Chapter 3: Singularities of toric varieties I

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Notation. We introduce the notation for affine toric varieties which we will use from now on. As in Chapter 2, we fix a **lattice**, i.e. a finitely generated free abelian group N, and let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. The corresponding duals will be denoted by M and $M_{\mathbb{R}}$.

Let σ be a rational, polyhedral convex cone in $N_{\mathbb{R}}$ (a **cone**, for short). We assume that σ is pointed, so $S_\sigma := M \cap \sigma^\vee$ is an integral, finitely generated semigroup, which is saturated in $(S_\sigma)^{\mathrm{gp}} = M$. We denote by U_σ the corresponding affine toric variety $\mathrm{Spec}(k[S_\sigma])$. We also put $T_N := \mathrm{Spec}(k[M])$.

In order to study U_σ we may reduce ourselves to the case when $N_{\mathbb{R}}$ is the linear span of σ. Indeed, if W is the linear span of σ, then $N' := N \cap W$ is a lattice with $W = N'_{\mathbb{R}}$, and such that N/N' is torsion-free. If we choose a splitting of $N' \hookrightarrow N$, and if σ' denotes σ as a cone in W, then we have $U_\sigma \simeq U_{\sigma'} \times (k^*)^{n-n'}$, where $n = \mathrm{rank}(N)$ and $n' = \mathrm{rank}(N')$.

In particular, whenever we study the singularities of U_σ it is enough to consider the case when $N_{\mathbb{R}}$ is the linear span of σ.

Two general properties of toric varieties are that they are Cohen-Macaulay and they have rational singularities. As we will see, under certain assumptions on $\mathrm{char}(k)$, an affine toric variety is the quotient of a smooth variety by the product of a torus and a finite group. Both assertions on singularities are particular cases of results of Hochster-Roberts and Boutot for quotients of smooth varieties by linearly reductive groups. We will review in this chapter this point of view, but in fact we will prove directly that toric varieties are Cohen-Macaulay (at the end of this chapter) and that they have rational singularities (later).

Cohen-Macaulay and rational singularities. We review now the definitions of Cohen-Macaulay rings and of rational singularities. By definition, a local Noetherian ring $(R, \mathfrak{m})$ is **Cohen-Macaulay** if there is a regular sequence in $\mathfrak{m}$ of length equal to $\dim(R)$. A scheme is Cohen-Macaulay if all its local rings are Cohen-Macaulay. We will use, in fact a different characterization of Cohen-Macaulay rings in terms of local cohomology, which we give below. For details and proofs we refer to [BH].

We recall first a few basic facts about local cohomology. If R is not necessarily local, and if $\mathfrak{a}$ is an ideal in R, then the **local cohomology** functors $\{H^i_{\mathfrak{a}}(-)\}_{i\geq 0}$ are the right derived functors of $H^0_{\mathfrak{a}}(-)$, where $H^0_{\mathfrak{a}}(P)$ is the set of all elements of P with support in the closed subset defined by $\mathfrak{a}$ in $\mathrm{Spec}(R)$. If U is the complement of this set in $\mathrm{Spec}(R)$, and if $\widetilde{P}$ is the quasicoherent sheaf corresponding to P, then we have a long exact sequence relating the cohomology of $\widetilde{P}$ on $\mathrm{Spec}(R)$ and on U with the local cohomology of P. Since the ambient variety is affine, we get an exact sequence

$$0 \to H^0_{\mathfrak{a}}(P) \to P \to H^0(U, \widetilde{P}) \to H^1_{\mathfrak{a}}(P) \to 0,$$

as well as isomorphisms

$$H^i_{\mathfrak{a}}(P) \simeq H^{i-1}(U, \widetilde{P})$$

for every $i \geq 2$.

If $x_1, \ldots, x_r$ are elements of $\mathfrak{a}$ which generate an ideal with the same radical of $\mathfrak{a}$, then the corresponding principal affine open subsets give an open cover of U, and we may compute the cohomology of $\widetilde{M}$ on U using the Čech complex associated to this cover. This shows that $H^i_{\mathfrak{a}}(R)$ is isomorphic to the ith cohomology module of the complex

$$0 \to C^0 \to C^1 \to \ldots \to C^r \to 0,$$

where $C^p = \oplus_{i_1<\ldots<i_p} R_{x_{i_1}\ldots x_{i_p}}$, and the maps are either zero or localization maps with suitable signs, as in the Čech complex. Note that it follows from this formula that local cohomology commutes with localization.

It is known that if R is local and if $\mathfrak{a}$ is the maximal ideal, then the maximal length of a regular sequence in $\mathfrak{a}$ is equal to the smallest i for which $H^i_{\mathfrak{a}}(R)$ is nonzero (see [BH]). Hence R is Cohen-Macaulay if and only if $H^i_{\mathfrak{a}}(R) = 0$ for all $i < \dim(R)$. It is this characterization via local cohomology, and the computation of local cohomology in terms of the Čech complex, which we will use to prove that toric varieties are Cohen-Macaulay.

