corollary follows from the theorem. □

Example 4.5. Note that $n_4 = 23$, hence Corollary 4.4 implies that over an algebraically closed field of characteristic 0, a general quartic hypersurface in $\mathbf{P}^n$, with $n \geq 23$, is unirational. One of the outstanding problems regarding unirationality concerns the behavior of smooth quartic hypersurfaces in $\mathbf{P}^4$. While there are examples of unirational such hypersurfaces due to Segre [Seg60], determining whether the general ones are unirational or not is an open problem.

Remark 4.6. A more recent result, due to Harris, Mazur, and Pandharipande [HMP98], says that in fact, if n is large enough, then *every* smooth hypersurface of degree d in $\mathbf{P}^n$ is unirational.

Proof of Theorem 4.3. We argue by induction on $d \geq 3$. We have seen in Lecture 3 that a smooth hypersurface X of degree $d \geq 3$ in $\mathbf{P}^n$ is unirational, as long as $n \geq 3$ and X contains one line. Therefore we may assume that we know the assertion in the theorem for $d - 1 \geq 3$ and we will show that we obtain the corresponding one for d.

Let $\mathbf{P}(W) \subseteq \mathbf{P}^n$ be a linear subspace that does not meet Λ and such that $\dim\big(\mathbf{P}(W)\big) = n - 1 - r$, where $r = n_{d-1}$. We consider the projection map $\varphi\colon \mathbf{P}^n \smallsetminus \Lambda \to \mathbf{P}(W)$ and proceed as in the case of cubic hypersurfaces. Let $\pi\colon \widetilde{\mathbf{P}^n} \to \mathbf{P}^n$ be the blow-up along Λ, with exceptional divisor E. We know that the rational map $\varphi \circ \pi$ extends to a morphism $f\colon \widetilde{\mathbf{P}^n} \to \mathbf{P}(W)$. Let $\widetilde{X}$ be the strict transform of X and $Y = \widetilde{X} \cap E$. Note that $\widetilde{X}$ is the blow-up of X along Λ, with exceptional divisor Y, hence both X and Y are smooth and irreducible. Recall that

$$\iota = (f, \pi)\colon \widetilde{\mathbf{P}^n} \hookrightarrow \mathbf{P}(W) \times \mathbf{P}^n$$

embeds $\widetilde{\mathbf{P}^n}$ as a projective subbundle over $\mathbf{P}(W)$. If $y\colon \operatorname{Spec}(K) \to \mathbf{P}(W)$ is a K-valued point of $\mathbf{P}(W)$, for some field extension K/k, then ι induces an embedding $f^{-1}(y) \hookrightarrow \mathbf{P}^n_K$ as the linear span $\langle \Lambda_K, y\rangle$. Let $g\colon \widetilde{X} \to \mathbf{P}(W)$ and $h\colon Y \to \mathbf{P}(W)$ be the restrictions of f.

Claim. There is a nonempty open subset $\mathcal{U}$ of $\mathbf{P}(W)$ such that for all K-valued points $y\colon \operatorname{Spec}(K) \to \mathcal{U}$, where K/k is a field extension, the fibers

$$g^{-1}(y) \hookrightarrow \langle \Lambda_K, y \rangle \simeq \mathbf{P}_K^{r+1} \quad \text{and} \quad h^{-1}(y) \hookrightarrow \Lambda_K$$

are hypersurfaces of degree $d-1$, with $g^{-1}(y)$ smooth and irreducible.

It is convenient to choose coordinates on $\mathbf{P}^n$ such that

$$\Lambda = (x_{r+1} = \ldots = x_n = 0) \quad \text{and} \quad \mathbf{P}(W) = (x_0 = \ldots = x_r = 0).$$

Let $f \in k[x_0, \ldots, x_n]$ be an equation of X. Since $\Lambda \subseteq X$, we can write $f = \sum_{i=r+1}^n x_i f_i$, with the f_i homogeneous polynomials of degree $d-1$. We also put $g_i = f_i(x_0, \ldots, x_r, 0, \ldots, 0)$ for all i.

Suppose that K/k is a field extension and we have a K-valued point y of $\mathbf{P}(W)$ given by $(\lambda_{r+1}, \ldots, \lambda_n)$, with $\lambda_i \in K$ for all i. In this case we have coordinates $a_0, \ldots, a_{r+1}$ on $\langle \Lambda_K, y \rangle \simeq \mathbf{P}_K^{r+1}$ such that the subspace Λ_K is defined by (a_{r+1}) and the intersection $\langle \Lambda_K, y \rangle \cap X_K$ is defined by

$$f(a_0, \ldots, a_r, a_{r+1}\lambda_{r+1}, \ldots, a_{r+1}\lambda_n) = a_{r+1} \cdot \sum_{i=r+1}^n \lambda_i f_i(a_0, \ldots, a_r, a_{r+1}\lambda_{r+1}, \ldots, a_{r+1}\lambda_n).$$

We see that if $\mathcal{U}$ is contained in the open subset

$$\{(\lambda_{r+1}, \ldots, \lambda_n) \mid \sum_i \lambda_i g_i \neq 0\},$$

then for every K-valued point y of $\mathcal{U}$, we have

$$\langle \Lambda_K, y \rangle \cap X_K = \Lambda_K + Z,$$

where Z is a hypersurface of degree $d-1$ in $\langle \Lambda_K, y \rangle$, not containing Λ_K, such that $Z \cap \Lambda_K$ is defined by $\sum_i \lambda_i g_i$.

