Chapter 4: Fans and toric varieties

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The toric variety corresponding to a fan. Let N be a lattice. A **fan** in N is a finite collection of pointed (rational, polyhedral, convex) cones in $N_{\mathbb{R}}$ such that

(1) Each face of a cone in Δ is also a cone in Δ.
(2) The intersection of any two cones in Δ is a face of both of them.

Note that giving a fan is equivalent with giving a set of pointed cones in $N_{\mathbb{R}}$ such that the intersection of any two of the cones is a face of each of them: the corresponding fan is given by taking all the faces of the given cones. The **support** $|\Delta|$ of the fan Δ is the union of all the cones in the fan.

If Δ is a fan in N, then each cone σ gives an affine toric variety $U_\sigma = \mathrm{Spec}(k[S_\sigma])$. If τ is a face of σ (we will denote this by $\tau \leq \sigma$), then the inclusion $\sigma^\vee \subseteq \tau^\vee$ induces a morphism $U_\tau \to U_\sigma$. Moreover, Remark 5 in Chapter 3 shows that if u in $\sigma^\vee \cap M$ is such that $\tau = \sigma \cap u^\perp$, then via the above map U_τ is identified with the principal open subset of U_σ defined by χ^u.

Moreover, note that if τ_1 and τ_2 are two faces of σ, then via the above canonical identifications we have $U_{\tau_1} \cap U_{\tau_2} = U_{\tau_1 \cap \tau_2}$ in U_σ. Indeed, if $\tau_1 = \sigma \cap u_1^\perp$ and $\tau_2 = \sigma \cap u_2^\perp$ for u_1, $u_2 \in M \cap \sigma^\vee$, then $\tau_1 \cap \tau_2 = \sigma \cap (u_1 + u_2)^\perp$. On the other hand, $U_{\tau_1} \cap U_{\tau_2}$ is the principal affine open subset defined by $\chi^{u_1}\chi^{u_2}$, which gives our assertion.

Suppose now that σ_1 and σ_2 are two cones in Δ. Since $\tau = \sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2, it follows that U_τ is canonically identified with open subsets in both U_{σ_1} and U_{σ_2}. Moreover, the above discussion shows that in the case of three cones in Δ these identifications are compatible. Therefore we can define $X = X(\Delta)$ as the union of the U_σ for $\sigma \in \Delta$, where U_{σ_1} and U_{σ_2} are glued along $U_{\sigma_1 \cap \sigma_2}$. Note that by construction we have $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2}$ in X.

The torus $T_N = U_{\{0\}}$ is embedded in each U_σ. If τ is a face of σ, then the inclusion $U_\tau \subseteq U_\sigma$ is compatible with the torus action. Therefore $T = T_N$ has an induced action on X which extends the natural action on itself. The **toric variety** $X(\Delta)$ is this variety X, together with the action of T and with the embedding $T \subseteq X$.

Lemma 1. *$X(\Delta)$ is separated for every fan Δ.*

Proof. We need to prove that the diagonal morphism $f\colon X \longrightarrow X \times X$ is a closed immersion. As $X \times X$ is covered by $U_{\sigma_1} \times U_{\sigma_2}$ for σ_1 and σ_2

in Δ, it is enough to show that if $\tau = \sigma_1 \cap \sigma_2$, then

$$U_\tau \longrightarrow U_{\sigma_1} \times U_{\sigma_2}$$

is a closed immersion. This follows from Remark 6 in Chapter 3: as $S_\tau = S_{\sigma_1} + S_{\sigma_2}$, the homomorphism $k[S_{\sigma_1}] \otimes_k k[S_{\sigma_2}] \to k[S_\tau]$ is surjective, which proves our assertion.

Let $\phi\colon N' \to N$ be a homomorphism of lattices. If Δ' and Δ are fans in N' and N, respectively, we say that ϕ gives a **morphism of fans** if for every cone σ' in Δ' there is a cone σ in Δ such that $\phi(\sigma') \subseteq \sigma$.

If ϕ satisfies this condition, then for every σ' and σ as above we get a morphism of k-algebras

$$k[\sigma^\vee \cap M] \to k[(\sigma')^\vee \cap M'],\ \chi^u \to \chi^{\phi^*(u)}.$$

We get a corresponding morphism of varieties $U_{\sigma'} \to U_\sigma \subseteq X(\Delta)$. It can be easily checked that the morphism $U_{\sigma'} \to X(\Delta)$ is independent of the choice of σ, and that these morphisms can be patched to give a morphism $\phi_*\colon X(\Delta') \to X(\Delta)$. In particular, by taking σ' and σ to be the zero cones, we see that ϕ_* restricts to the morphism of algebraic groups $T_{N'} \to T_N$ induced by ϕ. Moreover, ϕ_* is compatible with the corresponding actions via this morphism (as this is true for each of the induced maps between the affine varieties). If $f\colon X(\Delta') \to X(\Delta)$ is a morphism inducing a group homomorphism between the corresponding tori such that f is compatible with the actions of these tori, then we will call f a **morphism of toric varieties**.

Example 1. Let N be the quotient of the lattice with basis $e_0, \ldots, e_n$ by the sublattice spanned by $e_0 + \ldots + e_n$. For every i with $0 \leq i \leq n$, let σ_i be the cone spanned by $e_0, \ldots, \widehat{e_i}, \ldots, e_n$. The set consisting of $\sigma_0, \ldots, \sigma_n$ defines a fan Δ.

Note that we have

$$M = \left\{ \sum_{i=0}^{n} u_i e_i^* \mid \sum_{i=0}^{n} u_i = 0 \right\}.$$

If we put $t_i = \chi^{e_i^*}$, then $k[\sigma_i^\vee \cap M] = k[t_0/t_i, \ldots, t_n/t_n]$. This shows that $X(\Delta) = \mathbb{P}^n$. Moreover, the action of the torus $T_N = (k^*)^{n+1}/k^*$ on $\mathbb{P}^n$ is induced by $(\lambda_0, \ldots, \lambda_n) \cdot (x_0 : \ldots : x_n) = (\lambda_0 x_0 : \ldots : \lambda_n x_n)$.

Example 2. Let $X(\Delta)$ be the toric variety associated to fan Δ in N. For every integer $m > 1$ the group morphism $\phi_m\colon N \to N$ given by $\phi_m(v) = mv$ induces a morphism of toric varieties $(\phi_m)_*$. If $\mathrm{char}(k) = p = m$, then $(\phi_p)_*$ is the relative Frobenius morphism. In general, these morphisms play a similar role to the Frobenius morphism in positive characteristic.

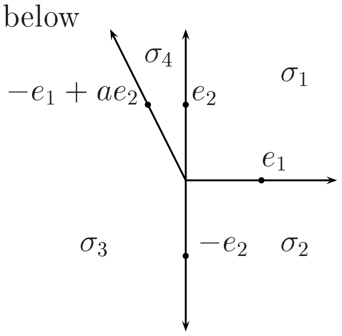
Exercise 1. Let Δ and Δ' be fans in N and N', respectively. Consider the fan in $N \times N'$

$$\Delta \times \Delta' := \{\sigma \times \sigma' \mid \sigma \in \Delta, \sigma' \in \Delta'\}.$$

Show that there is a canonical isomorphism $X(\Delta \times \Delta') = X(\Delta) \times X(\Delta')$.

Example 3. Let Δ be a fan in N such that all cones in Δ lie in a rational linear subspace W of $N_{\mathbb{R}}$ of codimension r. If N' is the lattice $N \cap W$, then we denote by Δ' the fan Δ as a fan in N'. After choosing a splitting of the inclusion $N' \hookrightarrow N$, we get an isomorphism $\Delta \simeq \Delta' \times \Delta''$, where Δ'' is the fan consisting of the zero cone in N/N'. This gives an isomorphism $X(\Delta) \simeq X(\Delta') \times (k^*)^r$.

Exercise 2. Consider the fan in $N = \mathbb{Z}^2$ with maximal cones $\sigma_1, \ldots, \sigma_4$ below



where a is a positive integer. The morphism $\phi \colon \mathbb{Z}^2 \to \mathbb{Z}$ given by $\phi(v_1, v_2) = v_1$ induces a morphism $\phi_* \colon X(\Delta) \to \mathbb{P}^1$. Show that $X(\Delta)$ is isomorphic over $\mathbb{P}^1$ with the Hirzebruch surface $\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathbb{O}_{\mathbb{P}^1}(a))$. Moreover, for every integer m we have a section of ϕ_* induced by $\psi \colon \mathbb{Z} \to \mathbb{Z}^2$, $\psi(v) = (v, mv)$.

