LECTURE 2. CUBIC HYPERSURFACES I: RATIONALITY OF CERTAIN CUBIC HYPERSURFACES

The main goal of this lecture is to show that if X is a smooth cubic surface (that is, $X \subseteq \mathbf{P}^3$ is a smooth degree 3 hypersurface) over an algebraically closed field k, then X is rational. In fact, we will prove something more general, showing that certain even-dimensional cubic hypersurfaces are rational.

We mention that the situation is much more subtle for cubic surfaces over non-algebraically closed fields (see, for example, [KSC04, Chapter 2]). However, we will not discuss this, at least for now.

As we will see later, Castelnuovo's rationality criterion says that a smooth projective surface X is rational if and only if $h^1(X, \mathcal{O}_X) = 0 = h^0(X, \omega_X^{\otimes 2})$. When X is a degree 3 hypersurface in $\mathbf{P}^3$, we have $\omega_X \simeq \mathcal{O}_X(-1)$ and both conditions in Castelnuovo's criterion are trivial to check. However, we prefer to give a direct geometric proof of rationality in this case, since the ingredients will also come up also in higher dimensions.

1. Lines on cubic surfaces

From now on we fix an algebraically closed field k of arbitrary characteristic. Our goal in this section is to prove the following two results about lines on cubic surfaces. We will give a partial generalization of these results to even-dimensional cubic hypersurfaces in Lecture 4.

Theorem 1.1. *If $X \subseteq \mathbf{P}^3$ is a hypersurface of degree 3, then X contains at least one line.*

Theorem 1.2. *If $X \subseteq \mathbf{P}^3$ is a smooth hypersurface of degree 3, then the following hold:*

a) *X contains precisely 27 lines, and*
b) *X contains at least 2 disjoint lines.*

In order to prove the above theorems, we begin by setting up an incidence correspondence. Let $G = G(2,4)$ denote the Grassmannian parametrizing lines in $\mathbf{P}^3$. This is a smooth variety of dimension 4. Let $\mathbf{P} \simeq \mathbf{P}^{19}$ be the projective space parametrizing all hypersurfaces of degree 3 in $\mathbf{P}^3$. Consider the incidence correspondence

$$M := \{(Y, L) \in \mathbf{P} \times G \mid L \subseteq Y\}.$$

We first show that M is closed in $\mathbf{P} \times G$. This is a standard fact, but we go over this in detail, since we will need later explicit equations. Let us consider, for example, the open subset $V \simeq \mathbf{A}^4 = \operatorname{Spec} k[a_1, a_2, b_1, b_2]$ of G, where a line is described as the linear span of the rows of the matrix

$$\begin{pmatrix} 1 & 0 & a_1 & a_2 \\ 0 & 1 & b_1 & b_2 \end{pmatrix}.$$

The hypersurface corresponding to $c = (c_\alpha)$, where $\alpha = (\alpha_0, \ldots, \alpha_3)$ with $\sum_i \alpha_i = 3$, is defined by

$$f_c = \sum_\alpha c_\alpha x^\alpha.$$

This contains the line corresponding to the above matrix if and only if

$$f_c\big(s(1, 0, a_1, a_2) + t(0, 1, b_1, b_2)\big) = 0 \quad \text{in} \quad k[s, t].$$

We can write

$$f_c(s, t, sa_1 + tb_1, sa_2 + tb_2) = \sum_{i=0}^{3} F_i(a, b, c)s^i t^{3-i},$$

where $a = (a_1, a_2)$ and $b = (b_1, b_2)$, and $M \cap (\mathbf{P} \times V)$ is defined in $\mathbf{P} \times V$ by (F_0, F_1, F_2, F_3).

Note that the two projections on $\mathbf{P} \times G$ restrict to two morphisms

$$p\colon M \to \mathbf{P} \quad \text{and} \quad q\colon M \to G.$$

Lemma 1.3. *The fibers of q are projective subspaces of $\mathbf{P}$, of codimension 4. In particular, M is irreducible, with* $\dim(M) = 19$.