In the global setting we have the following characterization: if X is a projective variety, then X is Cohen-Macaulay if any only if Serre duality holds on X, which is the case if and only if for some (every) ample line bundle L on X, we have $H^i(X, L^m) = 0$ for all $i < \dim(X)$ and all $m \ll 0$. We refer for this equivalence to [Ha].

Recall that a **resolution of singularities** of a variety X is a proper, birational morphism $\pi\colon X' \to X$ from a smooth variety X'. The variety X has **rational singularities** if there is such a resolution such that

$\pi_*(\mathcal{O}_{X'}) = \mathcal{O}_X$ and $R^i\pi_*(\mathcal{O}_{X'}) = 0$ for all $i > 0$. Note that given π, the first condition is equivalent with the fact that X is normal.

In positive characteristic this notion is not very well behaved. One does not know whether every variety has such a resolution, and more importantly, one does not know whether having the above property for one resolution implies it for all resolutions. However, both these problems disappear if $\mathrm{char}(k) = 0$.

Remark 1. It is known that if $\mathrm{char}(k) = 0$ and if X has rational singularities, then it is Cohen-Macaulay (see [KM], Theorem 5.10). However, this does not hold in general in positive characteristic.

Quotients by linearly reductive groups. We give a brief review of actions of linearly reductive groups. For more details and for proofs see the nice introduction in [LeP], Chapter 6.

Let G be a **linearalgebraicgroup**, i.e. an algebraic group which is an affine variety. A **finite dimensional representation** of G is a finite dimensional vector space V over k with an algebraic action $G \times V \longrightarrow V$ such that for every $g \in G$, multiplication by g gives a linear endomorphism of V. Equivalently, G acts on V via a morphism $G \longrightarrow GL(V)$. Given such a representation, we denote by V^G the set $\{v \in V \mid gv = v \,\text{for all}\, g \in G\}$.

The group G is **linearly reductive** if for every finite dimensional representation V of G, the inclusion $V^G \hookrightarrow V$ splits as a morphism of representations, i.e. there is $p\colon V \longrightarrow V^G$ such that $p(gv) = p(v)$ for all g in G and $v \in V$, and such that p is the identity on V^G. If such a splitting exists, then it is unique. In fact, one can check that if G is linearly reductive, then every inclusion of finite dimensional representations $W \hookrightarrow V$ splits, i.e. the category of finite dimensional representations is semisimple.

Example 1. If G is a finite group with $|G|$ not divisible by $\mathrm{char}(k)$, then G is linearly reductive. Indeed, if V is a finite dimensional representation of G, then we have $p\colon V \to V^G$ given by

$$p(v) = \frac{1}{|G|} \sum_{g \in G} gv.$$

Example 2. If G is a torus, then G is linearly reductive. More generally we will consider a finitely generated abelian group A and let $G = \mathrm{Spec}(k[A])$. There is an isomorphism $A \simeq \mathbb{Z}^d \times B$, where $B = \mathbb{Z}/m_1\mathbb{Z} \times \ldots \times \mathbb{Z}/m_r\mathbb{Z}$ for some positive integers $m_1, \ldots, m_r$. It follows that $G \simeq (k^*)^d \times \prod_{i=1}^r \mathrm{Spec}(k[t]/(t^{m_i} - 1))$. Therefore G is reduced if and only if $\mathrm{char}(k)$ does not divide any of the m_i, or equivalently, if it does not divide the order of the torsion group $\mathrm{tors}(A)$. In this case G is the product of a torus and a finite abelian group.

We assume from now on that G is reduced. In this case, every linear action of G on a finite dimensional vector space can be diagonalized. More precisely, if V is a representation of G with $\dim_k(V) = n$, then there is a decomposition $V = V_1 \oplus \ldots \oplus V_n$ into one-dimensional vector subspaces, and characters $\phi_1, \ldots, \phi_n \colon G \longrightarrow k^*$ of G such that $gv = \phi_i(g)v$ for every $g \in G$ and $v \in V_i$. The main point is that the endomorphism of V given by an element of G can be diagonalized, and since these endomorphisms commute, there is a basis which simultaneously diagonalizes them. See [Bo], Chapter 1, §4 for more details.

Moreover, as in the case of tori, a closed point of G corresponds to a group homomorphism $u \colon A \to k^*$, and the group of characters of G is isomorphic to A. If ϕ_a is the character corresponding to $a \in A$, and if u is a closed point of G, then $\phi_a(u) = u(a)$.

Example 3. The classical groups GL_n, SL_n, Sp_{2n} etc. are linearly reductive in characteristic zero. In fact, in every characteristic these groups are **reductive**, i.e. the maximal connected solvable normal subgroup is a torus. See [Bo] for this and for more about reductive groups. On the other hand, in characteristic zero a linear algebraic group G is linearly reductive if and only if it is reductive (see [Fo], p. 178).