Exercise 3. Let Δ and Δ' be fans in N' and N, respectively. If $\phi \colon N' \to N$ is a fan morphism, then the induced morphism of toric varieties ϕ_* is birational if and only if ϕ is a group isomorphism.

Exercise 4. If Δ is a fan in N, then a **subfan** of Δ is a subset of the cones in Δ which is still a fan. Show that if Δ' is a subfan of Δ, then the identity on N induces an open immersion $X(\Delta') \subseteq X(\Delta)$.

We have assumed that all the cones in our fans are pointed. If this is not the case, one can still define a toric variety as follows. Suppose that Δ is a set of cones in N such that every face of a cone in Δ is also in Δ, and such that the intersection of two cones in Δ is a face of bothe of them. If σ is a cone in Δ, let $W = \sigma \cap (-\sigma)$ be the largest linear subspace contained in σ. This is the smallest face of σ, so if τ is another cone in Δ, τ must contain W. It follows that for every τ in Δ we have $W = \tau \cap (-\tau)$.

As a matter of terminology, if W is nonzero we will call such Δ a **degenerate fan**. However, unless explicitely mentioned, we will assume that all our fans are nondegenerate.

If Δ is a possibly degenerate fan and if $p\colon N \to N' = N/N\cap W$ is the canonical projection, then the set of cones $\Delta' := \{p_{\mathbb{R}}(\sigma) | \sigma \in \Delta\}$ gives a fan in N' and we put $X(\Delta) := X(\Delta')$. Note that $X(\Delta)$ is covered by the open subsets $V_\sigma := \mathrm{Spec}(k[\sigma^\vee \cap M])$ and $V_\sigma \cap V_\tau = V_{\sigma\cap\tau}$. The torus T_N has a canonical action on $X(\Delta)$ induced via the surjection $T_N \to T_{N'} \subseteq X(\Delta')$. The dimension of $X(\Delta)$ is equal to the codimension of W.

Orbits and invariant subvarieties. We recall first some facts we have proved about orbits and invariant subvarieties of affine toric varieties. Let σ be a pointed cone in a lattice N of rank n and let U_σ be the corresponding affine toric variety. We have seen in Chapter 1 that with respect to the action of T_N on U_σ there are finitely many orbits, each of them corresponding to a face of $\sigma^\vee \cap M$.

Each such face is of the form $\sigma^\vee \cap \tau^\perp \cap M$ for some face τ of σ. Moreover, the group generated by $\sigma^\vee \cap \tau^\perp \cap M$ is $M \cap \tau^\perp$. If we denote by O_τ the orbit corresponding to the face τ of σ, then $O_\tau = \mathrm{Spec}(k[M \cap \tau^\perp])$. This is a torus embedded as a locally closed subset in U_σ as follows.

If we identify the set of closed points of U_σ with the set of semigroup morphisms $\phi\colon \sigma^\vee \cap M \to k$, then O_τ consists of those morphisms ϕ such that $\phi^{-1}(k^*) = \sigma^\vee \cap \tau^\perp \cap M$. At the level of k-algebras, the inclusion $O_\tau \subseteq U_\sigma$ is induced by the morphism

$$f\colon k[\sigma^\vee \cap M] \to k[\tau^\perp \cap M]$$

such that $f(\chi^u) = \chi^u$ if $u \in \sigma^\vee \cap \tau^\perp \cap M$ and $f(\chi^u) = 0$ otherwise.

The closure $V(\tau)$ of O_τ is $\mathrm{Spec}(k[\sigma^\vee \cap \tau^\perp \cap M])$. The inclusion $V(\tau) \hookrightarrow U_\sigma$ is induced by

$$g\colon k[\sigma^\vee \cap M] \to k[\sigma^\vee \cap \tau^\perp \cap M]$$

with $g(\chi^u) = \chi^u$ if $u \in \sigma^\vee \cap \tau^\perp \cap M$ and $g(\chi^u) = 0$ otherwise. Moreover, we have seen that every invariant subvariety of U_σ is equal to $V(\tau)$ for some face τ of σ. Note that $V(\tau)$ is again a toric variety with torus O_τ and with $\dim(V(\tau)) = n - \dim(\tau)$. The correspondence between the faces of σ and orbits (or invariant subvarieties) reverses inclusions.

We have seen that if σ' is a face of σ, then we have a canonical open immersion $U_{\sigma'} \subseteq U_\sigma$. If τ is a face of σ', then it is clear that the embeddings of O_τ in $U_{\sigma'}$ and U_σ are compatible with the inclusion of $U_{\sigma'}$ in U_σ. We deduce that

$$U_{\sigma'} = \coprod_{\tau \leq \sigma'} O_\tau. \tag{1}$$

Note also that $O_{\tau_1} \subseteq \overline{O_{\tau_2}}$ if and only if $V(\tau_1) \subseteq V(\tau_2)$ hence if and only if $\tau_2 \subseteq \tau_1$. This implies that

$$V(\tau) = \coprod_{\tau' \geq \tau} O_{\tau'}. \tag{2}$$

Example 4. Let $N = \mathbb{Z}^n$ and let σ be the cone generated by the elements of the standard basis $e_1, \ldots, e_n$. If we denote by $e_1^*, \ldots, e_n^*$ the dual basis and if we put $t_i = \chi^{e_i^*}$, then $U_\sigma = \mathrm{Spec}(k[t_1, \ldots, t_n]) = \mathbb{A}^n$.

If τ is the face of σ generated by $\{e_i \mid i \in I\}$ for some $I \subseteq \{1, \ldots, n\}$, then $V(\tau)$ is defined by the ideal $(t_i \mid i \in I)$. The orbit O_τ is

$$\{x = (x_1, \ldots, x_n) \in \mathbb{A}^n \mid x_i = 0 \,\text{for}\, i \in I \,\text{and}\, x_j \neq 0 \,\text{for}\, j \notin I\}.$$

We turn now to the global case. Suppose that Δ is a fan in N and consider the corresponding toric variety $X(\Delta)$. For each face τ of a cone σ in Δ we have a locally closed embedding $O_\tau \subseteq U_\sigma$ such that the corresponding embedding $O_\tau \subseteq X(\Delta)$ is independent of which σ we have chosen.

Note that if τ_1 and τ_2 are distinct cones in Δ, then $O_{\tau_1} \neq O_{\tau_2}$. Indeed, if $O_{\tau_1} = O_{\tau_2}$, then this orbit is contained in $U_{\tau_1} \cap U_{\tau_2} = U_{\tau_1 \cap \tau_2}$. Formula (1) applied to the face $\tau_1 \cap \tau_2$ of τ_1 implies that $\tau_1 \subseteq \tau_2$. We similarly get $\tau_2 \subseteq \tau_1$, which proves our assertion. Hence the orbits corresponding to distinct cones in the fan are distinct.

Since $X(\Delta)$ is covered by the U_σ we get the following decomposition into orbits

$$X(\Delta) = \coprod_{\tau \in \Delta} O_\tau.$$

It is clear that formula (1) remains true in this setting. Moreover, O_{τ_1} is contained in the closure of $\overline{O_{\tau_2}}$ if and only if $\tau_2 \subseteq \tau_1$. Indeed, if τ_2 is contained in τ_1, then we have seen that O_{τ_1} is contained in the closure of O_{τ_2} in U_{τ_1}. Conversely, if O_{τ_1} is contained in the closure of O_{τ_2} then $U_{\tau_1} \cap O_{\tau_2}$ is nonempty, hence τ_2 is contained in τ_1.

For every cone τ in σ, let us denote by $V(\tau)$ the closure of O_τ in $X(\Delta)$, so $\dim(V(\tau)) = n - \dim(\tau)$. Note that $V(\tau) \cap U_\sigma$ is nonempty if and only if τ is a face of σ, and in this case $V(\tau) \cap U_\sigma$ is the invariant subvariety corresponding to τ in U_σ. Using this, we deduce that (2) also remains true in this setting, by reducing the assertion to the case when we have only one ambient cone.

Moreover, every invariant irreducible subvariety of $X(\Delta)$ is equal to some $V(\tau)$: if V is such a variety, and if σ is a cone in Δ such that $V \cap U_\sigma \neq \emptyset$, it is enough to use the corresponding assertion for U_σ.

For every τ in Δ, the subvariety $V(\tau)$ of $X(\Delta)$ is a toric variety with the torus $O_\tau \subseteq V(\tau)$. Let us describe the fan giving $V(\tau)$. Consider the lattice N_τ which is the intersection of N with the linear span of τ and let $p\colon N \to N(\tau) := N/N_\tau$ be the canonical projection. We denote by $M(\tau)$ and M_τ the dual lattices of $N(\tau)$ and N_τ, respectively. Note that $M(\tau) = M \cap \tau^\perp$.