Proof. In order to prove the first assertion, we may assume that L is defined by (x_2, x_3), in which case $q^{-1}(L)$ consists of those degree 3 homogeneous polynomials in $x_0, \ldots, x_3$ whose coefficients of $x_0^3, x_0^2x_1, x_0x_1^2, x_1^3$ are 0. This is a linear subspace of $\mathbf{P}$ of codimension 4. The fact that M is irreducible now follows from the well-known proposition below, while the assertion about the dimension of M follows from the theorem on dimensions of fibers of morphisms. ☐

Proposition 1.4. *Let $f\colon X \to Y$ be a proper morphism of schemes of finite type over a field k. If Y is irreducible and all fibers of f are irreducible, of the same dimension, then X is irreducible.*

Proof. The assertion follows by using the theorem on dimensions of fibers of morphisms, see[Sha13, p. 77, Theorem 8]. Note that while the statement in *loc. cit.* requires X and Y to be projective, the proof only makes use of the fact that f is proper. ☐

Proof of Theorem 1.1. Note that we have the morphism $p\colon M \to \mathbf{P}$ between 19-dimensional varieties. The assertion in the theorem says that p is surjective. Since this is a morphism of projective varieties, it is closed, hence it is enough to show that it is dominant. This follows from the theorem on dimensions of fibers of morphisms if we show that some fiber of p is 0-dimensional; equivalently, we need to exhibit a cubic surface that contains finitely many lines. When $\mathrm{char}(k) \neq 3$, we give such a surface in Example 1.5 below. ☐

Example 1.5. Suppose that $\mathrm{char}(k) \neq 3$ and let X be the Fermat surface in $\mathbf{P}^3$ defined by the equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

Note that the hypothesis on the characteristic implies that X is smooth.

Up to reordering the variables, every line $L \subseteq X$ can be given by equations of the form

$$x_0 = \alpha x_2 + \beta x_3 \quad \text{and} \quad x_1 = \gamma x_2 + \delta x_3,$$

for some $\alpha, \beta, \gamma, \delta \in k$. This line lies on X if and only if

$$(\alpha x_2 + \beta x_3)^3 + (\gamma x_2 + \delta x_3)^3 + x_2^3 + x_3^3 = 0 \quad \text{in} \quad k[x_2, x_3].$$

This is equivalent to the following system of equations:

$$\alpha^3 + \gamma^3 = -1, \alpha^2\beta + \gamma^2\delta = 0, \alpha\beta^2 + \gamma\delta^2 = 0, \text{ and } \beta^3 + \delta^3 = -1.$$

If α, β, γ, δ are all nonzero, then it follows from the third equation that

$$\gamma = -\alpha\beta^2\delta^{-2},$$

and plugging in the second equation, we get

$$\alpha^2\beta + \alpha^2\beta^4\delta^{-4} = 0,$$

which implies $\beta^3 = -\delta^3$, contradicting the fourth equation.

Suppose now, for example, that $\alpha = 0$. We deduce from the second equation that $\gamma\delta = 0$. Moreover, $\gamma^3 = -1$ by the first equation, hence $\delta = 0$ and $\beta^3 = -1$ by the fourth equation. We thus get in this way the 9 lines with the equations

$$x_0 = \beta x_3 \quad \text{and} \quad x_1 = \gamma x_2,$$

where $\beta, \gamma \in k$ are such that $\beta^3 = -1 = \gamma^3$. After permuting the variables, we obtain 2 more sets of lines on X, hence in total we have 27 lines.

Note that there is a pair of disjoint lines on X: consider for example the pair of lines

$$L = (x_0 = \alpha x_2, x_1 = \alpha x_3) \quad \text{and} \quad L' = (x_0 = \beta x_2, x_1 = \beta x_3),$$

for some α and β such that $\alpha^3 = -1 = \beta^3$ and $\alpha \neq \beta$.

Example 1.6. In fact, it is easy to determine precisely how many pairs of disjoint lines there are on the Fermat cubic. More precisely, given any of the 27 lines on the cubic X in Example 1.5, there are precisely 10 of the other 26 lines that intersect this one. In particular, there are precisely 216 pairs of disjoint lines on X.

After relabeling the coordinates and rescaling them by cube roots of 1, we may assume that the line L that we consider has the equations

$$x_0 = -x_1 \quad \text{and} \quad x_2 = -x_3.$$

Consider now one of the remaining 26 lines, let it be L'. We have 3 cases to consider.