Suppose now that G is a linearly reductive group acting algebraically on the affine variety X. We have an induced action of G on $\mathcal{O}(X)$, and one can show that $\mathcal{O}(X)$ is a union of finite dimensional vector subspaces V_i preserved by the action of G. We can therefore glue the splittings $V_i \to V_i^G$ to get the **Reynolds operator**

$$\rho_X \colon \mathcal{O}(X) \to \mathcal{O}(X)^G.$$

The restriction of ρ_X to $\mathcal{O}(X)^G$ is the identity, and using the uniqueness of the above splittings one can show that ρ_X is a morphism of $\mathcal{O}(X)^G$-modules.

The Reynolds operator is the main ingredient in the following results about the ring of invariants $\mathcal{O}(X)^G$.

Theorem (Hilbert)**.** *If G is a linearly reductive group acting on an affine variety X, then the ring $\mathcal{O}(X)$ is finitely generated.*

Theorem (Hochster-Roberts)**.** *If G is a linearly reductive group acting on a smooth affine variety X, then the ring $\mathcal{O}(X)^G$ is Cohen-Macaulay.*

Theorem (Boutot)**.** *Suppose that* $\mathrm{char}(k) = 0$. *If G is a linearly reductive group acting on an affine variety X which has rational singularities, then* $\mathrm{Spec}(\mathcal{O}(X)^G)$ *has rational singularities.*

If G and X are as above, the variety $\mathrm{Spec}(\mathcal{O}(X)^G)$ is the **categorical quotient** of X by the action of G, and it is denoted by $X//G$. This means that we have a G-invariant morphism $\pi\colon X \longrightarrow X//G$ (i.e. $\pi(gx) = \pi(x)$ for $g \in G$ and $x \in X$) induced by the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$ such that for every G-invariant morphism $p\colon X \to Y$ to a variety Y, there is a unique $q\colon X//G \to Y$ with $q\pi = p$.

In general $X//G$ is not the space of orbits of X under the action of G. In fact, we have $\pi(x) = \pi(y)$ if and only if the closures of the orbits Gx and Gy intersect. One deduces that $X//G$ is in fact the space of orbits (with the quotient topology) if and only if all the orbits of G in X are closed. In this case one says that $X//G$ is a **geometric quotient**. A special case when this happens is when the action of G on X is **free**, i.e. when $gx \neq x$ for all x if g is not the unit element.

Affine toric varieties as quotients. We start with a group $G = \mathrm{Spec}(k[A])$, where A is a finitely generated abelian group with $|\mathrm{tors}(A)|$ not divisible by $\mathrm{char}(k)$. Let U_σ be the affine toric variety corresponding to pointed cone $\sigma \subseteq N_{\mathbb{R}}$, with the corresponding torus $T_N \subseteq U_\sigma$. We have a **toric action** of G on U_σ if there is a morphism of algebraic groups $G \longrightarrow T_N$ inducing an action of G on X.

Proposition 1. *If G has a toric action on U_σ, then the quotient $U_\sigma//G$ has a natural structure of affine toric variety.*

Proof. As in Chapter 1, one can show that the morphism of algebraic groups $G \to T_N$ is induced by a group morphism $p\colon M \to A$. The action of G on U_σ is given by the morphism of k-algebras $\Phi\colon k[S_\sigma] \to k[A] \otimes_k k[S_\sigma]$ defined by $\Phi(\chi^u) = \chi^{p(u)} \otimes \chi^u$ for every $u \in S_\sigma$. A regular function f on U_σ lies in $\mathcal{O}(U_\sigma)^G$ if and only if the following diagram is commutative

$$\begin{tikzcd} G \times U_\sigma \arrow[r] \arrow[d, "\mathrm{pr}_2"'] & U_\sigma \arrow[d, "f"] \\ U_\sigma \arrow[r, "f"] & k, \end{tikzcd}$$

where the top horizontal morphism gives the action of G on U_σ. This is equivalent with $\Phi(f) = 1 \otimes f$ in $k[A] \otimes k[S_\sigma]$. If we write $f =$

$\sum_{u\in S_\sigma} c_u\chi^u$ (with only finitely many nonzero c_u), then the above condition says that

$$\sum_u c_u\chi^{p(u)}\otimes\chi^u = \sum_u c_u 1\otimes\chi^u.$$

Therefore f is in $\mathcal{O}(U_\sigma)^G$ if and only if $c_u = 0$ whenever u is not in $L := \ker(p)$, so $\mathcal{O}(U_\sigma)^G = k[S_\sigma\cap L]$.

Note that the inclusion $L\hookrightarrow M$ induces $N\to L^* = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$, and if τ is the image of σ in $L^*_{\mathbb{R}}$, then $S_\sigma\cap L = \tau^\vee\cap L$. Therefore $S_\sigma\cap L$ is a finitely generated, integral semigroup, saturated in L (note, however, that its associated group might be strictly contained in L). Therefore $U_\sigma/\!/G = \mathrm{Spec}(k[\tau^\vee\cap L])$ is an affine toric variety.