The cones $\{p_{\mathbb{R}}(\sigma) \mid \sigma \in \Delta, \tau \leq \sigma\}$ form a fan $\Delta(\tau)$ in $N(\tau)$ whose corresponding toric variety is $V(\tau)$. Indeed, it is enough to note that if σ is a cone in Δ containing τ then $U_{p_{\mathbb{R}}(\sigma)} = \mathrm{Spec}(k[M \cap \sigma^\vee \cap \tau^\perp])$ and these sets are glued in the obvious way.

Each orbit O_τ is a torus, so it contains a distinguished point, namely the unit element which we denote by x_τ. If τ is a face of σ, then x_τ lies in U_σ and it is given by the morphism of semigroups $x_\tau\colon \sigma^\vee \cap M \to k$

$$x_\tau = \begin{cases} 0 & \text{if}\, u \notin \tau^\perp, \\ 1 & \text{otherwise.} \end{cases}$$

Note that there is a unique open orbit, namely $O_{\{0\}} = T_N$ and $V(\{0\}) = X(\Delta)$. The invariant prime divisors of $X(\Delta)$ are given by the subvarieties $V(\tau)$, where τ is a ray in Δ.

An orbit O_τ is closed if and only if τ is a maximal cone in Δ. Note also that the fixed points of the torus action on $X(\Delta)$ are the points x_τ, where τ is a cone in Δ of dimension $n = \mathrm{rank}(N)$.

Exercise 5. Consider the description of $\mathbb{P}^n$ as a toric variety in Example 1. If we consider $t_0, \ldots, t_n$ as homogeneous coordinates on $\mathbb{P}^n$, show that the codimension-one invariant subvariety corresponding to the ray generated by e_i is the hyperplane defined by $t_i = 0$.

Exercise 6. Let τ_1 and τ_2 be two cones in Δ. Show that if there is a cone in Δ containing both τ_1 and τ_2 and if τ is the smallest such cone, then we have scheme-theoretically $V(\tau_1) \cap V(\tau_2) = V(\tau)$. If there is no cone in Δ containing both τ_1 and τ_2, then $V(\tau_1) \cap V(\tau_2) = \emptyset$.

Exercise 7. Let U be a nonempty open subset in $X(\Delta)$.

(1) Show that U is T_N-invariant if and only if its complement F is. This is the case if and only if each irreducible component of F is T_N-invariant.

(2) Suppose that U is T_N-invariant, and let $\tau_1, \ldots, \tau_r$ be the cones in Δ such that $V(\tau_1), \ldots, V(\tau_r)$ are the irreducible components

of F. Show that the cones $\sigma \in \Delta$ such that no τ_i is contained in σ form a subfan Δ' of Δ with $U = X(\Delta')$. Therefore every T_N-invariant open subset of a toric variety has a natural structure of toric variety.

If τ is a cone in Δ different from $\{0\}$, then the inclusion $i \colon V(\tau) \hookrightarrow X(\Delta)$ is not a toric morphism. On the other hand, we have a retract of a neighbourhood of $V(\tau)$ in $X(\Delta)$ to $V(\tau)$.

Exercise 8. Let τ be a cone in a fan Δ in N, and let Σ be the set of cones σ in Δ such that both σ and τ are contained in some cone of Δ.

(1) Show that Σ is a subfan of Δ such that $V(\tau)$ is contained in $X(\Sigma)$.
(2) The complement of $X(\Sigma)$ in $X(\Delta)$ is the union of those $V(\tau')$ such that $V(\tau) \cap V(\tau') = \emptyset$.
(3) The projection $N \to N(\tau)$ induces a morphism of fans between Σ and the fan defining $V(\tau)$. The corresponding morphism of toric varieties $r \colon X(\Sigma) \to V(\tau)$ restricts to the identity on $V(\tau)$.
(4) If Δ is the fan defining $\mathbb{P}^n$, show that the above morphism r is the projection onto a projective linear subspace.

Exercise 9. Let $X(\Delta)$ be a toric variety. Show that if Δ' is the set of nonsingular cones in Δ, then Δ' is a subfan of Δ such that $X(\Delta')$ is the smooth locus of $X(\Delta)$. Equivalently, $V(\tau)$ is contained in the singular locus of $X(\Delta)$ if and only if τ is a singular cone.

Exercise 10. If τ is a cone in the fan Δ and if $M(\tau) = M \cap \tau^\perp$, show that the dimension of the tangent space of $X(\Delta)$ at x_τ is equal to $\operatorname{codim}(\tau) + r(\tau)$, where $r(\tau)$ denotes the number of elements of the Hilbert basis of $(\tau^\vee \cap M)/M(\tau)$.

Problem. *Given an n-dimensional toric variety $X = X(\Delta)$ give a (local) description of the Jacobian ideal of X. Describe, at least, its integral closure. Recall: if Ω_X is the cotangent sheaf on X, then the Jacobian ideal is the nth Fitting ideal* $\operatorname{Fitt}^n(\Omega_X)$. *Its support is the singular locus of X.*

We consider now the behaviour of orbits and invariant subvarieties under toric morphisms. Let Δ' and Δ be fans in N' and N, respectively, and let $\phi \colon N' \to N$ inducing a morphisms of fans.

Proposition 1. *With the above notation, let $\phi_* \colon X(\Delta') \to X(\Delta)$ be the induced morphism of toric varieties. If τ' is a cone in Δ' and if τ is the smallest cone in Δ containing $\phi(\tau')$, then $\phi_*(x_{\tau'}) = x_\tau$. Therefore $\phi_*(O_{\tau'}) \subseteq O_\tau$ and $\phi_*(V(\tau')) \subseteq V(\tau)$.*

Proof. It is enough to show that $\phi_*(x_{\tau'}) = x_\tau$. In order to check this we have to show that given u in M, the image $\phi^*(u)$ of u by the dual of ϕ lies in $(\tau')^\perp$ if and only if u lies in $\tau^\perp$.

If u is in $\tau^\perp$ we get $\phi^*(u)$ in $\tau'^\perp$ since $\phi(\tau') \subseteq \tau$. Conversely, if $\phi^*(u)$ lies in $(\tau')^\perp$ then $\phi(\tau') \subseteq \tau \cap u^\perp$. By the minimality in our choice of τ, it follows that $\tau \cap u^\perp$ can't be a proper face of τ, so $u \in \tau^\perp$.

Remark 1. Let ϕ, τ' and τ be as in the above proposition. The induced group homomorphism $\psi\colon N(\tau') \to N(\tau)$ gives a morphism of fans between the fans defining $V(\tau')$ and $V(\tau)$, respectively. Indeed, if σ' is a cone in Δ' containing τ', then we can find a cone σ in Δ such that $\phi(\sigma') \subseteq \sigma$. Since τ is the minimal cone containing $\phi(\tau')$, it follows that τ is contained in σ.

This gives a morphism of toric varieties $\psi_*\colon V(\tau') \to V(\tau)$ which is the restriction of ϕ_*.

Exercise 11. Consider the fan describing the Hirzebruch surface $\mathbb{F}_a$ as a toric variety. Show that the curves C_1 and C_2 corresponding to the rays generated by $-e_1 + ae_2$ and e_1, respectively, are fibers of the canonical projection $\pi\colon \mathbb{F}_a \to \mathbb{P}^1$. If C_3 and C_4 correspond to, respectively, e_2 and $-e_2$, then π induces isomorphisms of C_3 and C_4 with $\mathbb{P}^1$. Moreover, the normal bundles of C_3 and C_4 in $\mathbb{F}_a$ are given by $N_{C_3/\mathbb{F}_a} \simeq \mathcal{O}_{\mathbb{P}^1}(-a)$ and $N_{C_4/\mathbb{F}_a} \simeq \mathcal{O}_{\mathbb{P}^1}(a)$.

Exercise 12. With the notation in Proposition 1, let σ be a cone in Δ. Show that the set of cones σ' in Δ' such that $\phi(\sigma) \subseteq \sigma'$ form a subfan Σ of Δ' such that $X(\Sigma) = \phi_*^{-1}(U_\sigma)$.

We discuss now morphisms between two tori.