Case 1. The line L' has equations

$$x_0 = \alpha x_1 \quad \text{and} \quad x_2 = \beta x_3$$

for some α and β such that $\alpha^3 = -1 = \beta^3$ and $(\alpha, \beta) \neq (-1, -1)$. We have $L \cap L' \neq \emptyset$ if and only if there is a point $(a_0, a_1, a_2, a_3) \in \mathbf{P}^3$ with

$$a_0 = -a_1 = \alpha a_1 \quad \text{and} \quad a_2 = -a_3 = \beta a_3.$$

If $\alpha \neq -1$ (in which case we have two choices for α), then the condition is that $a_0 = a_1 = 0$ and $\beta = -1$. Similarly, if $\beta \neq -1$ (in which case we have two choices for β), we get the condition $a_2 = a_3 = 0$ and $\alpha = -1$. We thus conclude that in this case we have precisely 4 lines that intersect L.

Case 2. The line L' has equations

$$x_0 = \alpha x_2 \quad \text{and} \quad x_1 = \beta x_3$$

for some α and β such that $\alpha^3 = -1 = \beta^3$. In this case, $L \cap L' \neq \emptyset$ if and only if there is $(a_0, a_1, a_2, a_3) \in \mathbf{P}^3$ such that

$$a_0 = -a_1 = \alpha a_2 = -\beta a_3 \quad \text{and} \quad a_2 = -a_3.$$

In this case we see that $\alpha = \beta$ and conversely, if this is the case, we can find $(a_0, \ldots, a_3)$ as required. Therefore in this case we obtain 3 lines that intersect L.

Case 3. The line L' has equations

$$x_0 = \alpha x_3 \quad \text{and} \quad x_1 = \beta x_2$$

for some α and β such that $\alpha^3 = -1 = \beta^3$. This is analogous to Case 2, hence we get 3 more lines that intersect L.

By combining the above 3 cases, we see that 10 out of the remaining 26 lines on X intersect L. This means that precisely 16 do not intersect L, hence we have a total of $\frac{27 \cdot 16}{2} = 216$ pairs of disjoint lines on X.

We now know that every cubic surface in $\mathbf{P}^3$ contains lines. Of course, some cubic surfaces contain infinitely many lines (for example, the projective cones over elliptic curves in $\mathbf{P}^2$). We turn to smooth cubic surfaces. The fact that they have finitely many lines on them (in fact, precisely 27) will follow easily from the next proposition. Let $U \subseteq \mathbf{P}$ be the open subset that parametrizes smooth cubic surfaces.

Proposition 1.7. *The induced map $p^{-1}(U) \to U$ is finite and étale.*

Proof. Since the morphism is proper, we only need to prove that it is étale. After considering a suitable cover of G by affine charts, we see that it is enough to show that $p^{-1}(U) \cap q^{-1}(V) \to U$ is étale. We have seen that $p^{-1}(U) \cap q^{-1}(V)$ is cut out in $U \times V$ by 4 equations $F_0, \ldots, F_3$. With our previous notation for the coordinates on U and V, we see that we need to show that the Jacobian matrix

$$A = \frac{\partial(F_0, F_1, F_2, F_3)}{\partial(a_1, a_2, b_2, b_2)}(X, L)$$

has rank 4 for every $X \in U$ and $L \in V$ such that $L \subseteq X$. After a suitable change of coordinates, we may assume that L corresponds to the point $(a_1, a_2, b_1, b_2) = (0, 0, 0, 0)$. Recall that $F_0, \ldots, F_3$ were defined such that

$$\sum_{i=0}^{3} s^i t^{3-i} F_i(a, b, c) = f_c(s, t, sa_1 + tb_1, sa_2 + tb_2)$$

for all s and t, where f_c is the equation defining X. By differentiating this with respect to a_1, we obtain

$$\frac{\partial}{\partial a_1}\left(\sum_{i=0}^{3} s^i t^{3-i} F_i(a,b,c)\right)\Big|_{a=b=0} = s\frac{\partial f_c}{\partial x_2}(s,t,0,0).$$