Recall that for every y as above, the map ι embeds the fiber $f^{-1}(y)$ in $\mathbf{P}_K^n$ as the linear span $\langle \Lambda_K, y \rangle$, inducing an isomorphism $f^{-1}(y) \cap E_K \simeq \Lambda_K$. This further induces an isomorphism

$$g^{-1}(y) \smallsetminus Y_K = (f^{-1}(y) \cap \widetilde{X}_K) \smallsetminus E_K \simeq (\langle \Lambda_K, y \rangle \cap X_K) \smallsetminus \Lambda_K.$$

In particular, we see that g is dominant. We now choose $\mathcal{U}$ as above such that, in addition, for all K-valued points y of $\mathcal{U}$, the fiber $g^{-1}(y)$ is smooth, of dimension

$$(n-1) - \dim \mathbf{P}(W) = r,$$

and not contained in Y (this is possible since $\widetilde{X}$ is smooth and we are in characteristic 0). Note that since $g^{-1}(y)$ is a smooth hypersurface in $f^{-1}(y) \simeq \mathbf{P}_K^{r+1}$, it is irreducible. Given such y, we see that

$$g^{-1}(y) = \overline{g^{-1}(y) \smallsetminus Y_K} \hookrightarrow \langle \Lambda_K, y \rangle$$

is a hypersurface of degree $d-1$. Moreover, we see that if y corresponds to $(\lambda_{r+1}, \ldots, \lambda_n)$, then $h^{-1}(y) \hookrightarrow \Lambda_K$ is the hypersurface defined by $\sum \lambda_i g_i$. This completes the proof of the claim.

Let $\mathbf{P}^s$ be the projective space parametrizing the hypersurfaces of degree $d-1$ in Λ and $\mathcal{H} \hookrightarrow \mathbf{P}^s \times \Lambda$ the universal hypersurface of degree $d-1$ in Λ. More precisely, if $u_0, \ldots, u_s$ are the degree $d-1$ monomials in $k[x_0, \ldots, x_r]$, and $y_0, \ldots, y_s$ are the coordinates on $\mathbf{P}^s$, then $\mathcal{H}$ is defined by $\sum_{i=0}^{s} y_i u_i$. It follows from the above description of the fibers of h over $\mathcal{U}$ that we have a morphism $q\colon \mathcal{U} \to \mathbf{P}^s$ induced by a linear projection such that

$$h^{-1}(\mathcal{U}) \simeq \mathcal{U} \times_{\mathbf{P}^s} \mathcal{H} \quad \text{over} \quad \mathcal{U}.$$

Recall now that by hypothesis, $g_{r+1}, \ldots, g_n$ span the space of homogeneous degree $d-1$ polynomials in $k[x_0, \ldots, x_r]$; this implies that q is dominant.

Let G be the Grassmann variety parametrizing n_{d-2}-dimensional linear subspaces of Λ and $I \hookrightarrow \Lambda \times G$ the corresponding incidence correspondence. Since I is a projective bundle over G by Proposition 1.1, we deduce that I is rational. Consider the Cartesian diagram

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & I \\ \downarrow & & \downarrow \\ \mathcal{U} & \xrightarrow{q} & \mathbf{P}^s. \end{array}$$

Using the fact that q is the restriction of a projection to an open subset, we see that $\mathcal{V}$ is reduced and irreducible, and birational to the product of I with an affine space, hence it is rational. Note also that since the projection $I \to \mathbf{P}^s$ is dominant, the map $\mathcal{V} \to \mathcal{U}$ is dominant, hence the map

$$\rho\colon g^{-1}(\mathcal{U}) \times_{\mathcal{U}} \mathcal{V} \to g^{-1}(\mathcal{U}) \subseteq X$$

has dense image. In order to show that X is unirational, it is enough to show some irreducible component of $g^{-1}(\mathcal{U}) \times_{\mathcal{U}} \mathcal{V}$ that dominates X is a unirational variety.

It follows from definition that if L/k is any field extension, then giving an L-valued point z of $\mathcal{V}$ is equivalent to giving an L-valued point y of $\mathcal{U}$ and an n_{d-2}-dimensional L-subspace of $g^{-1}(y) \subseteq \Lambda_L$. We take L to be the function field of $\mathcal{V}$ and consider the following commutative diagram, with Cartesian squares:

$$\begin{array}{ccccc} Y' & \longrightarrow & h^{-1}(\mathcal{U}) \times_{\mathcal{U}} \mathcal{V} & \longrightarrow & h^{-1}(\mathcal{U}) \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & g^{-1}(\mathcal{U}) \times_{\mathcal{U}} \mathcal{V} & \longrightarrow & g^{-1}(\mathcal{U}) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}(L) & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{U}. \end{array}$$

Note that X' is a smooth hypersurface of degree $d-1$ in $\mathbf{P}_L^{r+1}$ and Y' is a hypersurface of degree $d-1$ in $\Lambda_L \subseteq \mathbf{P}_L^{r+1}$. The canonical morphism $\operatorname{Spec}(L) \to \mathcal{V}$ gives an n_{d-2}-dimensional linear subspace Λ' of Λ_L, contained in Y'. Moreover, since the morphism

$$\operatorname{Spec}(L) \to \mathcal{V} \to I$$

is dominant, it follows that the pair (Y', Λ') satisfies $(*)$ (see Remark 4.1), and therefore also the pair (X', Λ') satisfies $(*)$ (see Remark 4.2). We may thus apply the induction hypothesis to conclude that X' is unirational over L. If $\mathcal{Z}$ is the scheme-theoretic image of $X' \to g^{-1}(\mathcal{U}) \times_{\mathcal{U}} \mathcal{V}$, then $\mathcal{Z}$ is a variety over k that dominates X and and we have a Cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathcal{V}. \end{array}$$

Since $\mathcal{V}$ is rational, it is in particular unirational, and we may apply Proposition 2.6ii) in Lecture 3 to conclude that $\mathcal{Z}$ is unirational. This implies that X is unirational, completing the proof of the induction step. □

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