Proposition 2. *Let $\phi\colon N' \to N$ be a morphisms of lattices and let $f\colon T_{N'} \to T_N$ be the corresponding morphism of tori.*

1. *If ϕ is surjective, then there is a torus $T_{N''}$ and an isomorphism $T_{N'} \simeq T_N \times T_{N''}$ such that f corresponds to the projection onto the first factor. In particular, f is surjective, smooth and has connected fibers.*
2. *If ϕ is injective and* $\mathrm{coker}(\phi)$ *is finite, then f is finite, surjective, with* $\deg(\phi)$ *equal to the order of* $\mathrm{coker}(\phi)$. *Moreover, it is étale if and only if* $\mathrm{char}(k)$ *does not divide this order.*
3. *If ϕ is injective and* $\mathrm{coker}(\phi)$ *is free, then f is a closed immersion.*
4. *Any f admits a decomposition*

$$T_{N'} \xrightarrow{f_1} T_{N_1} \xrightarrow{f_2} T_{N_2} \xrightarrow{f_3} T_N,$$

where f_1 is as in (1), *f_2 is as in* (2) *and f_3 is as in* (3).

Proof. If ϕ is surjective, then we have an isomorphism $N' \simeq N \times N''$, where $N'' = \ker(\phi)$. This induces an isomorphism $T_{N'} \simeq T_N \times T_{N''}$ such that f corresponds to the projection onto the first factor.

If ϕ is injective and $\mathrm{coker}(\phi)$ is finite, then we get an exact sequence

$$0 \to M \hookrightarrow M' \to L \to 0$$

and L is finite with $|L| = |\mathrm{coker}(\phi)|$. We can choose a basis $e_1, \ldots, e_n$ of M' such that $a_1e_1, \ldots, a_ne_n$ is a basis of M for positive integers $a_1, \ldots, a_n$. Note that $|L| = \prod_i a_i$.

This gives isomorphisms $T_{N'} \simeq (k^*)^n$ and $T_N \simeq (k^*)^n$ such that f is given by $(t_1, \ldots, t_n) \to (t_1^{a_1}, \ldots, t_n^{a_n})$. This shows that f is finite, surjective and $\deg(f) = \prod_i a_i$. Moreover, f is étale if and only if $\mathrm{char}(k)$ does not divide any of the a_i.

If ϕ is injective and $\mathrm{coker}(\phi)$ is free, then we get an epimorphism $M \to M'$ which shows that f is a closed immersion.

An arbitrary morphism ϕ can be factored as $N \to N_1 \to N_2 \to N$ where N_1 is the image of ϕ and N_2 is such that N_2/N_1 is the torsion of N/N_1. The corresponding factoring of f satisfies our requirements.

We deduce the following

Corollary 1. *Let $\phi\colon N' \to N$ be a lattice homomorphism and let $f\colon T_{N'} \to T_N$ be the corresponding morphism of tori.*

1. *f is a closed immersion if and only if ϕ is a split monomorphism.*
2. *f is surjective if and only if it is dominant. This is the case if and only if the cokernel of ϕ is finite.*

We return to morphisms of toric varieties.

Corollary 2. *With the notation in Proposition 1, suppose that* $\mathrm{coker}(\phi)$ *is finite. Then $\phi_*(O_{\tau'}) = O_\tau$, and therefore the closure of $\phi_*(V(\tau'))$ is $V(\tau)$. If we assume that in addition ϕ_* is proper, then $\phi_*(V(\tau')) = V(\tau)$.*

Proof. The morphism of tori $O_{\tau'} \to O_\tau$ is induced by the group homomorphism $N(\tau') \to N(\tau)$. This has finite cokernel, since ϕ has this property. Therefore $\phi_*(O_{\tau'}) = O_\tau$ and the other assertions are easy consequences.

Corollary 3. *With the notation in Corollary 2, let $f = \phi_*$.*

1. *If ϕ has finite cokernel and f is proper, then f is surjective.*
2. *If ϕ is surjective, then $f^{-1}(T_N) \simeq T_N \times X(\Sigma)$ for a suitable fan Σ in the kernel of ϕ; moreover, the restriction of f to $f^{-1}(T_N)$ gets identified to the projection onto the first component.*

(3) *If ϕ is surjective and if f is proper, then $f_*(\mathcal{O}_{X(\Delta')}) = \mathcal{O}_{X(\Delta)}$.*
(4) *f is finite and surjective if and only if ϕ is a monomorphism with finite cokernel, and $\Delta' = \Delta$ as fans in $N'_{\mathbb{R}} = N_{\mathbb{R}}$.*

Proof. It follows from the previous corollary that if the cokernel of ϕ is finite, then T_N is contained in the image of f. Since f is proper, it follows that f is surjective.

Suppose now that ϕ is surjective. The inverse image of T_N by f is given by $X(\Delta'')$, where Δ'' is the set of all cones in Δ which lie in the kernel of ϕ. Let us denote by Σ the fan in $\ker(\phi)$ given by the same set of cones. Since $N'/\ker(\phi) \simeq N$, we can find an isomorphism $N' \simeq N \times \ker(\phi)$ such that ϕ gets identified to the projection onto the first component. Via this isomorphism we have $f^{-1}(T_N) \simeq T_N \times X(\Sigma)$, as required.

Note that by Proposition 2 f is a separable morphism. If it is proper, by factoring it as

$$X(\Delta') \to \mathcal{S}pec(f_*(\mathcal{O}_{X(\Delta')})) \to X(\Delta)$$

we see that in order to show that $f_*(\mathcal{O}_{X(\Delta')}) = \mathcal{O}_{X(\Delta)}$ it is enough to show that the generic fiber of f is connected. Since for every x in T_N we have $f^{-1}(x) \simeq X(\Sigma)$, the fiber over x is connected, which completes the proof of (3).

Assume now that f is finite and surjective. In particular, the restriction of f to $T_{N'}$ has finite fibers, so by Proposition 2 we see that ϕ is injective. As $X(\Delta)$ and $X(\Delta')$ must have the same dimension, we see that the cokernel of ϕ is finite. Therefore we may identify $N'_{\mathbb{R}}$ with $N_{\mathbb{R}}$ so ϕ is given by inclusion.

If σ' is a cone in Δ' and if σ is the smallest cone in Δ containing σ, then Corollary 3 shows that $f(V(\sigma')) = V(\sigma)$. Since f is finite, this shows that $\dim(\sigma') = \dim(\sigma)$.

We prove now by induction on $\dim(\sigma')$ that if σ' is in Δ' then σ is in Δ. This is clear if $\sigma' = \{0\}$. Given an arbitrary σ', by taking σ as above, the induction hypothesis shows that every facet of σ' is a facet of σ, hence the boundary of σ' is contained in the boundary of σ. This implies that in fact $\sigma' = \sigma$.

If Σ is the fan in N given by all the cones in Δ', we see that f factors through $X(\Sigma) \subseteq X(\Delta)$. As f is surjective, we see that $\Sigma = \Delta$, which shows that $\Delta' = \Delta$ in $N'_{\mathbb{R}} = N_{\mathbb{R}}$.

Conversely, suppose that ϕ is a monomorphism with finite cokernel such that $\Delta' = \Delta$ in $N'_{\mathbb{R}} = N_{\mathbb{R}}$. We see that for every cone $\sigma \in \Delta$ we have

$$f^{-1}(U_\sigma) = \mathrm{Spec}(k[\sigma^\vee \cap M']) \to U_\sigma = \mathrm{Spec}(k[\sigma^\vee \cap M]).$$

For every u in $\sigma \cap M'$ we have $(\chi^u)^d \in \sigma^\vee \cap M$, where $d = |M'/M|$. Therefore f is finite and surjective, as required.

Remark 2. With the notation in (4) above, note that if $\mathrm{char}(k)$ does not divide d, then the finite group $G = \mathrm{Spec}(k[M'/M])$ has a toric action on $f^{-1}(U_\sigma)$. This is induced by the inclusion $G \subseteq T_{N'}$ corresponding to the projection $M' \to M$. It follows from Proposition 1 in Chapter 3 (and its proof) that the restriction $f^{-1}(U_\sigma) \to U_\sigma$ is the quotient map by the action of G. Moreover, these local actions globalize to an action of G on $X(\Delta')$ such that $X(\Delta)$ is the quotient via f.

Example 5. Let $d_0, \ldots, d_n$ be positive integers. Consider the fan we used in the description of the projective space as a toric variety, i.e. its cones are generated by proper subsets of $e_0, \ldots, e_n$, where n of the vectors $e_0, \ldots, e_n$ are linearly independent, and the sum of all vectors is zero. However, we take the lattice to be generated by the vectors $(1/d_i)v_i$ for $0 \leq i \leq n$. The corresponding toric variety is the **weighted projective space** $\mathbb{P}(d_0, \ldots, d_n)$.