Therefore the first column of A corresponds to the list of coefficients of $s\frac{\partial f_c}{\partial x_2}(s,t,0,0)$. Similarly, the other 3 columns correspond to the list of coefficients of

$$s\frac{\partial f_c}{\partial x_3}(s,t,0,0), t\frac{\partial f_c}{\partial x_2}(s,t,0,0), t\frac{\partial f_c}{\partial x_3}(s,t,0,0).$$

Therefore the matrix A has rank less than 4 if and only if we can find $(\lambda_1,\lambda_2,\mu_1,\mu_2) \neq (0,0,0,0)$ such that

$$(\lambda_1 s + \mu_1 t)\frac{\partial f_c}{\partial x_2}(s,t,0,0) + (\lambda_2 s + \mu_2 t)\frac{\partial f_c}{\partial x_3}(s,t,0,0) = 0 \tag{1}$$

in $k[s,t]$.

Note that $\frac{\partial f_c}{\partial x_2}(s,t,0,0)$ and $\frac{\partial f_c}{\partial x_3}(s,t,0,0)$ are homogeneous polynomials of degree 2 in 2 variables, over an algebraically closed field, hence they split as products of linear polynomials. Therefore the condition in (1) says that they have a common factor. This means that there is a point $P = (p_0, p_1, 0, 0) \in L$ such that

$$\frac{\partial f_c}{\partial x_2}(P) = \frac{\partial f_c}{\partial x_3}(P) = 0. \tag{2}$$

On the other hand, since L is contained in the hypersurface X defined by f_c, we have $f_c(s,t,0,0) = 0$ for all s and t, hence

$$\frac{\partial f_c}{\partial x_0}(P) = \frac{\partial f_c}{\partial x_1}(P) = 0. \tag{3}$$

It follows from (2) and (3) that P is a singular point of X, contradicting the fact that X was a smooth hypersurface. □

We now prove the second main result of this section.

Proof of Theorem 1.2. Since $p^{-1}(U) \to U$ is finite and étale by Proposition 1.7, and U is connected, it follows that all fibers of this map have the same number of elements. At least when $\mathrm{char}(k) \neq 3$, we exhibited in Example 1.5 a point in U such that the fiber of p over this point has 27 elements. We thus obtain the first assertion in the theorem.

In order to prove the second assertion in the theorem, consider the morphism

$$W := p^{-1}(U) \times_U p^{-1}(U) = \{(X, L_1, L_2) \in U \times G \times G \mid L_1 \subseteq X, L_2 \subseteq X\} \longrightarrow U.$$

It follows from Proposition 1.7 that this morphism is finite and étale. In particular, W is smooth (but possibly disconnected). Our goal is to show that the subset

$$W' := \{(X, L_1, L_2) \in W \mid L_1 \cap L_2 = \emptyset\}$$

is a union of connected components of W.

Let Z be the closed subset of $G \times G$ consisting of pairs (L_1, L_2) such that $L_1 \cap L_2 \neq \emptyset$. If $g\colon Z \to G$ is induced by the first projection, then for every $L \in G$, we have

$$g^{-1}(L) = \{L' \in G \mid L \cap L' \neq \emptyset\}.$$

Note first that this is an irreducible subset of G, of dimension 3. Indeed, the closed subset

$$Q_L := \{(P, L') \in L \times G \mid P \in L'\}$$

of $L \times G$ has a morphism to L induced by the first projection, such that every fiber is isomorphic to $\mathbf{P}^2$. Since all these fibers are irreducible, of dimension 2, it follows using Proposition 1.4 that Q_L is irreducible, of dimension 3. The second projection gives a morphism $Q_L \to G$ whose image is $g^{-1}(L)$, hence $g^{-1}(L)$ is irreducible. Moreover, the general fiber of this map consists of 1 point, hence $\dim(g^{-1}(L)) = 3$.