Example 4. Suppose that $G = \mathrm{Spec}(k[A])$ acts linearly on a vector space $V = k^n$, which we identify with $\mathbb{A}^n_k$. We have seen that there is a decomposition $V = V_1\oplus\ldots\oplus V_n$ into one-dimensional vector subspaces on which G acts via characters $\chi_1,\ldots,\chi_n$. If χ_i corresponds to $a_i\in A$, then we have a morphism of groups $p\colon\mathbb{Z}^n\to A$ such that $p(e_i) = a_i$. After we choose nonzero elements in each V_i we get an isomorphism $\mathbb{A}^n\simeq\mathrm{Spec}(k[\mathbb{N}^n])$ such that the action of G is induced by p. We deduce from Proposition 1 and its proof that $\mathbb{A}^n/\!/G$ has a structure of toric variety given by $\mathbb{A}^n/\!/G\simeq\mathrm{Spec}(k[\mathbb{N}^n\cap\ker(p)])$.

We show now that every toric variety U_σ can be obtained, under certain assumptions on the characteristic, as a quotient as above. Suppose that σ spans $N_{\mathbb{R}}$. If $\tau_1,\ldots,\tau_d$ are the rays of σ, then let v_i be the generator of the semigroup $\tau_i\cap N\simeq\mathbb{N}$. Since the v_i generate $N_{\mathbb{R}}$, it follows that $\iota\colon M\to\mathbb{Z}^d$ given by $\iota(u) = (\langle u,v_1\rangle,\ldots,\langle u,v_d\rangle)$ is injective. Let $p\colon\mathbb{Z}^d\to A$ be its cokernel and let $G = \mathrm{Spec}(k[A])$. We assume that $\mathrm{char}(k)$ does not divide $|\mathrm{tors}(A)|$, so G is reduced. Note that p induces an embedding $G\hookrightarrow(k^*)^d$, so we get an effective toric action of G on $\mathbb{A}^d$ (recall that the action is called **effective** if the induced group morphism $G\to\mathrm{Aut}(\mathbb{A}^d)$ is injective). The action can be described as follows: if $\phi\colon A\to k^*$ corresponds to a point of G, then $\phi(x_1,\ldots,x_d) = (\phi(p(e_1))x_1,\ldots,\phi(p(e_d))x_d)$.

We clearly have $S_\sigma = \iota^{-1}(\mathbb{N}^d)$, which gives

Corollary 1. *With the above notation, if we consider the standard toric structure on $\mathbb{A}^n$, and the toric action of G on $\mathbb{A}^n$ induced by p, then U_σ is canonically isomorphic to $\mathbb{A}^n/\!/G$.*

Remark 2. Note that without any assumptions on $\mathrm{char}(k)$ the results of Proposition 1 and Corollary 1 hold if we consider G as a group scheme, and if we define $X//G$ as $\mathrm{Spec}(\mathcal{O}(X)^G)$, where a regular function f on X lies in $\mathcal{O}(X)^G$ if and only if the diagram in the proof of Proposition 1 is commutative.

Remark 3. If $N_{\mathbb{R}}$ is not the linear span of σ, then we can find an isomorphism $U_\sigma \simeq U_{\sigma'} \times (k^*)^r$, where σ' is a cone which spans the corresponding vector space. If G is associated to σ' as in Corollary 1, then we get a toric action of G on $\mathbb{A}^d \times (k^*)^r$ such that the quotient is isomorphic to U_σ.

Example 5. The toric variety U_σ is called **simplicial** if σ is generated by linearly independent vectors. In this case G is a finite group, so U_σ is the quotient of a smooth variety by a finite group. Therefore U_σ has quotient singularities. Recall that a variety X has **quotient singularities** if it is locally the quotient of a smooth variety by the action of a finite group; here locally means either the analytic topology (if we work over $\mathbb{C}$) or the étale topology.

Proposition 2. *If we write U_σ as $\mathbb{A}^d \times (k^*)^r//G$ as above, then this is a geometric quotient if and only if σ is simplicial.*

Proof. We use the notation in the proof of Proposition 1. Recall that the condition for being a geometric quotient is that all the orbits of the action of G are closed. To check this we may clearly assume that $r = 0$. If U_σ is simplicial, then G is finite, so the orbits are finite, hence closed.

Suppose now that U_σ is not simplicial, so $d > n = \mathrm{rank}(N)$. Note that the origin in $\mathbb{A}^d$ is a fixed point for the action of G. It follows that if the canonical projection $\pi\colon \mathbb{A}^d \to U_\sigma$ is a geometric quotient, then π has a zero-dimensional fiber. This contradicts the fact that $\dim(\mathbb{A}^d) > \dim(U_\sigma)$.

Remark 4. With the above notation, as the origin in $\mathbb{A}^n$ is fixed by the action of G, we see that the action is free if and only if G is trivial. Equivalently, this means that U_σ is smooth.