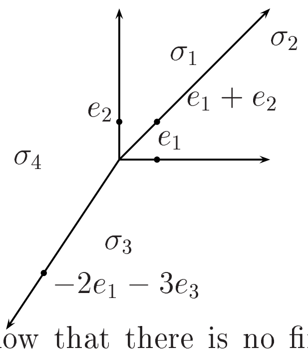
If N' is the lattice generated by $e_0, \ldots, e_n$, then we get a morphism $f \colon \mathbb{P}^n \to \mathbb{P}(d_0, \ldots, d_n)$. If $\mathrm{char}(k)$ does not divide any of the d_i, this expresses $\mathbb{P}(d_0, \ldots, d_n)$ as the quotient of $\mathbb{P}^n$ by the action of $\prod_{i=0}^n \mu_{d_i}$ given by $(\lambda_0, \ldots, \lambda_n) \cdot (x_0 : \ldots : x_n) = (\lambda_0 x_0 : \ldots : \lambda_n x_n)$. Here μ_ℓ denotes the group of ℓ^{th} roots of unity.

Exercise 13. Show that $\mathbb{P}(d_0, \ldots, d_n)$ is the (geometric) quotient of $\mathbb{A}^{n+1} \smallsetminus \{0\}$ by the action of k^* given by $\lambda \cdot (x_0, \ldots, x_n) = (\lambda^{d_0} x_0, \ldots, \lambda^{d_n} x_n)$.

A fan Δ is **simplicial** if all the cones in Δ are simplicial, i.e. they can be generated by linearly independent vectors. In this case also the corresponding toric variety is called simplicial.

We have seen that every simplicial affine toric variety can be written as the quotient of a smooth affine toric variety by the toric action of a finite group. This is not necessarily true for general simplicial toric varieties.

Exercise 14. Let X be the toric variety corresponding to the fan with maximal cones $\sigma_1, \ldots, \sigma_4$ below



Show that there is no finite surjective morphism of toric varieties $Y \to X$ with Y smooth.

One-parameter subgroups. Let $X(\Delta)$ be the toric variety assciated to the fan Δ in N. Recall that a one-parameter subgroup of the torus T_N corresponds to an element $v \in N$. If we denote by $f_v \colon k^* \to T_N$ the one-parameter subgroup corresponding to v, then $f_v = (\phi_v)_*$ where $\phi_v \colon \mathbb{Z} \to N$ is given by $\phi_v(1) = v$.

Proposition 3. *The one-parameter subgroup f_v can be extended to a morphism $\widetilde{f}_v \colon \mathbb{A}^1 \to X(\Delta)$ if and only if v lies in $|\Delta|$. Moreover, if this is the case and if the smallest cone in Δ containing v is σ, then $\widetilde{f}_v(0) = x_\sigma$.*

Proof. If $\widetilde{f}_v$ is an extension as above and if σ is such that $\widetilde{f}_v(0)$ lies in U_σ, then $\widetilde{f}_v(\mathbb{A}^1) \subseteq U_\sigma$ since $T \subseteq U_\sigma$. Therefore f_v can be extended as in the proposition if and only if there is an extension $\widetilde{f}_v \colon \mathbb{A}^1 \to U_\sigma$ for some cone σ in Δ. This is the case if and only if there is a k-algebra homomorphism $\beta \colon k[\sigma^\vee \cap M] \longrightarrow k[\mathbb{N}]$ such that the diagram

$$\begin{array}{ccc} k[\sigma^\vee \cap M] & \xrightarrow{\beta} & k[\mathbb{N}] \\ \downarrow & & \downarrow \\ k[M] & \xrightarrow{\alpha} & k[\mathbb{Z}], \end{array}$$

is commutative, where the vertical morphisms are inclusions, and where $\alpha(\chi^u) = t^{\langle u,v\rangle}$ (we put $t = \chi^1$). It is clear that there is such β if and only if $\alpha(k[\sigma^\vee \cap M]) \subseteq k[\mathbb{N}]$, i.e. if and only if $\langle u, v\rangle \geq 0$ for all $v \in \sigma \cap M$. This is equivalent with the fact that v lies in σ.

If this is the case, and if σ is the smallest cone in Δ which contains v, let I denote the ideal defining $\widetilde{f}_v(0)$. Since v lies in the relative interior of σ, we see that if $u \in \sigma^\vee \smallsetminus \sigma^\perp$, then $\langle u, v\rangle > 0$, so χ^u lies in I. Moreover, if u is in $\sigma^\perp$, then $\chi^u - 1$ lies in I. Therefore the generators of the ideal defining x_σ are contained in I, we deduce that $\widetilde{f}_v(0) = x_\sigma$.

Exercise 15. With the above notation, show that f_v can be extended to a morphism $\mathbb{P}^1 \to X(\Delta)$ if and only if both v and $-v$ lie in $|\Delta|$.

Remark 3. Proposition 3 gives a way to recover the fan Δ from the action of the torus on $X(\Delta)$.

Corollary 4. *Let $\tau \subseteq \sigma$ be two pointed cones in $N_{\mathbb{R}}$. The induced morphism of toric varieties $g\colon U_\tau \to U_\sigma$ is an open immersion if and only if τ is a face of σ.*

Proof. Suppose that g is an open immersion. By Proposition 8 in Chapter 2, in order to prove that τ is a face of σ it is enough to show that for every v_1 and v_2 in $\sigma \cap N$, such that $v_1 + v_2$ lies in τ, both v_1 and v_2 lie in σ.

Consider the one-parameter subgroups f_{v_1}, f_{v_2} and $f_{v_1+v_2}$ of T_N. We have extensions $\widetilde{f}_{v_1}, \widetilde{f}_{v_2}, \widetilde{f}_{v_1+v_2}\colon \mathbb{A}^1 \to U_\sigma$. Moreover, recall that the action of T_N on itself extens to a semigroup operation on U_σ so $\widetilde{f}_{v_1+v_2} = \widetilde{f}_{v_1} \cdot \widetilde{f}_{v_2}$. In particular, we have $\widetilde{f}_{v_1+v_2}(0) = \widetilde{f}_{v_1}(0) \cdot \widetilde{f}_{v_2}(0)$. On the other hand, as $v_1 + v_2$ lies in τ, it follows that $\widetilde{f}_{v_1+v_2}(0)$ lies in $g(U_\tau)$.

In general, if $V(\tau')$ is an invariant subvariety of U_σ, if x_1 is in $V(\tau')$ and x_2 is in U_σ, then $x_1 \cdot x_2$ is in $V(\tau')$. Indeed, this follows from the fact that $V(\tau')$ is the set of all $x\colon \sigma^\vee \cap M \to k$ such that $x(u) = 0$ if u is not in τ'.

As $g(U_\tau)$ is an open subset of U_σ stable under the torus action, we deduce that $\widetilde{f}_{v_1}(0)$ and $\widetilde{f}_{v_2}(0)$ lie in $g(U_\tau)$. Therefore f_{v_1} and f_{v_2} can be extended as morphisms $\mathbb{A}^1 \to U_\tau$ and therefore v_1 and v_2 lie in τ.

Exercise 16. If Δ is a fan in N, then any v in N determines a one-parameter subgroup $f_v\colon k^* \to T_N$, which induces an action of the torus k^* on $X(\Delta)$. Show that the set of fixed points of $X(\Delta)$ with respect to this action is the union of those subvarieties $V(\tau)$ such that v lies in N_τ.

Exercise 17. ([Boy]) If Δ is a fan in N, then the standard action of the torus T_N on itself can be extended to a morphism $X(\Delta) \times X(\Delta) \to X(\Delta)$ if and only if $X(\Delta)$ is affine.

Properness of toric morphisms. If $X(\Delta)$ is a complete variety, then by the Valuative Criterion for properness (see [Ha]) every one-parameter subgroup of the torus can be extended to a morphism $\mathbb{P}^1 \to X(\Delta)$. Proposition 3 implies that we must have $|\Delta| = N_{\mathbb{R}}$. We will show that the converse is also true, and moreover, a similar characterization holds for proper morphisms.

Let Δ' and Δ be fans in N' and N, respectively, and let $\phi\colon N' \to N$ be a morphism of fans. Note that this implies $\phi(|\Delta'|) \subseteq |\Delta|$.

Proposition 4. *The morphism ϕ_* is proper if and only if $\phi^{-1}(|\Delta|) = |\Delta'|$.*

Before proving the proposition, we give two lemmas.

Lemma 2. *If $\phi\colon N' \to N$ induces a morphism between the fans Δ' and Δ such that $\phi^{-1}(|\Delta|) = |\Delta'|$, then for every cone σ in Δ, the inverse image $\phi^{-1}(\sigma)$ is a union of cones in Δ'.*

Proof. Let v be in $\phi^{-1}(\sigma)$. We need to show that there is a cone τ' in Δ' such that v lies in τ' and $\tau' \subseteq \phi^{-1}(\sigma)$.