Since g has 3-dimensional, irreducible fibers, it follows that Z is irreducible and $\dim(Z) = 7$ (we use again Proposition 1.4). Consider now the closed subset

$$R = \{(X, L_1, L_2) \in \mathbf{P} \times Z \mid L_1 \subseteq Z, L_2 \subseteq Z\}$$

of $\mathbf{P} \times Z$. By projecting onto the second component, we obtain a morphism $h\colon R \to Z$ whose fibers are all projective spaces of codimension 7 in $\mathbf{P}$; indeed, if

$$L_1 = (x_0 = x_1 = 0) \quad \text{and} \quad L_2 = (x_0 = x_2 = 0),$$

then the cubic hypersurfaces containing L_1 and L_2 are those for which the coefficients of

$$x_2^3, x_2^2x_3, x_2x_3^2, x_3^3, x_1^3, x_1^2x_3, x_1x_3^2$$

vanish. We then conclude that R is irreducible of dimension $7 + (19 - 7) = 19$.

On the other hand, note that we have a closed embedding

$$R \hookrightarrow M \times_{\mathbf{P}} M.$$

By dimension considerations, it follows that if we restrict this embedding over U, the image is a connected component of W. The complement of this image is W', which shows that W' is a union of connected components of W.

The only thing left to note is that W' is nonempty: at least when $\mathrm{char}(k) \neq 3$, this follows from Example 1.5. Since the map $W' \to U$ is finite and étale, it is surjective, giving the second assertion in the theorem. In fact, we get more: since all fibers of $W' \to U$ have the same number of elements, it follows from Example 1.6 that for every smooth cubic surface X, we have precisely 216 pairs of disjoint lines on X. □

2. Rationality of some even-dimensional cubic hypersurfaces

We now prove a general rationality result about even-dimensional cubic hypersurfaces.

Theorem 2.1. *If $X \subseteq \mathbf{P}^{2m+1}$ is a cubic hypersurface over a field k, with $m \geq 1$, such that X contains two disjoint m-dimensional linear subspaces, then X is rational.*

Proof. Let Λ_1 and Λ_2 be the two disjoint m-dimensional linear subspaces of X.

Claim. For every point $Q \in \mathbf{P}^{2m+1} \smallsetminus (\Lambda_1 \cup \Lambda_2)$, there is a unique line L_Q such that $Q \in L_Q$ and $L_Q \cap \Lambda_1 \neq \emptyset$, $L_Q \cap \Lambda_2 \neq \emptyset$. Moreover, the map

$$\varphi\colon \mathbf{P}^{2m+1} \smallsetminus (\Lambda_1 \cup \Lambda_2) \to \Lambda_1 \times \Lambda_2,\ \varphi(Q) = (L_Q \cap \Lambda_1, L_Q \cap \Lambda_2)$$

is algebraic.

Proof of claim. After a suitable change of coordinates, we may assume that we have coordinates $x_0, \ldots, x_{2m+1}$ on $\mathbf{P}^{2m+1}$ such that

$$\Lambda_1 = (x_0 = \ldots = x_m = 0) \quad \text{and} \quad \Lambda_2 = (x_{m+1} = \ldots = x_{2m+1} = 0).$$

Suppose that $Q = (a_0, \ldots, a_{2m+1}) \not\in \Lambda_1 \cup \Lambda_2$.

Note that if the line L_Q contains Q, then $L_Q \not\subseteq \Lambda_1$ and $L_Q \not\subseteq \Lambda_2$. Suppose now that we have a line L, with $L \not\subseteq \Lambda_1$ and $L \not\subseteq \Lambda_2$ such that L intersects Λ_1 in a point $(0, \ldots, 0, \alpha_1, \ldots, \alpha_{m+1})$ and L intersects Λ_2 in a point $(\beta_1, \ldots, \beta_{m+1}, 0, \ldots, 0)$. Then we have

$$L = \{s(0, \ldots, 0, \alpha_1, \ldots, \alpha_{m+1}) + t(\beta_1, \ldots, \beta_{m+1}, 0, \ldots, 0) \mid (s,t) \in \mathbf{P}^1\},$$

hence $Q \in L$ if and only if there are $s, t \in k$ not both 0 such that

$$(t\beta_1, \ldots, t\beta_m, s\alpha_1, s\alpha_{m+1}) = (a_0, \ldots, a_{2m+1}) \quad \text{in} \quad \mathbf{P}^{2m+1}.$$

Note that by the assumption on Q, both s and t must be nonzero. We see that if $Q \in L$, then

$$L \cap \Lambda_1 = \{(0, \ldots, 0, a_{m+1}, \ldots, a_{2m+1})\} \quad \text{and} \quad L \cap \Lambda_2 = \{(a_0, \ldots, a_m, 0, \ldots, 0)\}.$$

Conversely, if L is the line generated by these two points, then $Q \in L$. This proves the existence and uniqueness of L_Q, as well as the fact that the map φ is algebraic.