Toric surface singularities. If $\mathrm{rank}(N) = 1$, and σ is a pointed cone in $N_{\mathbb{R}}$, then U_σ is isomorphic to $\mathbb{A}^1$ or to k^*. In particular, it is smooth.

Suppose now that $\mathrm{rank}(N) = 2$, and let σ be a pointed cone in $N_{\mathbb{R}}$ of maximal dimension. If σ is singular, then with an appropriate choice of basis e_1, e_2 for N, we may assume that σ is generated by e_2 and $me_1 - \ell e_2$ for some $0 < \ell < m$, with ℓ and m relatively prime. Indeed, every minimal generator along the two rays of σ is part of a basis of N, so we may take one of them to be e_2, and the other one $me_1 + be_2$ for a positive integer m and for some $b \in \mathbb{Z}$. If we replace e_1 by $e_1' = e_1 + ce_2$

for an integer c, then $me_1 + \ell e_2 = me'_1 + (b - cm)e_2$. This shows that b can be arbitrarily changed modulo m, so we may assume $b = -\ell$, $0 \leq \ell < m$. As U_σ is singular we must have $\ell > 0$, while ℓ and m are relatively prime as we have chosen the minimal generator along the ray.

We denote by e_1^* and e_2^* the dual basis of M. The minimal generators along the rays of $\sigma^\vee$ are e_1^* and $\ell e_1^* + me_2^*$. If we put $x = \chi^{e_1^*}$ and $y = \chi^{e_2^*}$, then $k[S_\sigma] = \oplus \ell \cdot x^i y^j$ where the sum is over those (i, j) in $\mathbb{N}^2$ such that $j \leq \frac{m}{\ell} i$.

With the notation before Corollary 1, the map ι is given by $\iota(e_1^*) = (0, m)$ and $\iota(e_2^*) = (1, -\ell)$. Therefore we may take $p\colon \mathbb{Z}^2 \to A = \mathbb{Z}/m\mathbb{Z}$ given by $p(a, b) = \overline{\ell a + b}$. Assuming that $\mathrm{char}(k)$ does not divide m, we may identify G with the group μ_m of mth roots of unity in k, such that the action on $\mathbb{A}^2$ is given by $\eta \cdot (t_1, t_2) = (\eta^\ell t_1, \eta t_2)$, for any element η of μ_m. It follows from Proposition 2 that U_σ is the geometric quotient of the affine plane by this action of μ_m.

Let us consider two particular cases. Suppose first that $\ell = 1$, so $k[S_\sigma]$ has a k-basis given by the monomials $x^i y^j$, with $j \leq mi$. It follows that $k[S_\sigma] = k[x, xy, \ldots, xy^m]$, and if we use the isomorphism $k[x, y] \simeq k[u^m, v/u]$ which takes x and y to u^m and v/u, respectively, then

$$k[S_\sigma] \simeq k[u^m, u^{m-1}v, \ldots, uv^{m-1}, v^m],$$

so U_σ is isomorphic to the affine cone over the rational normal curve in $\mathbb{P}^m$. In this case the action of μ_m on $\mathbb{A}^2$ is given by $\eta \cdot (t_1, t_2) = (\eta t_1, \eta t_2)$.

Consider now the case when $\ell = m - 1$, so the action of μ_m on $\mathbb{A}^2$ is given by $\eta \cdot (t_1, t_2) = (\eta^{-1} t_1, \eta t_2)$.

Exercise. Show that in this case $k[S_\sigma]$ is generated by x, xy and $x^{m-1}y^m$. By mapping u, v and w to these generators, we get an isomorphism $k[u, v, w]/(uw - v^m) \simeq U_\sigma$, so we see that U_σ is an A_{m-1}-**singularity**.

Let us consider also a non-simplicial example.

Exercise. Let N be the kernel of the morphism $q\colon \mathbb{Z}^4 \to \mathbb{Z}$ defined by $q(e_1) = q(e_2) = 1$, $q(e_3) = q(e_4) = -1$, so M is the quotient of the dual $\mathbb{Z}^4$ by the sublattice generated by $e_1^* + e_2^* - e_3^* - e_4^*$. If σ is the cone generated by $e_1 + e_3$, $e_1 + e_4$, $e_2 + e_3$ and $e_2 + e_4$, then $\sigma^\vee$ is generated by the classes of e_1^*, e_2^*, e_3^* and e_4^*, and moreover, these elements generate S_σ. Therefore $U_\sigma \simeq \mathrm{Spec}(k[t_1, \ldots, t_4]/(t_1t_2 - t_3t_4))$. Show that $U_\sigma \simeq \mathbb{A}^4/k^*$, where k^* acts on $\mathbb{A}^4$ by $\lambda \cdot (x_1, \ldots, x_4) = (\lambda x_1, \lambda^{-1} x_2, \lambda^{-1} x_3, \lambda x_4)$.