By hypothesis v lies in $|\Delta'|$. Let τ' be the smallest cone in Δ' which contains v. Since ϕ is a morphism of fans, we can find a cone τ in Δ such that $\phi(\tau') \subseteq \tau$. Suppose that τ is the smallest such cone. In this case $\phi(v)$ lies in the relative interior of τ, and as it also lies in σ, we have $\tau \subseteq \sigma$. Therefore $\tau' \subseteq \phi^{-1}(\sigma)$, which completes the proof.

Lemma 3. *Let $\phi\colon N' \to N$ be as in the previous lemma. If τ' is a cone in Δ', τ is the smallest cone in Δ containing $\phi(\tau')$, and $\psi\colon N'(\tau') \to N(\tau)$ is the induced morphism between the fans $\Delta'(\tau')$ and $\Delta(\tau)$ defining, respectively, $V(\tau')$ and $V(\tau)$, then $\psi^{-1}(|\Delta(\tau)|) = |\Delta'(\tau')|$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} N' & \xrightarrow{p'} & N'(\tau') \\ \Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\psi} \\ N & \xrightarrow{p} & N(\tau). \end{array}$$

We denote by $(N_\tau)_{\mathbb{R}}$ and $(N'_{\tau'})_{\mathbb{R}}$ the vector spaces corresponding to N_τ and $N'_{\tau'}$. To simplify the notation, we use the same notation for the group homomorphisms and for the corresponding linear maps.

If $w \in \psi^{-1}(|\Delta(\tau)|)$, then there is a cone σ in Δ containing τ such that $\psi(w)$ lies in $p(\sigma)$. We write $w = p'(v)$, so $\phi(v)$ lies in $\sigma + (N_\tau)_{\mathbb{R}} = (\sigma^\vee \cap \tau^\perp)^\vee$. We need to show that there is a cone σ' in Δ' containing τ', such that v is in $\sigma' + (N_{\tau'})_{\mathbb{R}}$.

If v_1 lies in the relative interior of τ', then $\phi(v_1)$ lies in the relative interior of τ, so

$$(\sigma^\vee \cap \tau^\perp)^\vee = (\sigma^\vee \cap (\phi(v_1))^\perp)^\vee = \sigma + \mathbb{R}(-\phi(v_1))$$

by Remark 7 in Chapter 2. Therefore we can find $\lambda > 0$ such that $\phi(v + \lambda v_1)$ lies in σ. Note that after replacing v by $v - \lambda v_1$ we may assume that $\lambda = 0$, and therefore that v is in $\phi^{-1}(\sigma)$.

Choose an element v_2 in τ'. It follows from the previous lemma that $\phi^{-1}(\sigma)$ is a union of cones in Δ'. Therefore we can find a cone σ' in Δ' which contains τ' and which contains also a point v_3 in the open line segment v_2). In this case v lies in $((\sigma')^\vee \cap (\tau')^\perp)^\vee = \sigma' + (N_{\tau'})_{\mathbb{R}}$, which proves the lemma.

Proof of Proposition 4. Suppose first that ϕ_* is proper. In order to show that $\phi^{-1}(|\Delta|) \subseteq |\Delta'|$, it is enough to consider v' in $\phi^{-1}(|\Delta|) \cap N'$. If $v = \phi(v')$, and if $f_{v'}$ and f_v are the one-parameter subgroups corresponding to v' and v, then by Proposition 3 we see that f_v admits an extension $\widetilde{f_v}\colon \mathbb{A}^1 \to X(\Delta)$. Consider the commutative diagram

$$\begin{array}{ccc} k^* & \xrightarrow{f_{v'}} & X(\Delta') \\ \downarrow & & \downarrow{\scriptstyle \phi_*} \\ \mathbb{A}^1 & \xrightarrow{f_v} & X(\Delta). \end{array}$$

By the Valuative Criterion for properness it follows that there is a morphism $\mathbb{A}^1 \to X(\Delta')$ such that the resulting two triangles in the above diagram are commutative. In particular, $f_{v'}$ admits an extension to $\mathbb{A}^1$, so Proposition 3 implies that v' lies in $|\Delta'|$. Therefore $\phi^{-1}(|\Delta|) = |\Delta'|$.

Conversely, let us assume that $\phi^{-1}(|\Delta|) = |\Delta'|$. Since we know that both $X(\Delta')$ and $X(\Delta)$ are separated varieties, the Valuative Criterion for properness says that in order to show that ϕ_* is proper we need to show that for every commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\alpha} & X(\Delta') \\ \downarrow & & \downarrow{\scriptstyle \phi_*} \\ \mathrm{Spec}(R) & \xrightarrow{\beta} & X(\Delta) \end{array}$$

we can find a morphism $\gamma\colon \mathrm{Spec}(R) \to X(\Delta')$ such that the resulting two tringles in the above diagram are commutative. Here R is a discrete valuation ring with field of fractions K.

Let τ' be the cone in Δ' such that $\alpha(\mathrm{Spec}(K))$ lies in $O_{\tau'}$. If τ is the smallest cone in Δ which contains in $\phi(\tau')$, then the morphism $V(\tau') \to V(\tau)$ induced by ϕ_* satisfies the same hypotheses as ϕ_* by Lemma 3. We may replace $X(\Delta')$ and $X(\Delta)$ by $V(\tau')$ and $V(\tau)$, respectively, and therefore assume that $\alpha(\mathrm{Spec}(K))$ lies in $T_{N'}$.

Suppose that β takes the closed point of R to a point in U_σ for some σ in Δ. On the other hand, the morphism $\alpha\colon \mathrm{Spec}(K) \to T_{N'}$ is given by a group homomorphism $\nu\colon M' \to K$. We have the commutative diagram

$$\begin{array}{ccc} \sigma^\vee \cap M & \longrightarrow & R \\ \Big\downarrow{\scriptstyle \phi^*} & & \Big\downarrow \\ M' & \xrightarrow{\nu} & K \end{array}$$

and we need to find a cone σ' in Δ' such that $\phi(\sigma') \subseteq \sigma$. $\nu((\sigma')^\vee \cap M') \subseteq R$. Denote by $\mathrm{ord}\colon K \to \mathbb{Z}$ the valuation on K corresponding to K. If $v = \mathrm{ord} \circ \nu \in N'$, then our assumption is that $\mathrm{ord} \circ \nu \circ \phi^*$ is non-negative on $\sigma^\vee \cap M$. Therefore $\phi(v)$ lies in σ and our hypothesis implies that there is σ' in M' such that v is in σ'. Therefore $\nu((\sigma')^\vee \cap M') \subseteq R$, which completes the proof.

Example 6. A fundamental example of a proper map is the blowing-up. We give now the interpretation in terms of fans for the blowing-up of a smooth toric variety along an invariant subvariety. Let $X(\Delta)$ be a smooth toric variety, and let $Y = V(\tau)$ for some cone τ in Δ of dimension r. As X is smooth, we may choose a basis $v_1, \ldots, v_n$ of N such that τ is spanned by $v_1, \ldots, v_r$ Let $v = v_1 + \ldots + v_r$. We replace every cone σ containing v by the by the cones $\sigma_1, \ldots, \sigma_r$, where σ_i is generated by the same rays as σ but with $\mathbb{R}_+ v_i$ replaced by $\mathbb{R}_+ v$. This gives a fan Δ' and the identity on N induces a morphism of toric varieties $f\colon X(\Delta) \to X(\Delta)$. We claim that this identifies $X(\Delta')$ with the blowing-up of $X(\Delta)$ along Y.

The assertion is local on $X(\Delta)$ so it is enough to restrict ourselves to U_σ for some σ in Δ. We may assume that σ contains τ: otherwise $Y \cap U_\sigma = \emptyset$ and f is the identity over U_σ. After replacing N with N_σ, we may also assume that σ is generated by $v_1, \ldots, v_n$.

The given basis for N identifies U_σ with $\mathbb{A}^n$ so $\mathcal{O}(U_\sigma) = k[t_1, \ldots, t_n]$. Since

$$\sigma_i^\vee = \{u = (u_1, \ldots, u_n) \mid u_j \geq 0 \,\text{for}\, j \neq i \,\text{and}\, u_1 + \ldots + u_r \geq 0\},$$

we see that $\mathcal{O}(U_{\sigma_i}) = k[t_1/t_i, \ldots, t_r/t_i, t_i, t_{r+1}, \ldots t_n]$.