By restricting φ to $X \smallsetminus (\Lambda_1 \cup \Lambda_2)$ we obtain a morphism

$$f\colon X \smallsetminus (\Lambda_1 \cap \Lambda_2) \to \Lambda_1 \times \Lambda_2.$$

We will show that f is birational; since $\Lambda_1 \times \Lambda_2$ is clearly rational, this implies that X is rational.

In order to prove that f is birational, we may pass to the algebraic closure of k and thus assume that k is algebraically closed. For every $Q_1 \in \Lambda_1$ and $Q_2 \in \Lambda_2$, consider the line $\overline{Q_1Q_2}$ spanned by Q_1 and Q_2. Note that for $(Q_1, Q_2) \in \Lambda_1 \times \Lambda_2$ general, the line $\overline{Q_1Q_2}$ is not contained in X. Indeed, the set of pairs (Q_1, Q_2) such that $\overline{Q_1Q_2}$ is contained in X is closed in $\Lambda_1 \times \Lambda_2$. If it is not a proper subset, then it follows from the above claim that every line that is not contained in $\Lambda_1 \cup \Lambda_2$ is contained in X, hence $X = \mathbf{P}^{2m+1}$, a contradiction.

It follows that if $(Q_1, Q_2) \in \Lambda_1 \times \Lambda_2$ is general, then $X \cap \overline{Q_1Q_2}$ is a subscheme of degree 3 of a line, containing both Q_1 and Q_2, hence it can contain at most one other point different from Q_1 and Q_2. In other words, a general fiber of f consists of at most 1 point. In characteristic 0, this is enough to show that f is birational.

In positive characteristic, we show that f is birational by an explicit computation of its inverse. As in the proof of the above claim, we choose coordinates on $\mathbf{P}^{2m+1}$ such that

$$\Lambda_1 = (x_0 = \ldots = x_m = 0) \quad \text{and} \quad \Lambda_2 = (x_{m+1} = \ldots = x_{2m+1} = 0).$$

Since Λ_1 and Λ_2 are contained in X, it follows that we can write an equation for X as

$$f = \sum_{i=0}^{m} \sum_{j=m+1}^{2m+1} x_i x_j \ell_{i,j}(x),$$

for some linear forms $\ell_{i,j}$. If $Q_1 = (0, \ldots, 0, \alpha_0, \ldots, \alpha_m)$ and $Q_2 = (\beta_0, \ldots, \beta_m, 0, \ldots, 0)$, then $\overline{Q_1 Q_2} \cap X$ is isomorphic to the subscheme of $\mathbf{P}^1$ (with homogeneous coordinates s, t) defined by $f(s\beta_0, \ldots, s\beta_m, t\alpha_0, \ldots, t\beta_m)$.

We can write

$$f(s\beta_1, \ldots, s\beta_{m+1}, t\alpha_1, \ldots, t\alpha_{m+1}) = st \cdot \sum_{i=0}^{m} \sum_{j=0}^{m} \beta_i \alpha_j \ell_{i,m+1+j}(s\beta_0, \ldots, s\beta_m, t\alpha_0, \ldots, t\alpha_m)$$

$$= st(A(\alpha, \beta)s + B(\alpha, \beta)t),$$

where A and B are bihomogeneous polynomials in α and β, of bidegrees $(1, 2)$ and $(2, 1)$, respectively. We thus see that the inverse map g of f is defined on the set

$$\{(\alpha, \beta) \in \Lambda_1 \times \Lambda_2 \mid A(\alpha, \beta) \neq 0 \text{ and } B(\alpha, \beta) \neq 0\}$$

by

$$g(\alpha, \beta) = \big(-\beta_0 B(\alpha, \beta), \ldots, -\beta_m B(\alpha, \beta), \alpha_0 A(\alpha, \beta), \ldots, \alpha_m A(\alpha, \beta) \big),$$

hence it is algebraic. □

Corollary 2.2. *If $X \subseteq \mathbf{P}^3$ is a smooth cubic surface over an algebraically closed field, then X is rational.*