M-graded rings and modules. Let σ be a cone whose linear span is $N_{\mathbb{R}}$, so $\sigma^\vee$ is a pointed cone. In this case we can find $v\colon M \to \mathbb{Z}$ such that $v(S_\sigma) \subseteq \mathbb{N}$ and $S_\sigma \cap v^\perp = \{0\}$. We can use v to associate an $\mathbb{N}$-grading to the standard M-grading on $k[S_\sigma]$: if $u \in S_\sigma$, then $\deg(\chi^u) = \langle u, v\rangle$. Hence the ideal

$$\mathfrak{m} = \oplus_{u \in S_\sigma \smallsetminus \{0\}} k\chi^u$$

corresponding to the unique fixed point under the torus action plays the role of the irrelevant ideal. Moreover, every M-graded $k[S_\sigma]$-module has a structure of $\mathbb{Z}$-graded module over the same ring.

In particular, familiar results in the $\mathbb{N}$-graded setting apply for M-graded modules over $k[S_\sigma]$. For example, the graded version of Nakayama's Lemma implies that if P is an M-graded $k[S_\sigma]$-module which is finitely generated and if $P/\mathfrak{m}P$ is zero, then P is zero. Similarly, if P is a finitely generated, M-graded projective module, then P is free. (In fact, Gubeladze has proved in [Gu] a conjecture of Anderson saying that in the previous assertion one does not need to assume that P is M-graded: in geometric language, all vector bundles on affine toric varieties are trivial; see also [Sw].)

Using similar arguments, we see that in order to prove that $k[S_\sigma]$ is Cohen-Macaulay it is enough to show that its localization at $\mathfrak{m}$ is (see [BH], Exercise 2.1.27 for the corresponding statement in the case of $\mathbb{N}$-graded rings). In light of the description of Cohen-Macaulay rings in terms of local cohomology, it follows that it is enough to check that $H^i_{\mathfrak{m}}(k[S_\sigma]) = 0$ for all $i < n = \mathrm{rank}(N)$.

Local cohomology of semigroup algebras. In this section we give the computation of the graded pieces in the local cohomology of a semigroup ring with respect to an invariant ideal, in terms of the cohomology of a suitable simplicial complex. As a consequence, we get a proof of the following

Theorem 1 (Hochster)**.** *Every affine toric variety is Cohen-Macaulay.*

Let S be an integral, finitely generated semigroup with $M = S^{\mathrm{gp}}$. We want to describe the graded pieces of $H^m_{\mathfrak{a}}(k[S])$ when $\mathfrak{a}$ is an invariant ideal of $R = k[S]$. Let us choose M-homogeneous generators $x_1 = \chi^{u_1}, \ldots, x_r = \chi^{u_r}$ for $\mathfrak{a}$ (or for an ideal with the same radical as $\mathfrak{a}$). The local cohomology $H^m_{\mathfrak{a}}(R)$ is the mth cohomology of the complex $K^\bullet$:

$$0 \to K^0 \to K^1 \to \ldots \to K^r \to 0,$$

where $K^m = \oplus_{i_1 < \ldots < i_m} R_{x_{i_1} \ldots x_{i_m}}$.

Note that this is an M-graded complex so we get an induced grading on the local cohomology. If u is in M, the component of degree u of

$$R_{x_{i_1}\ldots x_{i_m}} = k[S + \mathbb{N}(-u_{i_1}) + \ldots + \mathbb{N}(-u_{i_m})]$$

is nonzero if and only if $u \in S + \mathbb{N}(-u_{i_1}) + \ldots + \mathbb{N}(-u_{i_m})$; in this case it is equal to k.

Let Δ be the full simplicial complex on $\{1, \ldots, r\}$. For every u in M we define the simplicial complex Δ_u on $\{1, \ldots, r\}$ such that I is in Δ_u if and only if u is not in $S + \sum_{i\in I} \mathbb{N}(-u_i)$. We make the convention that if u is in S, then Δ_u is the void complex and if u is not in S, but for all j we have $u \in S + \mathbb{N}(-u_i)$, then Δ_u is the empty complex (recall that the void complex has zero cohomology in all degrees, unlike the empty complex which has nonzero cohomology in degree -1).

Denote by $C^\bullet(\Delta, \Delta_u)$ the complex computing the reduced relative simplicial cohomology of the pair (Δ, Δ_u). It follows from the previous discussion that we have a natural isomorphism $K_u^m \simeq C^{m-1}(\Delta, \Delta_u)$, and an inspection of the maps shows that in fact the complex $K_u^\bullet$ is isomorphic to the translated complex $C^\bullet(\Delta, \Delta_u)[-1]$. This gives an isomorphism between $H^m_{\mathfrak{a}}(k[S])_u$ and the reduced cohomology group $\widetilde{H}^{m-1}(\Delta, \Delta_u; k)$. On the other hand Δ has no reduced cohomology, so using the long exact sequence for the pair (Δ, Δ_u) we get the assertion in the following proposition.