On the other hand recall that the blowing-up of $\mathbb{A}^n$ along the subvariety Y defined by the ideal $(t_1, \ldots, t_r)$ is

$$\mathrm{Proj}_{\mathbb{A}^n} k[t_1, \ldots, t_n][Y_1, \ldots, Y_r]/(t_i Y_j - t_j Y_i \mid 1 \leq i < j \leq r).$$

This is covered by the opensubsets U_i defined by $Y_i \neq 0$ such that

$$\mathcal{O}(U_i) = k[Y_1/Y_i, \ldots, Y_n/Y_i, t_i, t_{r+1}, \ldots, t_n].$$

We get an isomorphism $U_{\sigma_i} \simeq U_i$ induced by $t_j/t_i \to Y_j/Y_i$ for all $j \neq i$. As these isomorphisms are mutually compatible, we see that $X(\Delta')$ gets identified with the blowing-up of $X(\Delta)$ along Y.

Remark 4. Consider a morphism of fans $\phi\colon N' \to N$ inducing a morphism of toric varieties $f\colon X(\Delta') \to X(\Delta)$. f is proper and birational if and only if ϕ is an isomorphism and every cone in Δ is a union of cones in Δ' (via the identification given by ϕ). We say that Δ' is a **refinement** of Δ.

Toric varieties from polytopes. Suppose first that P is a rational polytope in $N_{\mathbb{R}}$ such that the origin lies in P. Consider the cones over the faces of P which do not contain the origin. These are pointed cones by Lemma 10 in Chapter 2.

They form a fan: indeed, if τ_Q is the cone over the face Q of P, then there is a bijection between the faces of τ_Q and those of Q (with the convention that $\{0\}$ corresponds to the empty face of Q). Moreover, we have $\tau_{Q_1} \cap \tau_{Q_2} = \tau_{Q_1 \cap Q_2}$; indeed, v lies in τ_Q if and only if $\mathbb{R}_+ v$ intersects Q. Note also that $\dim(\tau_Q) = 1 + \dim(Q)$.

A particularly important construction of toric varieties starts with a rational polytope P in the dual space $M_{\mathbb{R}}$, with $\dim(P) = \mathrm{rk}(M)$. The corresponding fan is the **normal fan** of P. We give the definition of this fan more generally, in the case when P is an n-dimensional rational polyhedron.

If Q is a face of P, consider the cone

$$\sigma_Q := \{v \in N_{\mathbb{R}} \mid \langle u, v\rangle \le \langle u', v\rangle \text{ for all } u \in Q, u' \in P\}.$$

In other words, σ_Q is the dual to the "angle" at Q consisting of all vectors pointing from points of Q to points of P.

Proposition 5. *The set of cones* $\{\sigma_Q \mid Q \text{ is a face of } P\}$ *gives a fan* Δ_P *in* N *such that* $\dim(\sigma_Q) = \mathrm{codim}(Q)$*. The support* $|\Delta_P|$ *of* Δ_P *is equal to* σ*, where* σ *is such that we have a decomposition* $P = P' + \sigma^\vee$ *for some polytope* P'*.*

Proof. The above set of cones does not change if we replace P by a multiple λP, or if we translate P. Therefore we may assume that the origin lies in the interior of P, so the polar P° is a polytope.

We show that the above set of cones are precisely the cones over the faces of P° which do not contain the origin. If $Q = P$, then we have $\sigma_Q = \{0\}$, so from now on we assume that Q is a proper face.

Recall that we associate to Q the face of P°

$$Q^* = \{v \in P^\circ \mid \langle u, v\rangle = -1 \text{ for all } u \in Q\}$$

such that $\dim(Q^*) = n - 1 - \dim(Q)$. This induces a bijection between the proper faces of P and of P° which do not contain the origin.

We show that the cone τ_{Q^*} over Q^* is equal to σ_Q. The inclusion $Q^* \subseteq \sigma_Q$ follows from definitions. Suppose now that v is in $\sigma_Q \setminus \{0\}$. Since $\langle u, v\rangle \leq \langle u', v\rangle$ for all $u \in Q$ and $u' \in P$, we see that $\langle u, v\rangle$ is constant for u in Q. Moreover, as the origin lies in the interior of P, this constant c has to be negative. Therefore $(-1/c)v$ lies in Q^* hence v is in τ_{Q^*}. This shows that Δ_P is a fan.

Suppose now that $P = P' + \sigma^\vee$ as in the proposition. If v lies in some σ_Q, then for every u in Q, u' in P', u'' in $\sigma^\vee$ and every $t > 0$ we have

$$\langle u, v\rangle \leq \langle u', v\rangle + t\langle u'', v\rangle.$$

Therefore $\langle u'', v\rangle \geq 0$, so v lies in $(\sigma^\vee)^\vee = \sigma$.

Conversely, if v is in σ then

$$a := \min_{u\in P}\langle u, v\rangle = \min_{u\in P'}\langle u, v\rangle.$$

This shows that $Q := \{u \in P \mid \langle u, v\rangle = a\}$ is a face of P such that v lies in σ_Q.

If $P = P' + \sigma^\vee$ as in Proposition 5 and $\dim(\sigma^\vee) = \mathrm{rk}(M)$, then σ is a pointed cone and Δ_P gives a refinement of the fan consisting of σ. This induces a proper birational morphism $f\colon X(\Delta_P) \to U_\sigma$.

The most important case of the construction in Proposition 5 is when P is a polytope. In this case $|\Delta_P| = N_{\mathbb{R}}$, so the toric variety $X(\Delta_P)$ is complete. We will see in the next chapter that a toric variety has such a description precisely when it is projective.

As the cones in Δ_P correspond to faces of P, we get a description of the orbits and of the invariant subvarieties of $X(\Delta_P)$ in terms of the faces of P. More precisely, each face Q of P determines an orbit and an invariant subvariety of $X(\Delta_P)$ of dimension equal to $\dim(Q)$. In particular, the fixed points correspond to the vertices of P, and the codimension one invariant subvarieties correspond to the facets of P.

Exercise 18. Show that if P is the simplex in $\mathbb{R}^n$ with vertices at the origin and at the points $e_1^*, \ldots, e_n^*$, then Δ_P is the fan whose cones are generated by proper subsets of $\{e_1, \ldots, e_n, -\sum_{i=1}^n e_i\}$. Therefore $X(\Delta_P) = \mathbb{P}^n$.

Exercise 19. Show that if P is the hypercube in $\mathbb{R}^n$ with vertices $\pm e_1^* \pm \ldots \pm e_n^*$, then Δ_P is the fan over the polytope with vertices $\pm e_i$, so $X(\Delta_P) = \prod_{i=1}^n \mathbb{P}^1$.

Exercise 20. Show that if P and P' are rational polyhedra in N and N', respectively, show that $\Delta_{P\times P'} \simeq \Delta_P \times \Delta_{P'}$.

Remark 5. If P is a rational polyhedron in $M_{\mathbb{R}}$ with $\dim(P) < \mathrm{rk}(M)$, then the set of cones $\Delta_P = \{\sigma_Q \mid Q \text{ is a face of } P\}$ is a degenerate fan, which we still call the normal fan to P. Indeed, note first that we may translate P, and therefore we may assume that it contains the origin. If U' is the linear span of P, then $(U')^{\perp}$ is contained in every σ_Q. Moreover, we see that if P' denotes P as a polytope in U', then Δ_P consists of the pull-backs of the cones in $\Delta_{P'}$ via the projection $N_{\mathbb{R}} \to (U')^*$.

It follows that every rational polyhedron P in $M_{\mathbb{R}}$ determines a toric variety $X(\Delta_P)$. In fact, with the above notation we have $X(\Delta_P) = X(\Delta_{P'})$, so $\dim(X(\Delta_P)_{=} \dim(P)$.

The characterization of toric varieties and morphisms. We show now that every variety with a suitable torus action is the toric variety associated to a fan, and moreover, every morphism of toric varieties is induced by a (unique) morphism of the corresponding fans.

Theorem 1. *Let X be a separated variety with an open immersion of a torus T, such that the natural action of the torus on itself extends to an action of T on X. Then there is a fan Δ in a lattice N such that we have an isomorphism $X \simeq X(\Delta)$ inducing an isomorphism of the corresponding tori $T \simeq T_N$, and which is equivariant.*

We will use the affine case proved in Chapter 1 together with the following theorem of Sumihiro (see [Su]).

Theorem. *If T is a torus acting on a normal variety X, then every point of X has an open affine neighbourhood stable under the torus action.*

Proof of Theorem 1. By Sumihiro's theorem we can cover X by open affine sets $U_1, \ldots, U_r$ such that T induces an action on each U_i. Since each U_i intersects $T \subseteq X$ we must have $T \subseteq U_i$.

Let N be the group of one-parameter subgroups of T, so we have a canonical identification $T = T_N$. By Proposition 3 in Chapter 1 we see that for every i there is a semigroup S_i in $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ such that $S_i^{\mathrm{gp}} = M$ and such that the inclusion $T_N \subseteq U_i$ induces $\mathcal{O}(U_i) = k[S_i] \subseteq k[M]$. As X is normal, each S_i is saturated in M, so there are pointed cones σ_i in $N_{\mathbb{R}}$ such that $S_i = \sigma_i^{\vee} \cap M$. Thus we get canonical identifications $U_i = U_{\sigma_i}$ for $i = 1, \ldots, r$.