Proof. It follows from Theorem 1.2 that X contains two disjoint lines. We can thus apply Theorem 2.1 with $m = 1$ to conclude that X is rational. □

Remark 2.3. If $X \subseteq \mathbf{P}^3$ is a singular cubic surface over an algebraically closed field, which is reduced and irreducible, then by Example 3.7 in Lecture 1, X either has a point of multiplicity 3, in which case it is the cone over a smooth plane conic, or has a point of multiplicity 2, in which case it is rational.

Remark 2.4. If $X \subseteq \mathbf{P}^3$ is a smooth cubic surface, then by applying the argument in the proof of Theorem 2.1, we have a birational map $X \dashrightarrow \mathbf{P}^1 \times \mathbf{P}^1$. One can show that this is in fact a morphism, which is the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1$ at 5 points.

Remark 2.5. It is well-known (and easy to see) that the blow-up W of $\mathbf{P}^1 \times \mathbf{P}^1$ at one point is isomorphic to the blow-up of $\mathbf{P}^2$ at two points P and Q (such that the exceptional divisor on W corresponds to the strict transform of the line $\overline{PQ}$). The assertion in the previous remark thus implies that a smooth cubic surface X is isomorphic to the blow-up $g \colon Y \to \mathbf{P}^2$ at 6 points. It is easy to see that since

$$\omega_Y^{-1} \simeq g^* \mathcal{O}_{\mathbf{P}^3}(3)(-E_1 - \ldots - E_6)$$

is ample (where $E_1, \ldots, E_6$ are the components of the exceptional divisor), the 6 points are general, in the sense that they do not lie on a conic and no 3 of them lie on a line. Conversely, if Σ is a set of 6 points in $\mathbf{P}^2$ that are general,in the above sense, and $g\colon Y \to \mathbf{P}^2$ is the blow-up along Σ, then one can show that ω_Y^{-1} is very ample and embeds Y as a cubic surface in $\mathbf{P}^3$ (see [Har77, §V.4]).

Remark 2.6. We will see in Lecture 4 that if X is a hypersurface of degree $d \geq 2$ in $\mathbf{P}^n$ and if L is an r-dimensional linear subspace contained in the smooth locus of X, then $r \leq \frac{n-1}{2}$. Moreover, if $d \geq 3$ and $r = \frac{n-1}{2}$, then the hypersurface X contains in its smooth locus only finitely many r-dimensional subspaces. Using this, we will deduce that if $r \geq 1$ and $\mathbf{P}$ is the projective space parametrizing cubic hypersurfaces in $\mathbf{P}^{2r+1}$ (hence $\dim(\mathbf{P}) = \binom{2r+4}{3} - 1 = \frac{2}{3}(r+1)(r+2)(2r+3) - 1)$, then the subset of $\mathbf{P}$ consisting of smooth cubic hypersurfaces that contain two disjoint r-dimensional linear subspaces is an irreducible, constructible subset of $\mathbf{P}$, whose closure has dimension $(r+1)^2(r+4) - 1$.

In particular, we see that in the space of smooth cubic hypersurfaces in $\mathbf{P}^5$, we have an irreducible, codimension 2 closed subset F, whose general element consists of a hypersurface that contains two disjoint 2-planes, and therefore is rational. Similar arguments show that the set of smooth cubic hypersurfaces in $\mathbf{P}^5$ containing *one* 2-plane is an irreducible divisor D in the space of all smooth cubics, hence F is a divisor in D. We mention that Brendan Hassett constructed in [Has99] an infinite family of divisors in D and it is expected that the rational smooth cubic 4-folds correspond precisely to the points in these divisors in D. However, even the rationality of these cubic hypersurfaces is known in only a few cases.

Remark 2.7. Finally, we note that no examples of rational odd-dimensional smooth cubic hypersurfaces are known. Moreover, no examples of rational smooth hypersurfaces in $\mathbf{P}^n$ of degree $d \geq 4$ are known.

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