Proposition 3. *If S is a semigroup as above, for every u in M we have a canonical isomorphism*

$$H^m_{\mathfrak{a}}(k[S])_u \simeq \widetilde{H}^{m-2}(\Delta_u; k).$$

We assume from now on that S is saturated in M and let N be the dual lattice of M. Note that a repeated application of Remark 5 in Chapter 2 shows that $I \subseteq \{1, \ldots, r\}$ is in Δ_u if and only if u is not in

$$(S^\vee \cap \bigcap_{i\in I} u_i^\perp)^\vee.$$

Let $w_1, \ldots, w_s$ be the minimal generators of the one-dimensional faces of $S^\vee$ on which u is negative. We see that I is in Δ_u if and only if there is j such that $\langle u_i, w_j\rangle = 0$ for all i in I. For every j we define the set

$$I_j := \{i \le r \mid \langle u_i, w_j\rangle = 0\},$$

so Δ_u is the union of the full simplicial complexes on the sets I_j. Moreover, $I_{j_1} \cap \ldots \cap I_{j_\ell} \neq \emptyset$ if and only if there is i such that $\langle u_i, w_{j_1}\rangle = \ldots = \langle u_i, w_{j_\ell}\rangle = 0$. As the cohomology of a simplicial complex is equal to the cohomology of the nerve of a cover by simplices, we deduce

Corollary 2. *With the above notation, let Γ_u be the simplicial complex on $\{1, \ldots, s\}$ such that J is in Γ_u if and only if there is $i \leq r$ with $\langle u_i, w_j \rangle = 0$ for all $j \in J$ (we keep the convention that Γ_u is the void complex if and only if u is in S). Then*

$$H^m_{\mathfrak{a}}(k[S])_u \simeq \widetilde{H}^{m-2}(\Gamma_u)$$

for all m. In particular, the component of degree u of the local cohomology does not depend on u, but only on the set of one dimensional faces of $S^\vee$ on which u is negative.

Remark 5. If u' is in S, then multiplication by $\chi^{u'}$ gives a map from $H^m_{\mathfrak{a}}(k[S])_u$ to $H^m_{\mathfrak{a}}(k[S])_{u+u'}$. On the other hand, if a ray of $S^\vee$ is negative on $u+u'$, then it is negative also on u, so we get a map of simplicial complexes $\Gamma_{u+u'} \to \Gamma_u$. This induces a morphism in cohomology

$$\widetilde{H}^{m-2}(\Gamma_u; k) \to \widetilde{H}^{m-2}(\Gamma_{u+u'}; k)$$

which via the isomorphisms given by Corollary 2 is identified with the above multiplication map in local cohomology.

Let us suppose now that $\mathfrak{a}$ is a prime ideal, so there is a face F such that $\mathfrak{a} = \oplus_{u \in S \smallsetminus F} k\chi^u$. We need to choose $\chi^{u_1}, \ldots, \chi^{u_r}$ in $\mathfrak{a}$ which generate an ideal whose radical is equal to $\mathfrak{a}$. This means that for a face G of S such that no u_i is in G, we have $G \subseteq F$.

For this, consider the faces $G_1, \ldots, G_r$ of S which are not contained in F and which are minimal with this property. For every $i \leq r$, let u_i be in $G_i \smallsetminus F$ (by minimality of G_i, we get that u_i lies in the relative interior of G_i).

With the notation in Corollary 2, we see that $J \subseteq \{1, \ldots, s\}$ is in Γ_u if and only if $S \cap \bigcap_{j \in J} w_j^\perp$ is not contained in F. Note that the facets $S \cap w_j^\perp$ of S are precisely those for which u does not lie in the corresponding supporting half-space. It follows that Γ_u can be identified with the simplicial complex on the set of such facets, such that J is in Γ_u if and only if the intersection of the facets in J is not contained in F.

Exercise. Consider the case when $M = \mathbb{Z}^4/(e_1 + e_2 - e_3 - e_4)$ and S is the semigroup generated by the classes of e_1, e_2, e_3 and e_4. We have an isomorphism $k[S] \simeq k[t_1, \ldots, t_4]/(t_1t_2 - t_3t_4)$ and let $\mathfrak{a}$ be the prime ideal generated by the classes of t_1 and t_3. Given the class u in M of $(u_1, \ldots, u_4)$, show that the simplicial complex Γ_u is the void complex or it is contractible, unless we are in one of the following two cases.

(1) Γ_u is the empty simplicial complex. This happens if and only if $u_1 + u_3 < 0$, $u_2 + u_3 \geq 0$, $u_1 + u_4 \geq 0$ and $u_2 + u_4 \geq 0$.

In this case the only nonzero local cohomology in degree u is $H^1_{\mathfrak{a}}(k[S])_u \simeq k$.

(2) Γ_u consists of the disjoint union of two points. This is the case if and only if $u_2 + u_3 < 0$, $u_1 + u_4 < 0$ and $u_2 + u_4 \geq 0$. If these inequalities are satisfied, then the only nonzero local cohomology in degree u is $H^2_{\mathfrak{a}}(k[S])_u \simeq k$.