Since X is separated, for every i and j the intersection $U_i \cap U_j$ is again affine. Applying the previous argument we see that there is a pointed cone $\tau_{i,j}$ in N such that $\mathcal{O}(U_i \cap U_j) = k[\tau_{i,j}^{\vee} \cap M]$.

Since we have $\mathcal{O}(U_i)$, $\mathcal{O}(U_j) \subseteq \mathcal{O}(U_i \cap U_j)$, it follows that $\tau_{i,j} \subseteq \sigma_i \cap \sigma_j$. In fact, we must have equality. Indeed, if v lies in $\sigma_i \cap \sigma_j$, then by Proposition 3 the corresponding one-parameter subgroup f_v of T_N admits extensions $f'_v \colon \mathbb{A}^1 \to U_i$ and $f''_v \colon \mathbb{A}^1 \to U_j$. As X is separated, these extensions must agree and therefore the image has to lie in $U_{\tau_{i,j}}$. Proposition 3 implies that v is in $\tau_{i,j}$ so $\tau_{i,j} = \sigma_i \cap \sigma_j$.

Corollary 4 implies that $\tau_{i,j}$ is a face of both σ_i and σ_j so by taking all the faces of $\sigma_1, \ldots, \sigma_r$ we get a fan Δ. Moreover, the identifications $U_i = U_{\sigma_i}$ glue to give a canonical isomorphism $X \simeq X(\Delta)$ which satisfies our requirements.

Proposition 6. *Let Δ' and Δ be fans in N' and N, respectively. For every morphism of toric varieties $h \colon X(\Delta') \to X(\Delta)$, there is a unique morphism of fans $\phi \colon N' \to N$ such that $h = \phi_*$.*

Proof. By definition of a toric morphism, h induces a morphism of algebraic groups $g \colon T_{N'} \to T_N$. It follows from Proposition 4 in Chapter 1 that there is a unique $\phi \colon N' \to N$ such that $g = \phi_*$, as a morphism between the two tori. In order to prove our assertion it is enough to show that for every cone σ' in Δ' there is σ in Δ such that $\phi(\sigma') \subseteq \sigma$. Indeed, once we have this then $h = \phi_*$ as they agree on the dense subset $T_{N'}$, and as $X(\Delta')$ and $X(\Delta)$ are separated.

Since h is equivariant with respect to the tori actions via g, we see that there is σ in Δ such that $h(O_{\sigma'}) \subseteq O_\sigma$. We show that this cone satisfies our requirement. Indeed, if v' is in σ', then it defines a one-parameter subgroup $f_{v'}$ of $T_{N'}$ which by Proposition 3 extends to $\widetilde{f_{v'}} \colon \mathbb{A}^1 \to X(\Delta')$ such that $\widetilde{f_{v'}}(0) = x_{\sigma'}$.

On the other hand, $g \circ f_{v'}$ is the one-parameter subgroup of T_N correponding to $\phi(v')$. As it can be extended to $h \circ \widetilde{f_{v'}} \colon \mathbb{A}^1 \to X(\Delta)$, another application of Proposition 3 shows that $\phi(v')$ lies in Δ and $h(\widetilde{f_{v'}}(0)) = x_\tau$, where τ is the smallest cone in Δ containing $\phi(\sigma)$. As we know that $h(x_{\sigma'})$ lies in O_σ, it follows that $\tau = \sigma$, hence $\phi(\sigma') \subseteq \sigma$.

Remark 6. We can rephrase the above results by saying that the category of toric varieties is equivalent with the category of pairs (N, Δ) where Δ is a fan in N.

Exercise 21. Let Δ' be a refinement of the fan Δ in N. If $f \colon X(\Delta') \to X(\Delta)$ is the corresponding morphism of toric varieties, show that the largest open subset U of $X(\Delta)$ such that $f^{-1}(U) \to U$ is an isomorphism is given by $X(\Sigma)$, where Σ is the fan consisting of those cones in Δ which lie also in Δ'.

Exercise 22. Show that if $f\colon X(\Delta') \to X(\Delta)$ is a proper birational morphism of toric varieties, then for every irreducible invariant subvariety Y of $X(\Delta)$ there is an irreducible invariant subvariety Y' of $X(\Delta')$ such that f induces a proper, birationl morphism $Y' \to Y$.

Exercise 23. Consider the fans Δ' and Δ in the lattices N' and N, respectively. A morphism of groups $\phi\colon N' \to N$ induces a rational map f from $X(\Delta')$ to $X(\Delta)$.

(1) Show that the largest open subset of $X(\Delta')$ on which f is defined is $X(\Sigma)$, where Σ is the set of cones σ' in Δ' for which there is σ in Δ such that $\phi(\sigma') \subseteq \sigma$.
(2) Suppose that $\phi(|\Delta'|) \subseteq |\Delta|$. Show that there is a toric variety Y and a proper, birational morphism $g\colon Y \to X(\Delta')$ such that $f \circ g$ is a morphism, and such that (Y, g) are universal with these properties.
(3) If f is the rational morphism from $\mathbb{P}^2$ to itself given by $(x\colon y\colon z) \to (yz\colon xz\colon xy)$, describe the corresponding g and $f \circ g$ as morphisms of toric varieties.

Solutions and hints for selected exercises

Exercise 9. It is enough to show that if σ is a singular cone in Δ, then x_τ is a singular point of $X(\Delta)$. For this we may replace $X(\Delta)$ by the open subset U_τ, in which case the assertion follows from Proposition 9 in Chapter 1.

Exercise 10. We may replace $X(\Delta)$ by U_τ. After we reduce to the case when τ has dimension equal to $\mathrm{rk}(N)$, the formula follows from Proposition 9 in Chapter 1.

Exercise 17. We use Proposition 3. Suppose that we have a morphism $\Phi\colon X(\Delta) \times X(\Delta) \to X(\Delta)$ extending the multiplication on T_N. It is enough to show that there is only one maximal cone in Δ. If σ_1 and σ_2 are two smaximal cones, let τ be the cone in Δ such that $\Phi(x_{\sigma_1}, x_{\sigma_2})$ is in O_τ. Let v_1 and v_2 be in the relative interior of σ_1 and σ_2, respectively. If f_{v_1}, f_{v_2} and $f_{v_1+v_2}$ are the one-parameter subgroups corresponding to v_1, v_2 and $v_1 + v_2$, respectively, then we have $\Phi(f_{v_1}(t), f_{v_2}(t)) = f_{v_1+v_2}(t)$ for every t in k^*. It follows that if $\widetilde{f}_{v_1}$ and $\widetilde{f}_{v_2}$ are the corresponding extensions as morphisms $\mathbb{A}^1 \to X(\Delta)$, then $\Phi(x_{\sigma_1}, x_{\sigma_2})$ is equal to x_τ. Therefore $v_1 + v_2$ lies in τ for every v_1 and v_2 in the relative interiors of σ_1 and σ_2, respectively, so σ_1 and σ_2 are both contained in τ, a contradiction.

Exercise 21. The largest such U is invariant under the torus action, so by Exercise 7 it is of the form $X(\Sigma)$ for some subfan Σ of Δ. Moreover, $f^{-1}(X(\Sigma)) = X(\Sigma')$ where Σ' is the set of cones σ' in Δ' that are contained in some cone in Σ. Since $X(\Sigma') \to X(\Sigma)$ is an isomorphism if and only if every cone in Σ is in Σ', the assertion follows.

Exercise 22. If τ is the cone in Δ such that $Y = V(\tau)$, choose a cone τ' in Δ' contained in τ and having the same dimension as τ. The subvariety $Y' = V(\tau')$ of $X(\Delta')$ satisfies the required conditions.

Exercise 23. The largest open subset on which f is defined is invariant under the torus action, so by Exercise 7 it is of the form $X(\Sigma')$ for some subfan Σ' of Δ'. It is clear that Σ is a subfan of Σ', while the reverse inclusion follows from Proposition 6. This gives (1).

Let $\widetilde{\Delta}$ be the fan consisting of cones of the form $\sigma' \cap \phi^{-1}(\sigma)$, where σ' and σ are cones in Δ' and Δ, respectively. Since $\phi(|\Delta'|) \subseteq |\Delta|$, it follows that the induced morphism $g\colon X(\widetilde{\Delta}) \to X(\Delta')$ is proper, and it is clear that $f \circ g$ is a morphism. The universal property follows using (1).

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