Proof of Theorem 1. We need to show that if S is saturated in $M = S^{\mathrm{gp}}$, then $k[S]$ is Cohen-Macaulay. We may assume that S has no nonzero invertible elements. If $\mathfrak{a}$ is the ideal corresponding to the zero face of S, then it is enough to prove that $H^m_{\mathfrak{a}}(k[S]) = 0$ for all $m < n$, where $n = \dim(S)$.

For u in M, we have to show that $H^m_{\mathfrak{a}}(k[S])_u$ is zero for $m < n$. We may clearly assume that u is not in S. If $\sigma = \mathbb{R}_+S$ is the cone generated by S, then σ is a poined cone, so it can be written as the cone over a polytope P in an affine hyperplane H (see the exercise in Chapter 2, p.13). Note that each face τ of σ is the cone over the face $\tau \cap H$ of P.

Let $\tau_1, \ldots, \tau_s$ be the facets of σ for which u does not lie in the corresponding supporting half-space. We see that Γ_u can be identified with the nerve of the cover of $Q := \bigcup_{i=1}^s (\tau_i \cap H)$ by the contractible sets $\tau_i \cap H$. Therefore the singular cohomology of Q is isomorphic to the simplicial cohomology of Γ_u, hence

$$H^m_{\mathfrak{a}}(k[S])_u \simeq \widetilde{H}^{m-2}(Q; k). \qquad (1)$$

We have to consider two cases. First, suppose that $-u$ lies in σ and let τ be the smallest face of S containing $-u$. We see that u does not belong to the supporting half-space determined by a facet of σ if and only if $-u$ does not belong to that facet. If $\tau = \sigma$, then Q is equal to the boundary $\partial(P)$ of P, which is homeomorphic to an $(n-2)$-dimensional sphere. Therefore $H^m_{\mathfrak{a}}(k[S])_u$ is isomorphic to k if $m = n$, and it is zero, otherwise.

If $\tau \neq \sigma$, note that if w is in $\sigma \smallsetminus \{0\}$, then w does not lie in the union of the facets τ_i not containing $-u$ if and only if the smallest face containing w contains τ. Therefore $\partial(P) \smallsetminus Q$ is the union of the relative interiors of the proper faces of P which contain $\tau \cap H$. One deduces from this that $\partial(P) \smallsetminus Q$ is contractible (and nonempty): if u' is a point in the relative interior of $\tau \cap H$ and if w lies in $\partial(P) \smallsetminus Q$, then the whole line segment $u'w$ is contained in $\partial(P) \smallsetminus Q$. Using (1) and Alexander duality for $Q \subset \partial(P)$ we get $H^m_{\mathfrak{a}}(k[S])_u = 0$ for all m.

Suppose now that $-u$ does not lie in S. It follows that the cone spanned by σ and u is pointed, so we may assume that some positive multiple u' of u lies in the hyperplane H (but not in P). We claim that in this case Q is homeomorphic with an $(n-2)$-dimensional ball, so $H^m_{\mathfrak{a}}(k[S])_u = 0$ for all m.

To check this assertion, we may assume that H is a linear space and that u' is the origin. Then Q is the union of the facets "visible from the origin": these are the points x on the boundary of P such that the line segment $0x$ intersects P only at x. If we write the cone spanned by P as the cone over a polytope P' in an affine hyperplane H' of H, then projecting from the origin to H' gives a homeomorphism between Q and P'. This proves our claim and completes the proof of the theorem.

Remark 6. Combining the above proof with Remark 5 we get also a description of the top cohomology module: this is the graded dual $\mathrm{Hom}^*_k(\oplus_{u\in \mathrm{Int}(S)} k\chi^u, k)$. Equivalently, this can be rephrased by saying that the canonical module of $k[S]$ is

$$\omega_{k[S]} \simeq \oplus_{u\in \mathrm{Int}(S)} k\chi^u.$$

We refer to [BH] for graded canonical modules and for the connection with local cohomology.

References

[Bo] A. Borel, Linear algebraic groups. Second edition. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics 39, 1998.

[Fo] J. Fogarty, *Invariant theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1969

[Gu] J. Gubeladze, Anderson's conjecture and the maximal monoid class over which projective modules are free, Math. USSR Sbornik **63** (1989), 165–180.

[Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York, 1977.

[KM] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties*, Cambridge University Press, Cambridge, 1998, Cambridge Tracts in Mathematics.

[LeP] J. Le Potier, Fibrs vectoriels sur les courbes algbriques, With a chapter by Christoph Sorger. Publications Mathmatiques de l'Universit Paris 7—Denis Diderot **35**, 1995.

[Sw] Gubeladze's proof of Anderson's conjecture in *Azumaya algebras, actions, and modules* (Bloomington, IN, 1990), 215–250, Contemp. Math., 124, Amer. Math. Soc., Providence, RI, 1992.